Problem I: Linear Algebra for QM

(a)

Given two vectors written in the $\{\hat{e}_1, \hat{e}_2\}$ basis set :

$$\begin{aligned} \vec{A} &= 7\hat{e}_1 + 6\hat{e}_2 \\ \vec{B} &= -2\hat{e}_1 + 16\hat{e}_2 \end{aligned} \tag{1}$$

and given another basis set:

$$\begin{aligned} \hat{e}_q &= \frac{1}{2}\hat{e}_1 + \frac{\sqrt{3}}{2}\hat{e}_2 \\ \hat{e}_p &= -\frac{\sqrt{3}}{2}\hat{e}_1 + \frac{1}{2}\hat{e}_2 \end{aligned} \tag{2}$$

- Show that $\hat{e}_p$ and $\hat{e}_q$ are orthonormal.
- Determine the new components of $\vec{A}$ and $\vec{B}$ in the $\{\hat{e}_q, \hat{e}_p\}$ basis set.

(b)

If the states $\{|1\rangle, |2\rangle, |3\rangle\}$ form an orthonormal basis and if the operator $\hat{G}$ has the properties:

$$\begin{aligned} \hat{G}|1\rangle &= 2|1\rangle - 4|2\rangle + 7|3\rangle \\ \hat{G}|2\rangle &= -2|1\rangle + 3|3\rangle \\ \hat{G}|3\rangle &= 11|1\rangle + 2|2\rangle - 6|3\rangle \end{aligned} \tag{3}$$

What is the matrix representation of $\hat{G}$ in the $|1\rangle, |2\rangle, |3\rangle$ basis?

(c)

Given the matrix:

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \tag{4}$$

Find the eigenvalues and the normalized eigenvectors of A.

(d)

Find the eigenvalues and the normalized eigenvectors of the Matrix:

$$A = \begin{pmatrix} 1 & 2 & 4 \\ 2 & 3 & 0 \\ 5 & 0 & 3 \end{pmatrix} \tag{5}$$

Are the eigenvectors orthogonal? Comment on this.

(e)

If the states $\{|1\rangle, |2\rangle, |3\rangle\}$ form an orthonormal basis and if the operator $\hat{K}$ has the following properties:

$$\begin{aligned} \hat{K}|1\rangle &= 2|1\rangle \\ \hat{K}|2\rangle &= 3|2\rangle \\ \hat{K}|3\rangle &= -6|3\rangle \end{aligned} \tag{6}$$

- Write an expression for $\hat{K}$ in terms of its eigenvalues and eigenvectors (projection operators). Use this expression to derive the matrix representing $\hat{K}$ in the $\{|1\rangle, |2\rangle, |3\rangle\}$ basis.
- What is the expectation or average value of $\hat{K}$, defined as $\langle\alpha|\hat{K}|\alpha\rangle$, in the state:

$$|\alpha >= \frac{1}{\sqrt{83}}(-3|1\rangle + 5|2\rangle + 7|3\rangle) \tag{7}$$

(f)

Given the matrix:

$$M = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \tag{8}$$

- Find the eigenvalues and the normalized eigenvectors of M.
- The projection operator of eigenstate $|i\rangle$ is given by $\hat{P}_i = |i\rangle\langle i|$. Construct the projection operator for the 3 obtained eigenvalues.
- Verify that the matrix can be written in terms of its eigenvalues and eigenvectors.

(g)

Given the matrix representation of the operators A and B:

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \tag{9}$$

- Are A and B hermitian operators?
- Prove that $[A, B] = 0$.
- Of the sets of operators: $\{A\}, \{B\}, \{A, B\}, \{A^2, B\}$ which form a Complete Set of Commuting Observables (C.S.C.O.).

(h)

Given the operators A and B defined by:

$$A\phi_1 = \phi_1 \qquad A\phi_2 = 0 \qquad A\phi_3 = -\phi_3 \tag{10}$$

$$B\phi_1 = \phi_3 \qquad B\phi_2 = \phi_2 \qquad B\phi_3 = \phi_1 \tag{11}$$

- Write the matrix representation of operators A and B in the $\{\phi_1, \phi_2, \phi_3\}$ basis.

