Corso di Laurea in Fisica - UNITS

ISTITUZIONI DI FISICA

PER IL SISTEMA TERRA

Born of the Sound Wave Equation

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What is a **wave**? - 3

Small perturbations of a **stable** equilibrium point → **Linear restoring force** → **Harmonic Oscillation**

Coupling of harmonic oscillators → the disturbances can **propagate**, superpose and stand

WAVE: organized **propagating imbalance**, satisfying differential equations of motion

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 y}{\partial t^2}$$

General form of LWE

Towards sound wave equation...

- Consider a source causing a perturbation in the gas medium rapid enough to cause a pressure variation and not a simple molecular flux.

- The regions where compression (or rarefaction), and thus the density variation of the gas, occurs are larger compared to the mean free path (average distance that gas molecules travel without collisions), otherwise flow would smear the perturbation.

- The perturbation fronts are planes and the displacement induced in the gas, X, depends only on x & t (and not on y, z).

Equilibrium state

The conventional unit for pressure is bar=$10^5 N/m^2$ and the pressure at the equilibrium is: 1atm=1.0133bar

The pressure perturbations associated to the sound wave passage are tipically of the order of 10^{-7}bar, thus very small if compared to the value of pressure at the equlibrium.

One can thus assume that:

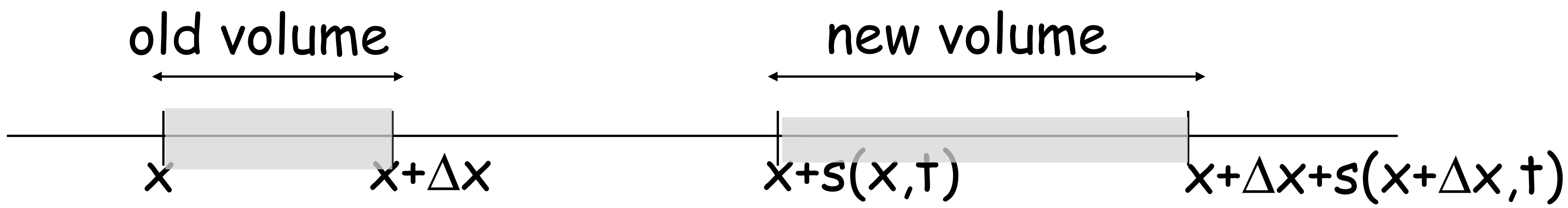
$P=P_0+\Delta P$ $\rho=\rho_0+\Delta\rho$

where ΔP and $\Delta\rho$ are the values of the (small) perturbations of the pressure and density from the equlibrium.

Sound wave equation – 1

The gas moves and causes density variations

Let us consider the displacement field, s(x,t) induced by sound



and considering a unitary area perpendicular to x, direction of propagation, one has that the quantity of gas enclosed in the old and new volume is the same

$$\rho_0 \Delta x = \rho\left[x + \Delta x + s(x+\Delta x) - x - s(x)\right]$$

where, since Δx is small, $s(x+\Delta x) \approx s(x) + \frac{\partial s}{\partial x}\Delta x$

$$\rho_0 \Delta x = (\rho_0 + \Delta\rho)\left[\Delta x + \frac{\partial s}{\partial x}\Delta x\right] = \rho_0 \Delta x + \rho_0 \frac{\partial s}{\partial x}\Delta x + \Delta\rho\Delta x + \ldots$$

thus, neglecting the second-order term, one has:

$$\Delta\rho = -\rho_0 \frac{\partial s}{\partial x}$$

relation between the variation of displacement along x with the density variation. The minus sign is due to the fact that, if the variation is positive the volume increases and the density decreases.

If the displacement field is constant the gas is simply translated without perturbation.

