Un altro modo di valutare l'irrilevanza delle collisioni tra stelle. Come abbiamo visto sopra, queste non sono punti materiali, ma hanno un certo diametro. Ragioniamo per ordini di grandezza assumendo

$$R_* = R_\odot = 7\times 10^{10}\ \text{cm} = 2.3\times 10^{-8}\ \text{pc}.$$

Consideriamo quindi N stelle di raggio R_* in un volume di dimensione caratteristica R: si ha urto geometrico quando la distanza tra due centri e` minore di $2R_*$. Ciascuna stella ha a disposizione un volume cilindrico di base $\sigma_* \equiv 4\pi R_*^2$ e lunghezza $\lambda_g$, in cui $\lambda_g$ puo` essere interpretato come il cammino libero medio. Imponendo che il volume totale spazzato da tutte le stelle sia uguale al volume totale a disposizione si ha:

$$N\lambda_g\sigma_* = \frac{4\pi}{3}R^3,$$

ovvero:

$$\frac{\lambda_g}{R_*} = \left(\frac{R}{R_*}\right)^3 \frac{1}{3N}$$

Se per una galassia si assume R = 10^4 pc ed N = 10^{11} si ha $\lambda_g/R_* = 10^{23}$. Per un ammasso globulare, con R = 10 pc ed N = 10^5 si ha $\lambda_g/R_* = 10^{20}$. Quindi in situazioni usuali possiamo evitare di preoccuparci delle collisioni geometriche tra stelle.

$$j_R = \int L\,\phi(L)\,dL = L_* \int_0^\infty \frac{L}{L_*}\,\phi_*\left(\frac{L}{L_*}\right)^{\alpha} e^{-L/L_*}\, d\left(\frac{L}{L_*}\right) =$$

$$= L_*\,\phi_* \int_0^\infty x^{\alpha+1} e^{-x}\, dx \qquad \text{con} \quad \frac{L}{L_*} \equiv x$$