 [(h)] continued on next page...

- Give the form of the most general matrix representing an operator which commutes with A. Same question for A^2 and B^2.
- Do A^2 and B form a C.S.C.O.? Give a basis of common eigenvectors.

Solutions

(a)

- The scalar product of $\hat{e}_p$ and $\hat{e}_q$ is given by:

$$\begin{aligned}\langle \hat{e}_p | \hat{e}_q \rangle &= \frac{1}{2} \times \left(-\frac{\sqrt{3}}{2}\right) + \frac{\sqrt{3}}{2}\left(\frac{1}{2}\right) \\ &= -\frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{4} = 0\end{aligned}$$

  Similarly: $\langle \hat{e}_p | \hat{e}_p \rangle = \langle \hat{e}_q | \hat{e} q \rangle = 1$

- To obtain $\vec{A}$ and $\vec{B}$ in the basis $\{\hat{e}_q, \hat{e}_p\}$ we apply the following formulas: $\vec{A} = A_q \hat{e}_q + A_p \hat{e}_p$ and $\vec{B} = B_q \hat{e}_q + B_p \hat{e}_p$, where:

$$\begin{aligned}
&A_q = \langle \hat{e}_q | \vec{A} \rangle = 7 \times \left(\frac{1}{2}\right) + 6 \times \left(\frac{\sqrt{3}}{2}\right) = \frac{7}{2} + 3\sqrt{3} \\
&A_p = \langle \hat{e}_p | \vec{A} \rangle = 7 \times \left(\frac{-\sqrt{3}}{2}\right) + 6 \times \left(\frac{1}{2}\right) = 3 - \frac{7\sqrt{3}}{2} \\
&B_q = \langle \hat{e}_q | \vec{B} \rangle = (-2) \times \frac{1}{2} + 16 \times \left(\frac{\sqrt{3}}{2}\right) = -1 + 8\sqrt{3} \\
&B_p = \langle \hat{e}_p | \vec{B} \rangle = (-2) \times \left(\frac{-\sqrt{3}}{2}\right) + 16 \times \left(\frac{1}{2}\right) = -\sqrt{3} + 8
\end{aligned}$$

(b)

The matrix elements of operator $\hat{G}$ (which we shall note G_{ij}) on the basis of states $\{|1\rangle, |2\rangle, |3\rangle\}$ are given by $\hat{G}_{ij}|j\rangle = \sum_i |i\rangle G_{ij}$ so that the expansion coefficients of the transformed basis vector $|j\rangle$ make the j^{th} column of matrix G:

$$G = \begin{pmatrix} 2 & -4 & 7 \\ -2 & 0 & 3 \\ 11 & 2 & -6 \end{pmatrix}$$

In the case of an orthonormal basis we have also: $G_{ij} = \langle i | \hat{G} | j \rangle$. Hereby we briefly illustrate how to compute the first few elements. We should keep in mind that any vector of the basis satisfies the relations: $\langle i|i \rangle = 1$ and $\langle i|j \rangle = 0$ for $i \neq j$. The first element is given by:

$$\langle 1 | \hat{G} | 1 \rangle = \langle 1 | (2|1\rangle - 4|2\rangle + 7|3\rangle) = 2\underbrace{\langle 1|1\rangle}_{=1} - 4\underbrace{\langle 1|2\rangle}_{=0} + 7\underbrace{\langle 1|3\rangle}_{=0} = 2 \times 1 = 2$$

Proceeding the same way for the second element, we get:

$$\langle 1 | \hat{G} | 2 \rangle = \langle 1 | (-2|1\rangle + 3|3\rangle) = -2\langle 1|1\rangle + 3\langle 1|3\rangle = -2 \times 1 = -2$$

(c)

Let us first solve the characteristic equation of matrix A: $det|A - \lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{vmatrix} = 0$$

$$\implies (1-\lambda)[(1-\lambda)^2 - 1] + (-1)[(1-\lambda) - 1] + 1[1 - (1-\lambda)] = 0$$
$$\implies (1-\lambda)(-\lambda)(2-\lambda) + \lambda + \lambda = 0$$
$$\implies -\lambda[(\lambda-1)(\lambda-2) - 2] = -\lambda[\lambda^2 - 3\lambda + 2 - 2] = 0$$
$$\implies -\lambda^2(\lambda - 3) = 0$$