Sound wave equation - 2

Density variations cause pressure variations

The pressure in the medium is related to density with a relationship of the kind **$P=f(\rho)$**,
that at the equilibrium is **$P_0=f(\rho_0)$**.

$$P = P_0 + \Delta P = f(\rho) = f(\rho_0 + \Delta\rho) \approx f(\rho_0) + \Delta\rho f'(\rho_0) = P_0 + \Delta\rho\kappa$$

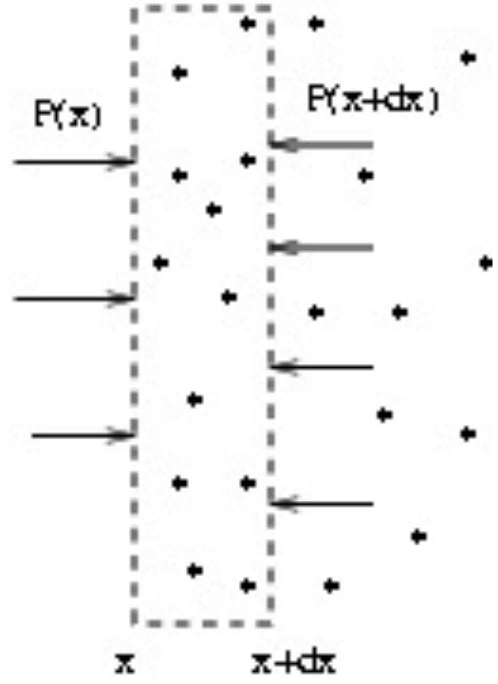
and neglecting second-order terms:

$$\Delta P = \kappa \Delta\rho$$

$$\kappa = f'(\rho_0) = \left(\frac{dP}{d\rho}\right)_0$$

Sound wave equation - 3

Pressure variations generate gas motion



The gas in the volume is accelerated by the different pressure exerted on the two sides...

$$P(x,t) - P(x+\Delta x,t) \approx -\frac{\partial P}{\partial x}\Delta x = -\frac{\partial (P_0 + \Delta P)}{\partial x}\Delta x = -\frac{\partial \Delta P}{\partial x}\Delta x$$

$$= \rho_0 \Delta x \frac{\partial^2 s}{\partial t^2} \quad \text{for Newton's 2nd law}$$

thus:

$$\rho_0 \frac{\partial^2 s}{\partial t^2} = -\frac{\partial \Delta P}{\partial x}$$

Sound wave equation

Using 1, 2 and 3 we have

$$\rho_0 \frac{\partial^2 s}{\partial t^2} = -\frac{\partial \Delta P}{\partial x} = -\frac{\partial\left(\kappa \Delta \rho\right)}{\partial x} = -\frac{\partial\left[\kappa\left(-\rho_0 \frac{\partial s}{\partial x}\right)\right]}{\partial x}$$

thus:

$$\frac{1}{\kappa}\frac{\partial^2 s}{\partial t^2} = \frac{\partial^2 s}{\partial x^2}$$

i.e. the typical wave equation, describing a perturbation traveling with velocity $v = \sqrt{\kappa}$

Sound wave velocity - isothermal

From the sound wave equation

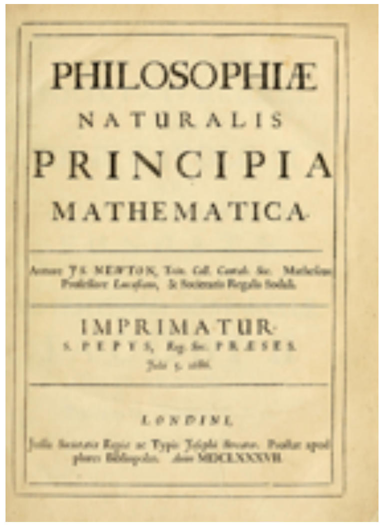
$$v = \sqrt{\kappa} = \sqrt{\left(\frac{dP}{d\rho}\right)_0}$$