uso la funzione gamma

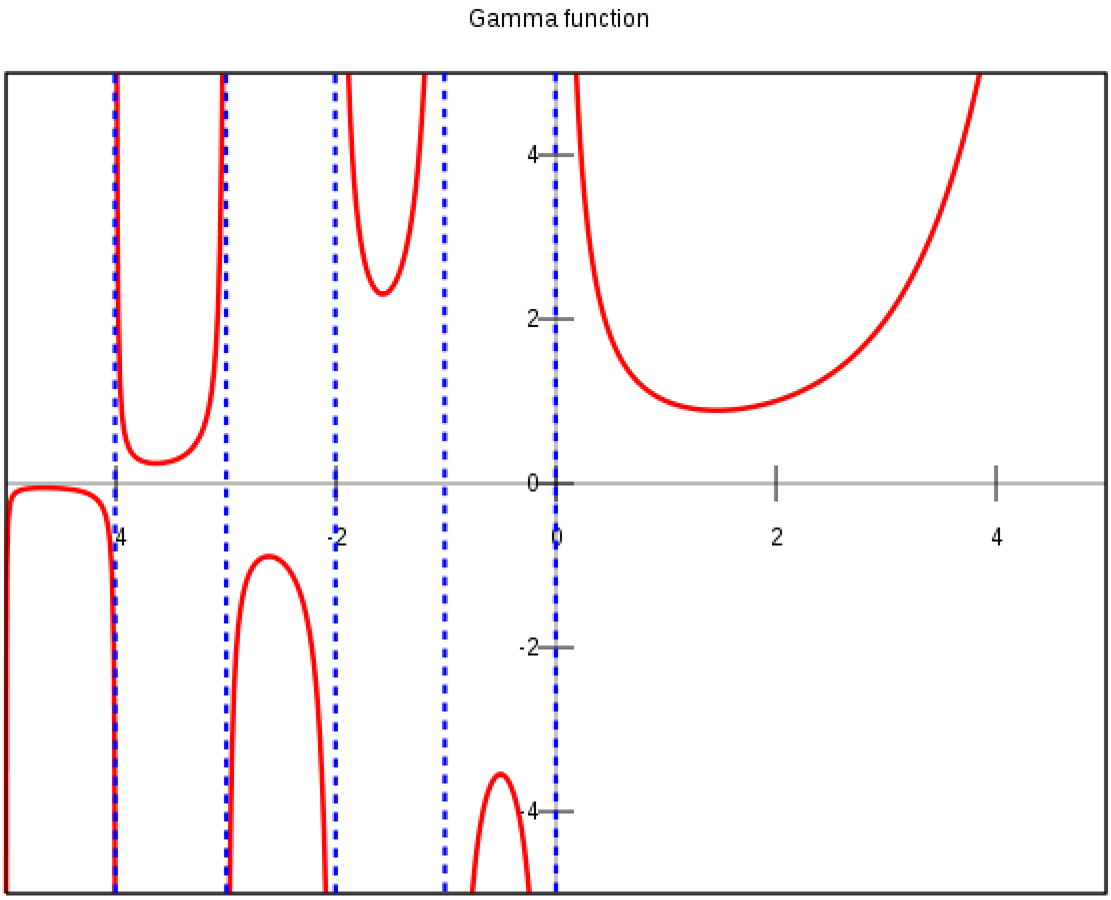
solo se z è intero

$$\int_0^\infty x^{z-1} e^{-x}\, dx \equiv \Gamma(z) = (z-1)!$$

qui $\alpha+1 \cong -1.1+1 \cong -0.1$ $\qquad \alpha+1 \equiv z-1 \rightarrow z = \alpha+2$

$$\boxed{j_R = L_*\,\phi_*\,\Gamma(\alpha+2)}$$

La funzione GAMMA lungo l'asse reale



Piano fondamentale

conseguenza del teorema del viriale:

$$\boxed{M \propto \sigma^2 \cdot r}$$

e del comportamento del rapporto $\frac{M}{L}$

Se $\boxed{\frac{M}{L} \propto L^{\theta}}$

- $M \propto \frac{M}{L} \cdot L \propto \sigma^2 \cdot r \rightarrow L \propto \frac{\sigma^2 r}{M/L}$
- $I \propto \frac{L}{r^2} \propto \frac{\sigma^2 r}{M/L} \cdot \frac{1}{r^2} \propto \frac{\sigma^2}{r \cdot M/L}$
- $r \propto \sigma^2 \, I^{-1} \, (M/L)^{-1}$
- $r \propto \sigma^2 \, I^{-1} \, L^{-\theta} \quad / \cdot r^{2\theta}$
- $r^{1+2\theta} \propto \sigma^2 \, I^{-1} \underbrace{L^{-\theta} \cdot r^{2\theta}}_{\left(\frac{L}{r^2}\right)^{-\theta} \equiv I^{-\theta}}$
- $r^{1+2\theta} \propto \sigma^2 \, I^{-1} \cdot I^{-\theta} \propto \sigma^2 \, I^{-(1+\theta)}$

$$\Rightarrow \boxed{r \propto \sigma^{\frac{2}{1+2\theta}} \cdot I^{-\frac{1+\theta}{1+2\theta}}}$$

Se $\theta \sim 0.3 \rightarrow r \propto \sigma^{1.25} \cdot I^{-0.81}$

$$\boxed{\log r = 1.25 \log \sigma - 0.81 \log I + \text{cost.}}$$

Anche: $L \propto \frac{\sigma^2 r}{M/L} \propto \frac{\sigma^2 r}{L^{\theta}} \Rightarrow L^{1+\theta} \propto \sigma^2 \cdot r$

$\log L = \frac{1}{1+\theta} \log(\sigma^2 r) + \text{cost.}$ $\boxed{L \propto (\sigma^2 \cdot r)^{1/1+\theta}}$ $\qquad \frac{1}{1+\theta} \sim 0.8$