This means that the eigenvalues of matrix A are given by: $\lambda_1 = \lambda_2 = 0$ and $\lambda_3 = 3$. We turn now to the evaluation of the eigenvectors. For $\lambda_3 = 3$, we have: $(A - I\lambda_3)|3\rangle = (A - 3I) = 0$. By taking:

$$|3\rangle = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

we can write:

$$\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

this leads to the system:

$$\begin{cases} -2a + b + c = 0 & (1) \\ a - 2b + c = 0 & (2) \\ a + b - 2c = 0 & (3) \end{cases}$$

Substracting equation (2) from equation (1) leads to: $a = b$ and replacing this equality in (3) we get: $c = a$. However $|3\rangle$ is normalized so that: $a^2 + b^2 + c^2 = 1 \implies a = \frac{1}{\sqrt{3}}$. Finally we get:

$$|3\rangle = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Now let's consider the case $\lambda_1 = \lambda_2 = 0$ with eigenvectors $|c_1\rangle$ and $|c_2\rangle$, the characteristic equation in this case is: $(A - 0I)|c_{1/2}\rangle = A|c_{1/2}\rangle = 0$. This leads to:

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\implies a + b + c = 0$. In this case we could choose a and b and from the obtained equation, c could be determined. As an example:

$$a = 1, b = 0 \implies c = -1 \implies |c_1\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

and choosing:

$$a = 0, b = 1 \implies c = -1 \implies |c_2\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

The obtained eigenvectors aren't orthonormal:

$$\langle c_1 | c_2 \rangle = \frac{1}{2}(1.0 + 0.1 + (-1)(-1)) = \frac{1}{2}$$

We could obtain orthonormality by the Schmidt orthogonalization method:

$$|\bar{c}_2\rangle = |c_2\rangle - \frac{1}{2}|c_1\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 - 1/2 \\ 1 \\ -1 + 1/2 \end{pmatrix}$$

Finally, by normalizing $|\bar{c}_2\rangle$ we get:

$$|\bar{c}_2\rangle = \frac{1}{\sqrt{6}} \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}$$

We note that the choice of constants a, b and c is completely arbitrary and will lead to vectors which are linear combinations of the vectors determined here.

- Try the choice $a = b = 1$ then $a = -b = 1$. Show that the obtained vectors are linear combinations of the ones obtained previously.

(d)

The characteristic equation for matrix A is:

$$det|A - \lambda I| = \begin{vmatrix} 1-\lambda & 2 & 4 \\ 2 & 3-\lambda & 0 \\ 5 & 0 & 3-\lambda \end{vmatrix} = 0$$

$$\implies (1-\lambda)(3-\lambda)^2 + (-2)(2)(3-\lambda) + 4(-5)(3-\lambda) = 0$$
$$\implies (3-\lambda)[(1-\lambda)(3-\lambda) - 4 - 20] = 0$$
$$\implies (3-\lambda)(\lambda^2 - 4\lambda - 21) = (3-\lambda)(\lambda+3)(\lambda-7) = 0$$

The eigenvalues are given by: $\lambda_1 = 3, \lambda_2 = -3$ and $\lambda_3 = 7$.
Let us find the eigenvector of eigenvalue $\lambda_1 = 3$.

$$A|1\rangle = \lambda_1 |1\rangle = 3|1\rangle$$

with

$$|1\rangle = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \tag{12}$$

We could write:

$$\begin{pmatrix} 1 & 2 & 4 \\ 2 & 3 & 0 \\ 5 & 0 & 3 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 3 \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

Which we could write as a system:

$$\begin{cases} -2a + 2b + 4c = 0 \\ 2a + 0.b + 0.c = 0 \\ 5a + 0.b + 0.c = 0 \end{cases}$$

which give: $a = 0$ and $b = -2c$. Since the eigenvectors must be normalized to 1 we have:

$$\begin{aligned} \langle 1|1\rangle = 1 \implies (a^* b^* c^*) \begin{pmatrix} a \\ b \\ c \end{pmatrix} = a^2 + b^2 + c^2 = 1 \\ (-2c)^2 + c^2 = 1 \implies c = \frac{1}{\sqrt{5}} \implies |1\rangle = \frac{1}{\sqrt{5}} \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} \end{aligned} \tag{13}$$