Newton computed the derivative of the pressure assuming that the heat is moving from one to another region in a such rapid way that the temperature cannot vary, isotherm, PV=constant i.e. P/ρ=constant, thus

$$v = \sqrt{\left(\frac{dP}{d\rho}\right)_0} = \sqrt{(\mathrm{constant})_0} = \sqrt{\left(\frac{P}{\rho}\right)_0}$$

called **isothermal sound velocity**

I. Newton, "Philosophiæ Naturalis Principia Mathematica", 1687; 1713; 1728..



Sound wave velocity - adiabatic

Laplace correctly assumed that the heat flux between a compressed gas region to a rarefied one was negligible, and, thus, that the process of the wave passage was adiabatic PV^{γ}=constant, P/ρ^{γ}=constant, with γ, ratio of the specific heats: C_p/C_v

$$v = \sqrt{\left(\frac{dP}{d\rho}\right)_0} = \sqrt{\left(\frac{\gamma}{\rho}\mathrm{constant}\,\rho^{\gamma}\right)_0} = \sqrt{\gamma\left(\frac{P}{\rho}\right)_0}$$

called adiabatic sound velocity

P. S. Laplace, "Sur la vitesse du son dans l'air et dans l'eau"
Annales de chimie, 1816, 3: 238-241.

Sound velocity in the air

Using the ideal gas law

$PV=nRT=NkT$

one can write the velocity on many ways:

$$v = \sqrt{\frac{\gamma P}{\rho}} = \sqrt{\frac{\gamma PV}{\rho V}} = \sqrt{\frac{\gamma nRT}{m}} = \sqrt{\frac{\gamma NkT}{Nm_{mol}}} = \sqrt{\frac{\gamma KT}{m_{mol}}} = \sqrt{\frac{\gamma RT}{weight_{mol}}}$$

showing that it depends on **temperature only**. If the "dry" air is considered (biatomic gas γ=7/5) one has:

$v=20.05\ T^{1/2}$ or

$v=331.4+0.6T_c$ m/s (temperature measured in Celsius)

Bulk modulus

It corresponds to the "spring constant" of a spring, and gives the magnitude of the restoring agency (pressure for a gas, force for a spring) in terms of the change in physical dimension (volume for a gas, length for a spring). Defined as an "intensive" quantity:

$$B = -\frac{\Delta P}{\Delta V / V} = -V\frac{dP}{dV}$$

and for an adiabatic process (from the 1st principle of thermodynamics applied to an ideal gas):

$$B = \gamma\ P$$

Sound speed

Sound velocity depends on the compressibility of the medium.

If the medium has a bulk modulus B and density at the equilibrium is ρ, the sound speed is: $\mathbf{v = (B/\rho)^{1/2}}$

that can be compared with the velocity of transversal waves on a string:

