Problem I: Linear Algebra for QM

(a)

Given two vectors written in the $\{\hat{e}_1, \hat{e}_2\}$ basis set :

$$\begin{aligned} \vec{A} &= 7\hat{e}_1 + 6\hat{e}_2 \\ \vec{B} &= -2\hat{e}_1 + 16\hat{e}_2 \end{aligned} \tag{1}$$

and given another basis set:

$$\begin{aligned} \hat{e}_q &= \frac{1}{2}\hat{e}_1 + \frac{\sqrt{3}}{2}\hat{e}_2 \\ \hat{e}_p &= -\frac{\sqrt{3}}{2}\hat{e}_1 + \frac{1}{2}\hat{e}_2 \end{aligned} \tag{2}$$

- Show that $\hat{e}_p$ and $\hat{e}_q$ are orthonormal.
- Determine the new components of $\vec{A}$ and $\vec{B}$ in the $\{\hat{e}_q, \hat{e}_p\}$ basis set.

(b)

If the states $\{|1>, |2>, |3>\}$ form an orthogonal basis and if the operator $\hat{G}$ has the properties:

$$\begin{aligned} \hat{G}|1> &= 2|1> -4|2> +7|3> \\ \hat{G}|2> &= -2|1> +3|3> \\ \hat{G}|3> &= 11|1> +2|2> -6|3> \end{aligned} \tag{3}$$

What is the matrix representation of $\hat{G}$ in the $|1>, |2>, |3>$ basis?

(c)

Given the matrix:

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \tag{4}$$

Find the eigenvalues and the normalized eigenvectors of A.

(d)

Find the eigenvalues and the normalized eigenvectors of the Matrix:

$$A = \begin{pmatrix} 1 & 2 & 4 \\ 2 & 3 & 0 \\ 5 & 0 & 3 \end{pmatrix} \tag{5}$$

Are the eigenvectors orthogonal? Comment on this.

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(e)

If the states $\{|1>, |2>, |3>\}$ form an orthonormal basis and if the operator $\hat{K}$ has the following properties:

$$\begin{aligned} \hat{K}|1> &= 2|1> \\ \hat{K}|2> &= 3|2> \\ \hat{K}|3> &= -6|3> \end{aligned} \tag{6}$$

- Write an expression for $\hat{K}$ in terms of its eigenvalues and eigenvectors (projection operators). Use this expression to derive the matrix representing $\hat{K}$ in the $\{|1>, |2>, |3>\}$ basis.
- What is the expectation or average value of $\hat{K}$, defined as $<\alpha|\hat{K}|\alpha>$, in the state:

$$|\alpha>= \frac{1}{\sqrt{83}}(-3|1> +5|2> +7|3>) \tag{7}$$

(f)

Given the matrix:

$$M = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \tag{8}$$

- Find the eigenvalues and the normalized eigenvectors of M.
- The projection operator of eigenstate $|i>$ is given by $\hat{P}_i = |i><i|$. Construct the projection operator for the 3 obtained eigenvalues.
- Verify that the matrix can be written in terms of its eigenvalues and eigenvectors.

(g)

Given the matrix representation of the operators A and B:

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \tag{9}$$

- Are A and B hermitian operators?
- Prove that $[A, B] = 0$.
- Of the sets of operators: $\{A\}, \{B\}, \{A, B\}, \{A^2, B\}$ which form a Complete Set of Commuting Observables (C.S.C.O.).

(h)

Given the operators A and B defined by:

$$A\phi_1 = \phi_1 \qquad A\phi_2 = 0 \qquad A\phi_3 = -\phi_3 \tag{10}$$

$$B\phi_1 = \phi_3 \qquad B\phi_2 = \phi_2 \qquad B\phi_3 = \phi_1 \tag{11}$$

- Write the matrix representation of operators A and B in the $\{\phi_1, \phi_2, \phi_3\}$ basis.
- Give the form of the most general matrix representing an operator which commutes with A. Same question for A^2 and B^2.
- Do A^2 and B form a C.S.C.O.? Give a basis of common eigenvectors.