Similarly for $|2\rangle$ one can write: $A|2\rangle = \lambda_2 |2\rangle = -3|2\rangle \implies$:

$$\begin{pmatrix} 1 & 2 & 4 \\ 2 & 3 & 0 \\ 5 & 0 & 3 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = -3 \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$\implies \begin{cases} 4a + 2b + 4c = 0 \\ 2a + 6b = 0 \\ 5a + 6c = 0 \end{cases}$$

From these equations one can write: $b = -a/3$ and $c = -5/6a$. By applying normalization we get:

$$\begin{aligned} &\langle 2|2\rangle = 1 \implies a^2 + b^2 + c^2 = 1 \\ &\implies a^2 + (-a/3)^2 + (-5/6a)^2 = 1 \implies a = \sqrt{\frac{324}{585}} = \frac{6}{\sqrt{65}} \end{aligned}$$

Finally we get: $|2\rangle = \frac{1}{\sqrt{65}} \begin{pmatrix} 6 \\ -2 \\ -5 \end{pmatrix}$.

Similarly for $|3\rangle$ we find: $|3\rangle = \frac{1}{\sqrt{45}} \begin{pmatrix} 4 \\ 2 \\ 5 \end{pmatrix}$.

One can notice that: $\langle 1|2\rangle = \frac{-1}{\sqrt{325}} \neq 0$, $\langle 1|3\rangle = \frac{1}{\sqrt{225}} \neq 0$, $\langle 2|3\rangle = \frac{-5}{\sqrt{2925}} \neq 0$

This is **OK** since A is not hermitian and therefore its eigenvectors needn't to be orthonormal.

(e)

- Using the closure relation $\sum_c |c\rangle \langle c| = 1$, we can write:

$$\begin{aligned} \hat{K} &= \sum_c \hat{K} |c\rangle \langle c| \\ &= \hat{K}|1\rangle \langle 1| + \hat{K}|2\rangle \langle 2| + \hat{K}|3\rangle \langle 3| \end{aligned}$$

  with:

$$\hat{K}|1\rangle = 2|1\rangle \qquad \hat{K}|2\rangle = 3|2\rangle \qquad \hat{K}|3\rangle = -6|3\rangle$$

Any matrix representing an operator written in its basis of own eigenvectors is diagonal with the eigenvalues on the diagonal $\Longrightarrow$:

$$K = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -6 \end{pmatrix}$$

Let's verify this by first computing $|1\rangle\langle 1|$, $|2\rangle\langle 2|$ and $|3\rangle\langle 3|$:

$$|1\rangle\langle 1| = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$|2\rangle\langle 2| = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$|3\rangle\langle 3| = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

which gives:

$$K = 2\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + 3\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} - 6\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -6 \end{pmatrix}$$

- In order to evaluate $\langle K\rangle_\alpha = \langle\alpha|\hat{K}|\alpha\rangle$ we proceed by matrix multiplication:

$$\langle\alpha|\hat{K}|\alpha\rangle = \frac{1}{\sqrt{83}}\begin{pmatrix} -3 & 5 & 7 \end{pmatrix}\begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -6 \end{pmatrix}\frac{1}{\sqrt{83}}\begin{pmatrix} -3 \\ 5 \\ 7 \end{pmatrix}$$

$$= \frac{1}{83}\begin{pmatrix} -3 & 5 & 7 \end{pmatrix}\begin{pmatrix} -6 \\ 15 \\ 42 \end{pmatrix} = -\frac{201}{83}$$