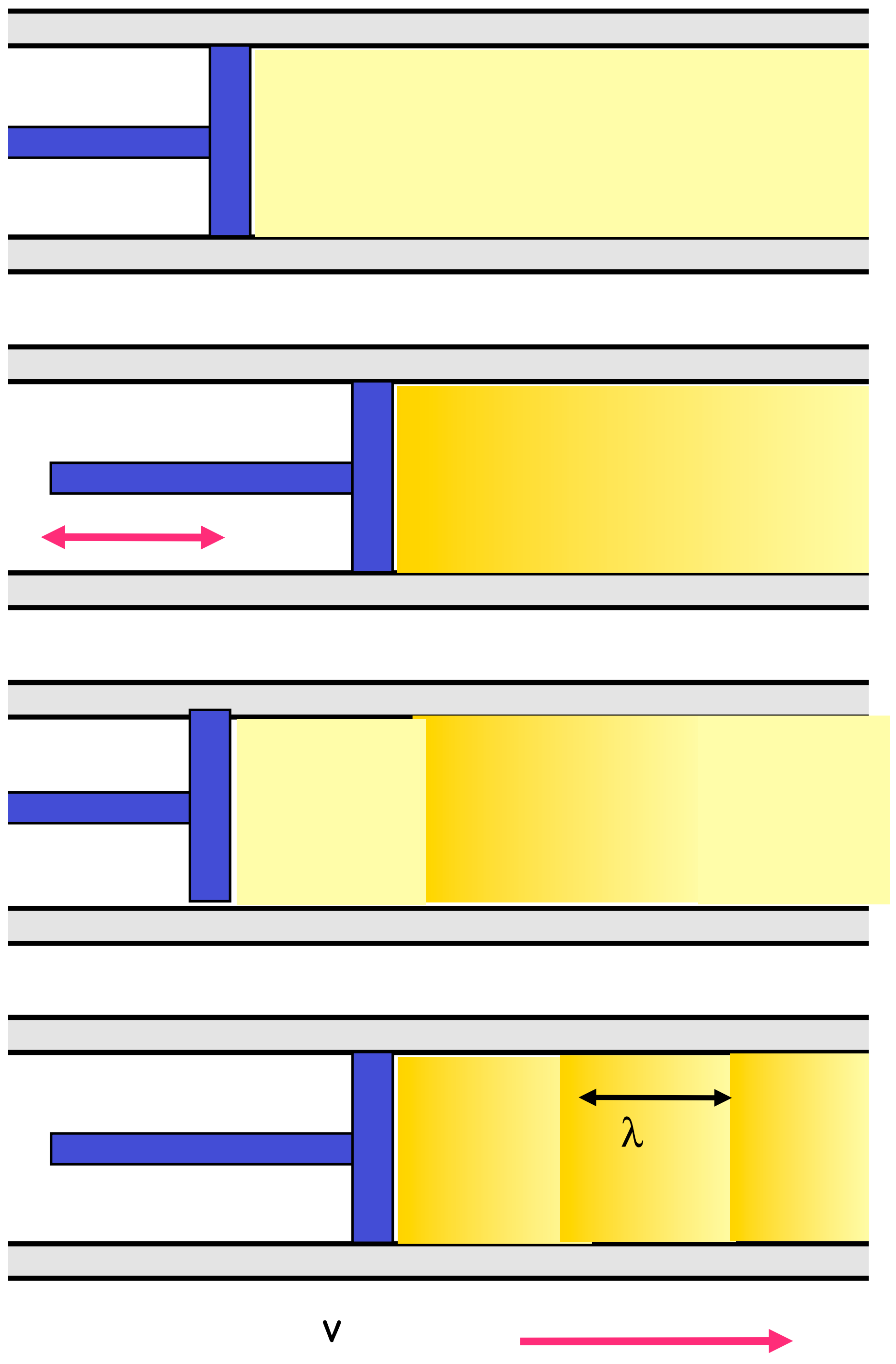
$$v = (F/\mu)^{1/2}$$

Thus, velocity depends on the elastic of the medium (B or F) and on inertial (ρ or μ) properties

Harmonic sound waves

If the source of a longitudinal wave (eg tuning fork,loudspeaker) oscillates with SHM the resulting disturbance will also be harmonic

Consider this system →



As the piston oscillates backwards and forwards regions of compression and rarefaction are set up.

The distance between successive compressions or rarefactions is λ.