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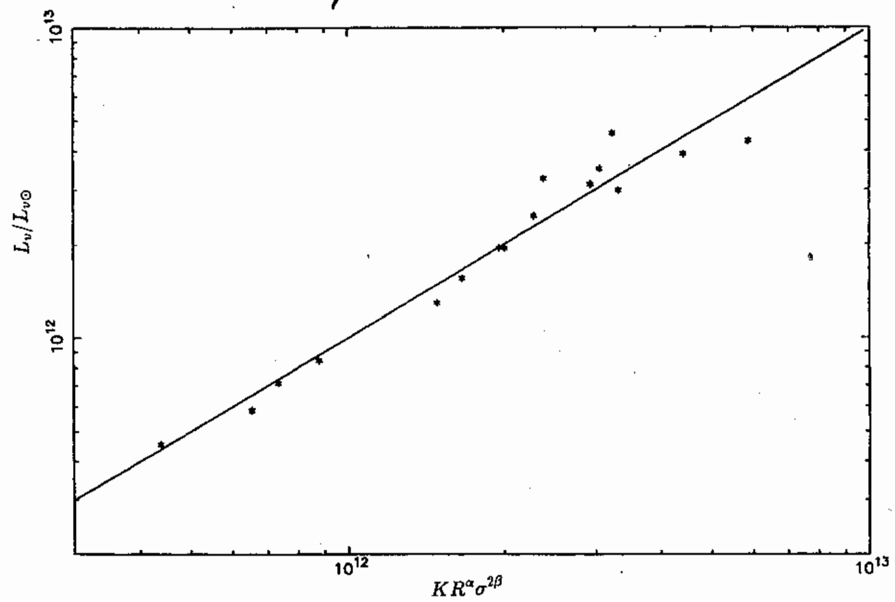


Figure 3. Relation between luminosity L and the product $R^{\alpha}\sigma^{2\beta}$. Note the excellent fit ($\alpha=0.89$, $\beta=0.64$ with a constant factor $K=4\times10^{8}$), for which the χ^2 per degree of freedom is improved by a factor of 8 compared to the previous cases.

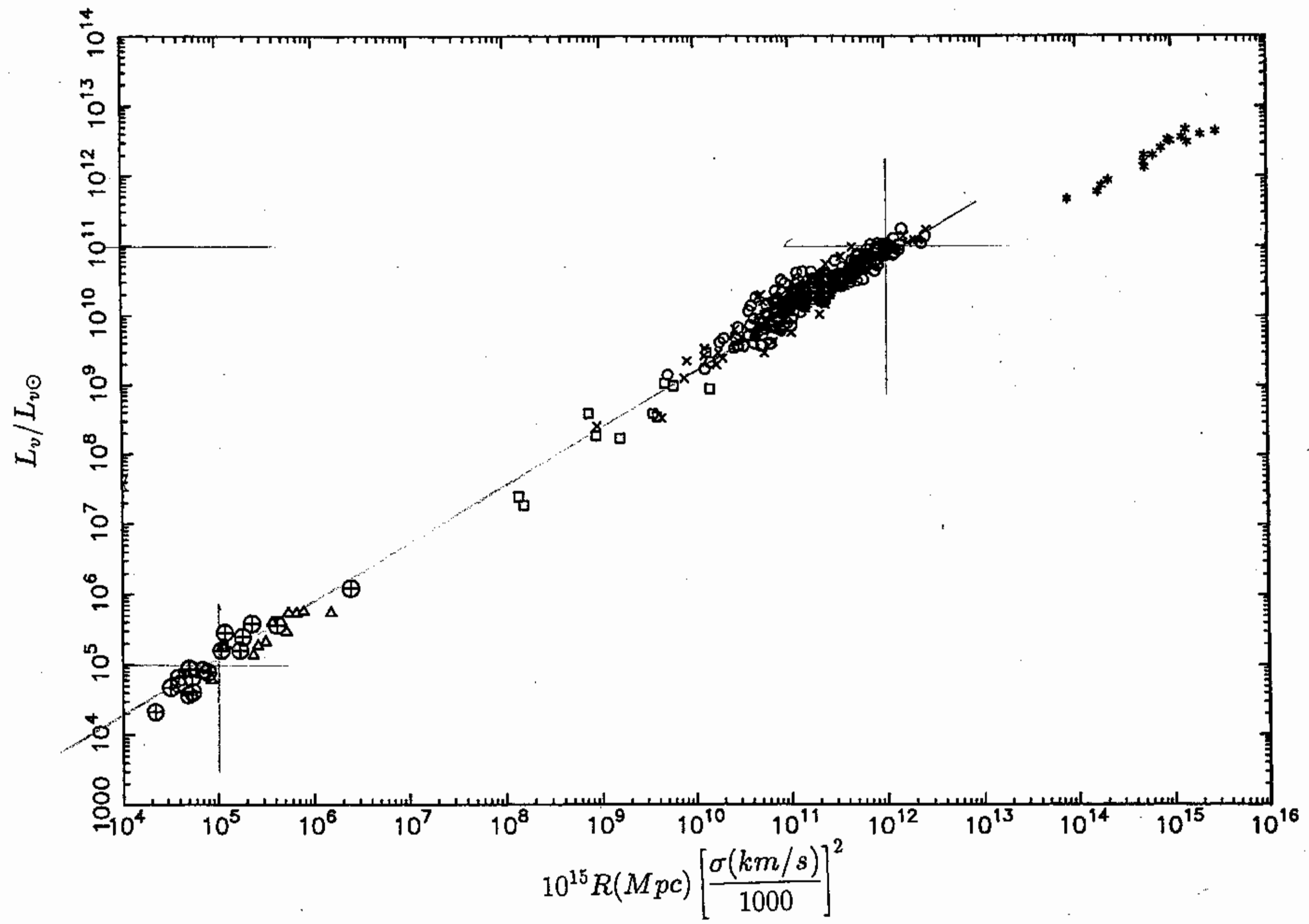
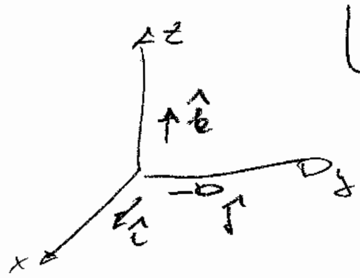


