Classification of Partial Differential Equations (PDE)

Second-order PDEs of two variables are of the form:

$$a\frac{\partial^2 f(x,y)}{\partial x^2} + b\frac{\partial^2 f(x,y)}{\partial x \partial y} + c\frac{\partial^2 f(x,y)}{\partial y^2} + d\frac{\partial f(x,y)}{\partial x} + e\frac{\partial f(x,y)}{\partial y} = F(x,y)$$

$b^2 - 4ac < 0$	elliptic	LAPLACE equation
$b^2 - 4ac = 0$	parabolic	DIFFUSION equation
$b^2 - 4ac > 0$	hyperbolic	WAVE equation

Elliptic equations produce **stationary and energy-minimizing** solutions

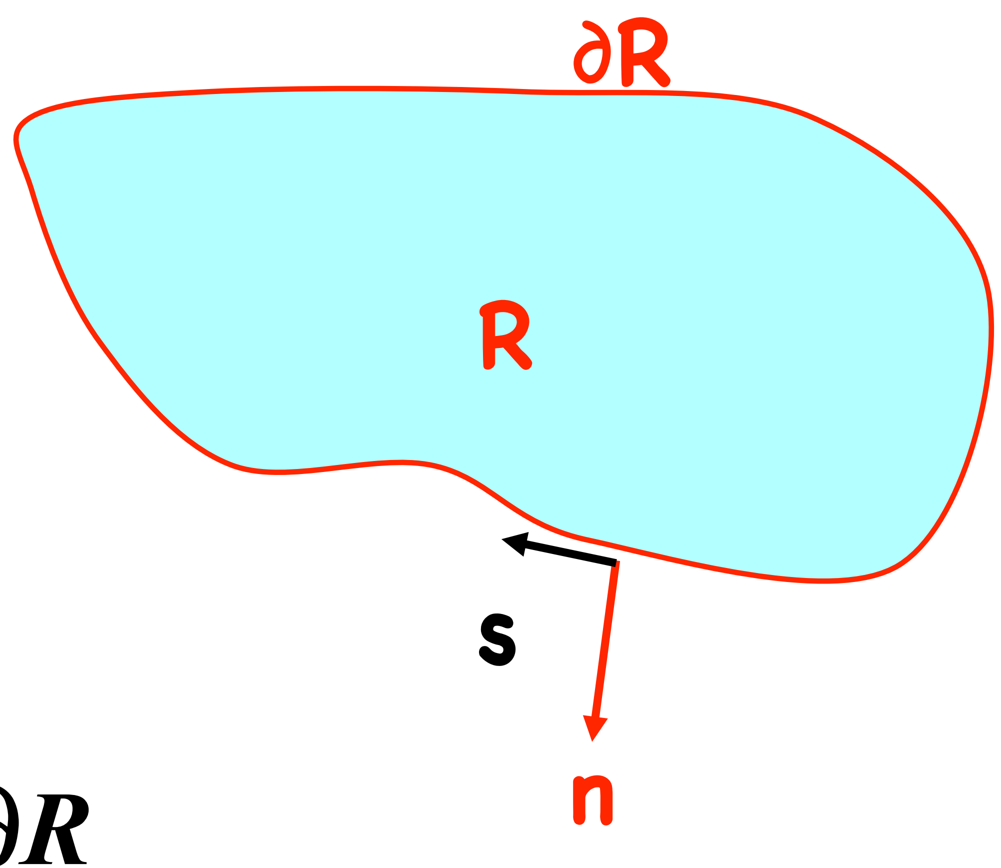
Parabolic equations a **smooth-spreading flow** of an initial disturbance

Hyperbolic equations a **propagating disturbance**

Boundary and Initial conditions

Initial conditions: starting point for propagation problems

Boundary conditions: specified on domain boundaries to provide the interior solution in computational domain



$$\begin{cases} \text{(i) Dirichlet condition: } u = f \text{ on } \partial R \\ \text{(ii) Neumann condition: } \dfrac{\partial u}{\partial n} = f \text{ or } \dfrac{\partial u}{\partial s} = g \text{ on } \partial R \\ \text{(iii) Robin (mixed) condition: } \dfrac{\partial u}{\partial n} + ku = f \text{ on } \partial R \end{cases}$$