Harmonic sound waves

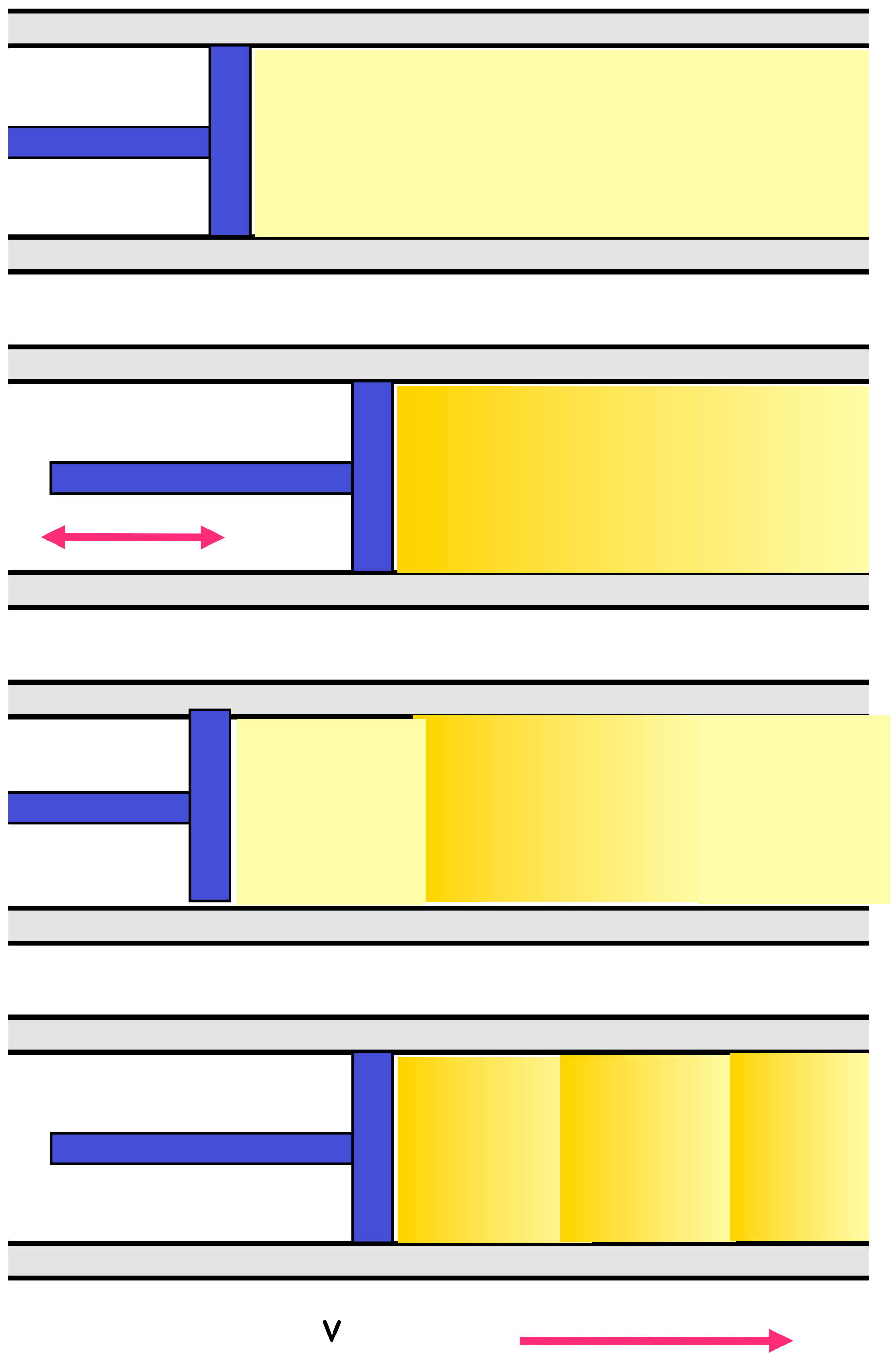
Any small region of the medium moves with SHM, given by

$$s(x,t) = s_m \cos(kx - \omega t)$$

s_m = max displacement from equilibrium

The change of the pressure in the gas, ΔP, measured relative to the equilibrium pressure