Figure 4. The 'fundamental plane' seen edge-on for different systems. Crossed circles: globular clusters with individual stellar spectra; triangles: globular clusters with integrated spectra; squares: dwarf and low-luminosity ellipsoidal galaxies (Bender & Nieto 1990; Bender et al. 1991); crosses: elliptical galaxies (Djorgovski & Davis 1987); circles: elliptical galaxies (Faber et al. 1989); stars: galaxy clusters (West et al. 1989; Struble & Rood 1991).

Vettori

$$\bar r = x\hat i + y\hat j + z\hat k$$

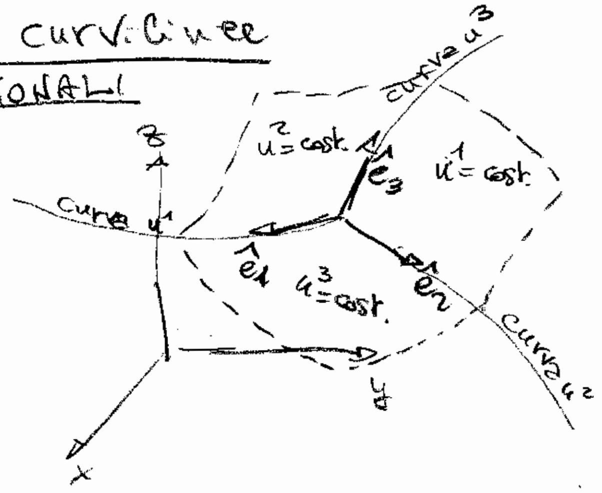
nel sistema cartesiano ortogonale



$$\bar v = \dot{\bar r} = \dot x\,\hat i + \dot y\,\hat j + \dot z\,\hat k$$

$$\bar a = \ddot{\bar r} = \ddot x\,\hat i + \ddot y\,\hat j + \ddot z\,\hat k$$

Passiamo ora alle coordinate curvilinee ORTOGONALI



da $x, y, z \longrightarrow u^1, u^2, u^3$

$$ds^2 = dx^2 + dy^2 + dz^2 = g_{ij}\,du^i du^j$$

Ricordiamo che

$$\bar x_i \equiv \frac{\partial \bar r}{\partial u^i} = \left|\frac{\partial \bar r}{\partial u^i}\right| \cdot \hat e_i$$