Elliptic PDEs

Steady-state two-dimensional heat conduction equation is prototypical elliptic PDE

Laplace equation - homogeneous (no source)

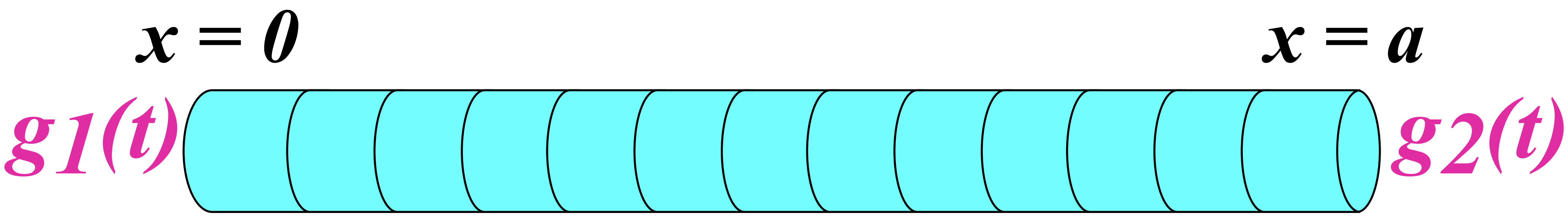
$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

Poisson equation - with source (e.g. heat)

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = f(x, y)$$

Heat Equation: parabolic PDE

Heat transfer in a one-dimensional rod



$$\frac{\partial u}{\partial t} = d\,\frac{\partial^2 u}{\partial x^2}, \qquad 0 \le x \le a, \quad 0 \le t \le T$$

$$\text{I.C.s} \qquad u(x,0) = f(x) \qquad 0 \le x \le a$$

$$\text{B.C.s} \quad \begin{cases} u(0,t) = g_1(t) \\ u(a,t) = g_2(t) \end{cases} \qquad 0 \le t \le T$$

Wave Equation: hyperbolic PDE

$b^2 - 4ac = 0 - 4(1)(-c^2) > 0$: Hyperbolic

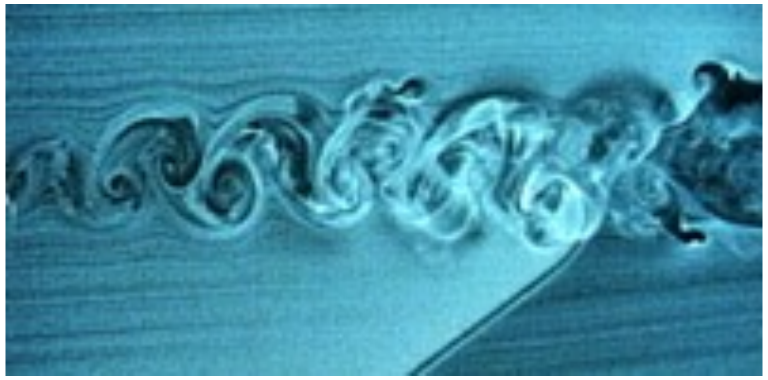
$$\boxed{\frac{\partial^2 u}{\partial t^2} = v^2 \frac{\partial^2 u}{\partial x^2}}, \quad 0 \le x \le a, \quad 0 \le t$$

$$\text{I.C.s} \quad \begin{cases} u(x,0) = f_1(x) \\ u_t(x,0) = f_2(x) \end{cases} \quad 0 \le x \le a$$

$$\text{B.C.s} \quad \begin{cases} u(0,t) = g_1(t) \\ u(a,t) = g_2(t) \end{cases} \quad t > 0$$

Coupled PDE

Navier-Stokes Equations



$$
\begin{cases}
\dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} = 0 \\[2ex]
\dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial x} + v\dfrac{\partial u}{\partial y} = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial x} + \nu\left(\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}\right) \\[2ex]
\dfrac{\partial v}{\partial t} + u\dfrac{\partial v}{\partial x} + v\dfrac{\partial v}{\partial y} = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial y} + \nu\left(\dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2}\right)
\end{cases}
$$