$$\Delta P = \Delta P_m \sin(kx - \omega t)$$



Harmonic sound waves

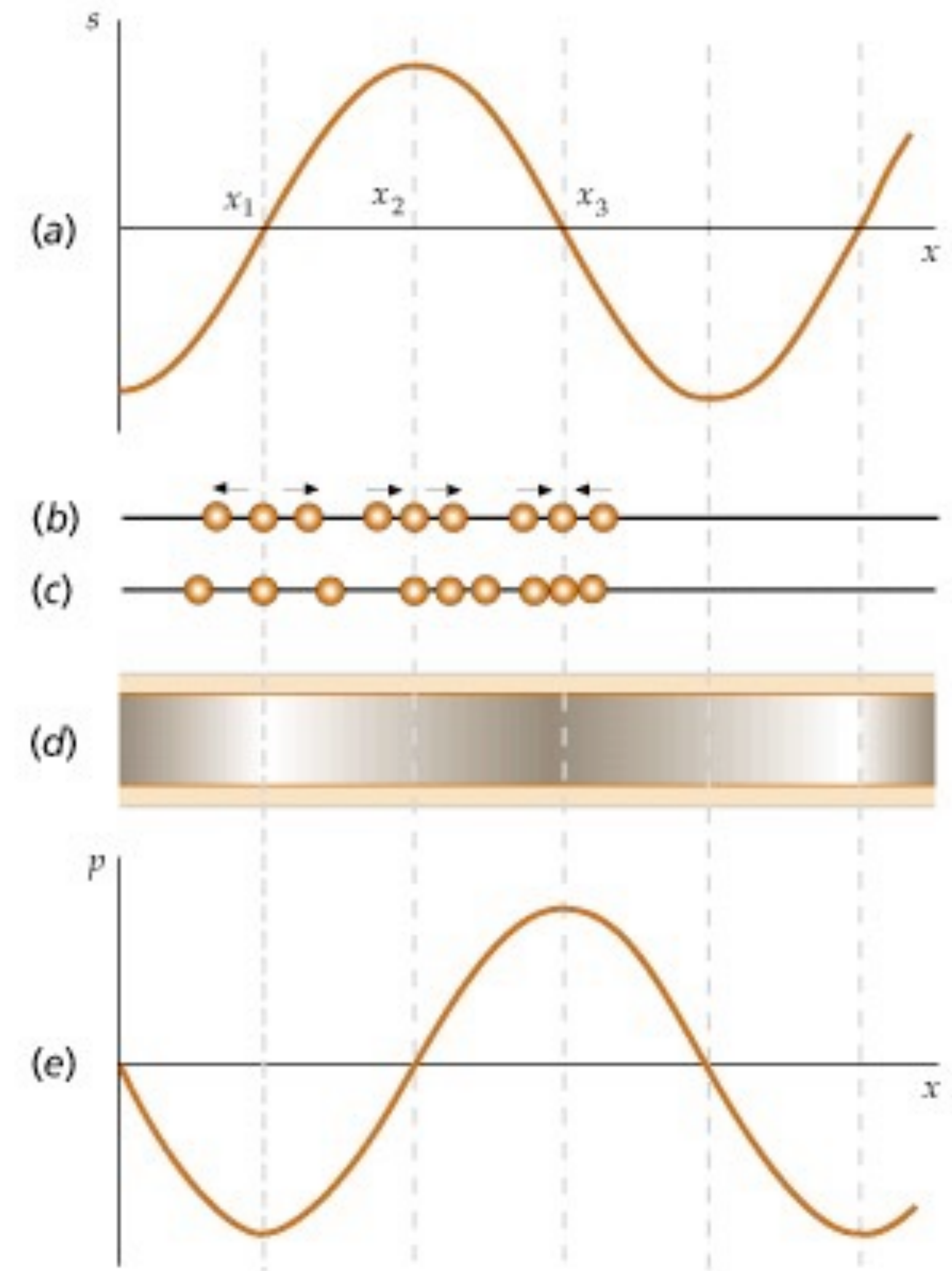
$$\Delta P = \Delta P_m \sin(kx - \omega t)$$