con $\hat e_i$ versore tangente alla curva della coordinata u^i

Sarà $$g_{ij} \equiv \bar x_i \cdot \bar x_j = \frac{\partial \bar r}{\partial u^i}\cdot\frac{\partial \bar r}{\partial u^j} = \left|\frac{\partial \bar r}{\partial u^i}\right|\left|\frac{\partial \bar r}{\partial u^j}\right| \hat e_i \cdot \hat e_j$$

Se le coord. curvilinee sono a due a due ortogonali, $\hat e_i \cdot \hat e_j$ sarà nullo se $i \neq j$, per cui il tensor metrico g_{ij} risulterà avere elementi $\neq 0$ solo sulla diagonale principale, per cui

$$ds^2 = g_{11}(du^1)^2 + g_{22}(du^2)^2 + g_{33}(du^3)^2 = \underbrace{\underbrace{\left|\frac{\partial \bar r}{\partial u^1}\right|^2}_{h_1^2}(du^1)^2}_{ds_1^2} + \underbrace{\left|\frac{\partial \bar r}{\partial u^2}\right|^2}_{h_2^2}(du^2)^2 + \underbrace{\left|\frac{\partial \bar r}{\partial u^3}\right|^2}_{h_3^2}(du^3)^2$$

con $h_i \equiv \left|\dfrac{\partial \bar r}{\partial u^i}\right|$ $\qquad ds_1^2$ + analoghe

L'elemento di volume sarà

$$dV = \sqrt{g}\;du^1 du^2 du^3 = \sqrt{g_{11}\cdot g_{22}\cdot g_{33}}\;du^1 du^2 du^3 =$$

$$= \sqrt{h_1^2 h_2^2 h_3^2}\;du^1 du^2 du^3 = h_1 h_2 h_3\, du^1 du^2 du^3$$

Gradiente

$$\boxed{\overline{\nabla}\Phi = \frac{\partial \Phi}{\partial x}\hat{\imath} + \frac{\partial \Phi}{\partial y}\hat{\jmath} + \frac{\partial \Phi}{\partial z}\hat{k}}$$

Coord. cartesiane ortogonali

Il gradiente è il vettore con direzione e modulo della massima variazione spaziale della funzione Φ.
La componente del $\overline{\nabla}\phi$ nella direzione normale alla famiglia di superfici $u^1 =$ costante è data dalla sua proiezione nella direzione del versore $\hat{e}_1$, $\perp$ alla sup. $u^1 =$ cost.

$$\hat{e}_1 \cdot \overline{\nabla}\Phi = \frac{\Delta \Phi}{\Delta s_1} = \frac{\Delta \Phi}{h_1 \Delta u^1} \overset{\lim}{\Rightarrow} \frac{1}{h_1}\frac{\partial \phi}{\partial u^1}$$

nella direzione del versore $\hat{e}_1$, per cui la componente del gradiente sarà

$$\frac{1}{h_1}\frac{\partial \phi}{\partial u^1}\cdot \hat{e}_1$$

Nel complesso il gradiente di $\phi(u^i)$ sarà

$$\boxed{\overline{\nabla}\Phi = \frac{1}{h_i}\frac{\partial \phi}{\partial u^i}\hat{e}_i}$$

(somma su $i = 1, 2, 3$ vettoriale)

Divergenza

Useremo il teorema della divergenza (o di Gauss)