-We could also apply the following formula: $\langle K\rangle_\alpha = \sum_i |\langle i|\alpha\rangle|^2\alpha_i$, where α_i is the component of state α on the basis vector i. We first compute the probabilities:

$$|\langle 1|\alpha\rangle| = \left|\frac{-3}{\sqrt{83}}\right|^2 = \frac{9}{83}$$

$$|\langle 2|\alpha\rangle| = \left|\frac{5}{\sqrt{83}}\right|^2 = \frac{25}{83}$$

$$|\langle 3|\alpha\rangle| = \left|\frac{7}{\sqrt{83}}\right|^2 = \frac{49}{83}$$

which gives:

$$\langle K\rangle_\alpha = 2\times\frac{9}{83} + 3\times\frac{25}{83} - 6\times\frac{49}{83} = \frac{-201}{83}$$

(f)

- $|M - \lambda I| = 0$ therefore:

$$\begin{vmatrix} -\lambda & 1 & 0 \\ 1 & -\lambda & 1 \\ 0 & 1 & -\lambda \end{vmatrix} = 0$$

$$-\lambda(\lambda^2 - 1) - 1(-\lambda) = \lambda(\lambda^2 - 2) = 0$$

The eigenvalues of M are therefore: $\lambda_1 = 0$ $\quad \lambda_2 = \sqrt{2}$ $\quad \lambda_3 = -\sqrt{2}$
The eigenvectors are found as usual:

$$|1\rangle = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, |2\rangle = \begin{pmatrix} 1 \\ \sqrt{2} \\ -1 \end{pmatrix}, |3\rangle = \begin{pmatrix} 1 \\ -\sqrt{2} \\ -1 \end{pmatrix}$$

- The projection operators are:

$$\hat{P}_1 = |1\rangle \langle 1| = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \times \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & -1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

$$\hat{P}_2 = |2\rangle \langle 2| = \frac{1}{2} \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix} \times \frac{1}{2} \begin{pmatrix} 1 & \sqrt{2} & 1 \end{pmatrix} = \frac{1}{4} \begin{pmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 2 & \sqrt{2} \\ 1 & \sqrt{2} & 1 \end{pmatrix}$$

$$\hat{P}_3 = |3\rangle \langle 3| = \frac{1}{2} \begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix} \times \frac{1}{2} \begin{pmatrix} 1 & -\sqrt{2} & 1 \end{pmatrix} = \frac{1}{4} \begin{pmatrix} 1 & -\sqrt{2} & 1 \\ -\sqrt{2} & 2 & -\sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{pmatrix}$$

One can clearly observe that:

$$M = \lambda_1 \hat{P}_1 + \lambda_2 \hat{P}_2 + \lambda_3 \hat{P}_3$$

or:

$$M = 0\hat{P}_1 + \sqrt{2}\hat{P}_2 - \sqrt{2}\hat{P}_3$$

(g)

- Generally a matrix C is hermitian if: $transpose(C^*) = C$, where C^* is the complex conjugate of C. The matrices A and B are real, therefore we only have to check if $transpose(A) = A$ and $transpose(B) = B$. Since A and B are symmetric this implies that $transpose(A) = A$ and $transpose(B) = B \implies A$ and B represent hermitian operators.

- $[A, B] = AB - BA$

$$AB = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}$$

$$BA = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}$$

$$\implies AB - BA = 0 \implies [A, B] = 0$$

- Let us find the eigenvalues of matrices A and B first. Since A is diagonal the elements of the diagonal represent its eigenvalues $\implies \lambda_{A1} = 1, \lambda_{A2} = \lambda_{A3} = -1$. Matrix B is block diagonal, from the B_{11} block we see $b_1 = 1$ with the same eigenvector $|1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ as A. For the 2×2 block B_{ij} with $i, j = 2, 3$, we obtain:

$$\begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = 0 \implies \lambda^2 - 1 = 0 \implies \lambda_{B2} = 1, \lambda_{B3} = -1$$

The relative eigenvectors are: $|2\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $|3\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.
These are still eigenvectors of A (you can try to prove it). Let's keep in mind that a complete set of commuting observables (CSCO) is a set of commuting operators whose eigenvalues completely specify the state of a system. We turn now to check which of the 4 cases constitutes a C.S.C.O. . Since A and B have both a degenerate eigenvalue (1 for A and -1 for B) they are therefore not a C.S.C.O. . However if we examine simultaneously the set of common eigenvectors $\{|1\rangle, |2\rangle, |3\rangle\}$ of A and B in which each vector is characterized by a set of eigenvalues (λ_A, λ_B), we obtain:

$$|1\rangle = |1, 1\rangle, \quad |2\rangle = |-1, 1\rangle, \quad |3\rangle = |-1, -1\rangle$$