The pressure amplitude ΔP_m is proportional to the displacement amplitude s_m via

$$\Delta P_m = \rho v \omega s_m$$

ωs_m is the maximum longitudinal velocity of the medium in front of the piston

ie a sound wave may be considered as either a displacement wave or a pressure wave (90° out of phase)

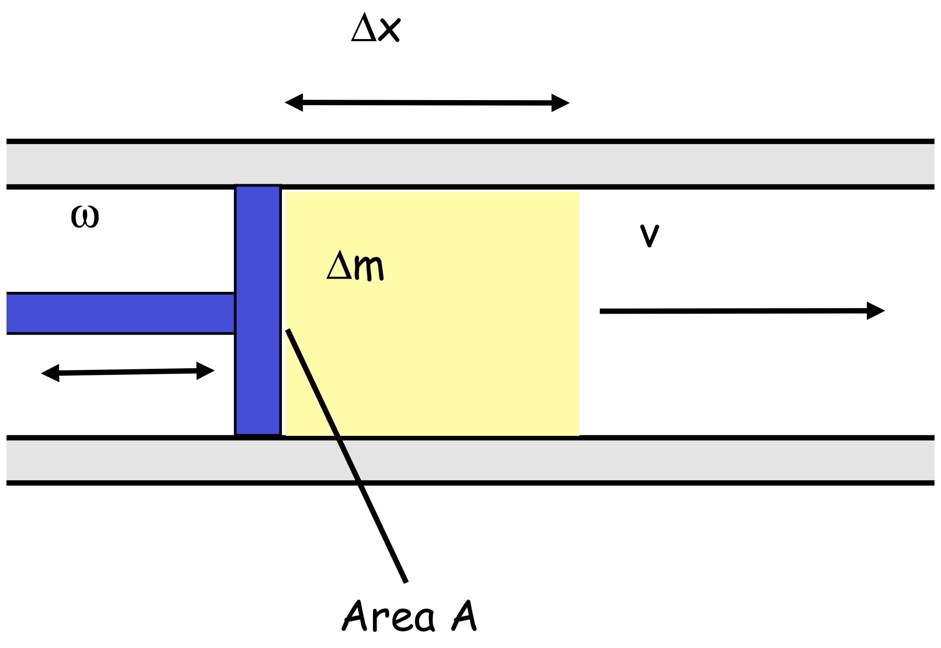


Tipler fig 15-10

Energy and Intensity of HSW

Consider a layer of air mass Δm and width Δx in front of a piston oscillating with frequency ω.

The piston transmits energy to the air.



In a system obeying SHM $KE_{ave} = PE_{ave}$ and $E_{ave} = KE_{max}$

$$\Delta E = \frac{1}{2}\Delta m\,(\omega s_m)^2$$

$$= \frac{1}{2}(\rho\, A\,\Delta x)(\omega s_m)^2$$

volume of layer

Energy and Intensity of HSW

Power = rate at which energy is transferred to each layer

$$\text{Power} = \frac{\Delta E}{\Delta t} = \frac{1}{2}\rho A\left(\frac{\Delta x}{\Delta t}\right)(\omega s_m)^2 = \frac{1}{2}\rho A v(\omega s_m)^2$$

velocity to right

$$\text{Intensity} = \frac{\text{Power}}{\text{area}} = \frac{1}{2}\rho v(\omega s_m)^2$$

$$= \frac{\Delta P_m^2}{2\rho v}$$

where $\Delta P_m = \rho v \omega s_m$

Intensity in decibels

The human ear detects sound on an approximately logarithmic scale. We define the **sound intensity level (SIL)** of a sound by:

$$SIL = 10 \log\left(\frac{I}{I_0}\right)$$

where I is the intensity of the sound, I_0 is the threshold of hearing ($\sim 10^{-12}$ W m^{-2}), and it is measured in decibels (dB).

Examples (just indicative, not frequency and distance dependent):

jet plane	150dB	conversation	50dB
rock concert	120dB	whisper	30dB
busy traffic	80dB	breathing	10dB