Which are distinct from one another. This means that each common eigenvectors of A and B is characterized by a unique combination (λ_A, λ_B). We can deduce from that that $\{A, B\}$ constitutes a C.S.C.O.. The condition $[A, B] = 0$ is necessary but not sufficient so that A and B constitute a C.S.C.O. . Even though $\left[A^2, B\right] = 0$, the last set $\{A^2, B\}$ does not constitute a C.S.C.O. since the eigenvalues of A^2 are $\lambda_{A^21} = \lambda_{A^22} = \lambda_{A^23} = 1$. This means that the state of the system cannot be determined unambiguously by examining the set of eigenvalues (λ_A^2, λ_B)

(h)

- $$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} ; B = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

  $$A^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} , B^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

  these matrices are real and hermitian, they can be diagonalized and they therefore represent observables.

- -Let M be an operator commuting with A. M cannot have any non-null matrix elements between $|\phi_1\rangle$ and $|\phi_2\rangle$, $|\phi_2\rangle$ and $|\phi_3\rangle$ as well as $|\phi_1\rangle$ and $|\phi_3\rangle$. The matrix representing M is therefore diagonal:

  $$[M, A] = 0 \iff M = \begin{pmatrix} m_{11} & 0 & 0 \\ 0 & m_{22} & 0 \\ 0 & 0 & m_{33} \end{pmatrix}$$

  We will demonstrate this by taking two eigenvalues a_1 and a_2 of $\hat{A}$ with $a_1 \neq a_2$:

  $$\hat{A} |\psi_1\rangle = a_1 |\psi_1\rangle$$
  $$\hat{A} |\psi_2\rangle = a_2 |\psi_2\rangle$$

  Since $[M, A] = MA - AM = 0 \implies \langle\psi_2| MA - AM |\psi_1\rangle = (a_2 - a_1) \langle\psi_2|M|\psi_1\rangle = 0$
  however $a_2 \neq a_1$, which gives finally:

  $$\langle\psi_2|M|\psi_1\rangle = 0$$

  -Let N be an operator which commutes with A^2. The matrix representing N can have elements between $|\phi_1\rangle$ and $|\phi_3\rangle$ (eigenvectors of A^2 with the same eigenvalue) but none between $|\phi_2\rangle$ and $|\phi_1\rangle$ or $|\phi_3\rangle$, therefore:

  $$[N, A^2] = 0 \iff N = \begin{pmatrix} n_{11} & 0 & n_{13} \\ 0 & n_{22} & 0 \\ n_{31} & 0 & n_{33} \end{pmatrix}$$

  - Since B^2 is the identity operator any 3×3 matrix commutes with B^2.

- Let 's evaluate $[A^2, B] = A^2B - BA^2$:

  $$A^2B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

  $$BA^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

  $\implies [A^2, B] = 0 \implies A^2$ and B could form a C.S.C.O. if by specifying simultaneously the eigenvalues of operators A and B one can determine a unique common eigenvector.

Note that $|\phi_2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ is a common eigenvector of A and B with eigenvalues 0 and 1 respectively ($\hat{A}\phi_2 = 0\phi_2$ and $\hat{B}\phi_2 = 1\phi_2$).

Since the $\{|\phi_1\rangle, |\phi_3\rangle\}$ subspace is degenerate for A^2 we can diagonalize B in it with the corresponding matrix given by: $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and eigenvalues ± 1. The corresponding eigenvectors are:

$$|u_1\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, |u_2\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

So that each the 3 common eigenvectors $(|u_1\rangle, |u_2\rangle, |\phi_2\rangle)$ has a distinct set of eigenvalues. This means that A and B share a common eigenvector basis in which each basis vector is characterized by a unique set of eigenvalues $(\lambda_{A^2}, \lambda_B)$.