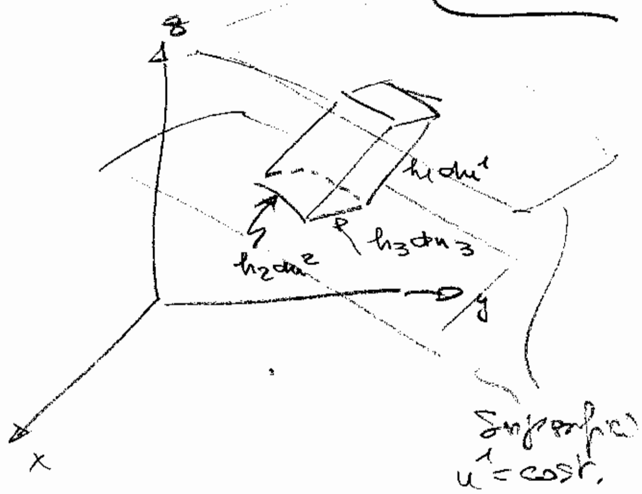
$$\int_{Vol.} \overline{\nabla}\cdot\overline{F}\, d\tau = \int_{Sup.} \overline{F}\cdot d\overline{\sigma}$$

che, per un volume $\int d\tau$ che tende a zero, si scrive anche

$$\overline{\nabla}\cdot\overline{F}(u^i) = \lim_{\int d\tau \to 0} \frac{\int \overline{F}\cdot d\overline{\sigma}}{\int d\tau}$$

in cui abbiamo visto che l'elemento di volume è $h_1 h_2 h_3\, du^1 du^2 du^3$.

Consideriamo un elemento di volume infinitesimo per valutare il flusso attraverso due facce $u^1 =$ costante del vettore $\overline{F}$. Chiamiamo F_1 la proiezione di $\overline{F}$ nella direzione (= del versore $\vec{e}_1$) della coord. u^1. Il flusso attraverso l'areola $h_2 du^2 \, h_3 du^3$ corrispondente a un certo valore di u^1 sarà



$F_1 h_2 h_3 \, du^2 du^3$.

Per avere il flusso devo fare la differenza tra i valori su $u^1 = cost.$ e $u^1 + du^1 = cost.$ (trascuro gli infinitesimi di ordine superiore al primo)

$$\left[F_1 h_2 h_3 + \frac{\partial}{\partial u^1}(F_1 h_2 h_3) du^1\right] du^2 du^3 - F_1 h_2 h_3 \, du^2 du^3 =$$

$$= \frac{\partial}{\partial u^1}(F_1 h_2 h_3) \, du^1 du^2 du^3$$

Aggiungo i risultati analoghi per le altre due coppie di superfici e ottengo

$$\int \overline{F} \cdot d\overline{\sigma} = \left[\frac{\partial}{\partial u^1}(F_1 h_2 h_3) + \frac{\partial}{\partial u^2}(F_2 h_3 h_1) + \frac{\partial}{\partial u^3}(F_3 h_2 h_1)\right] du^1 du^2 du^3$$

divido per l'elemento di volume $h_1 h_2 h_3 \, du^1 du^2 du^3$ e ottengo:

$$\boxed{\overline{\nabla} \cdot \overline{F}(u^i) = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u^1}(F_1 h_2 h_3) + \frac{\partial}{\partial u^2}(F_2 h_3 h_1) + \frac{\partial}{\partial u^3}(F_3 h_2 h_1)\right]}$$

Poichè il laplaciano $\nabla^2 \Phi$ è la divergenza del gradiente, e le componenti del $\nabla \Phi$ nella direzione $\vec{e}_i$ sono $\frac{1}{h_i}\frac{\partial \Phi}{\partial u^i}$, ottengo:

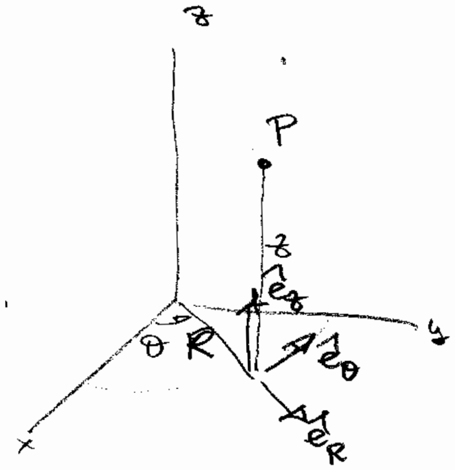
$$\boxed{\nabla^2 \Phi(u^i) = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u^1}\left(\frac{h_2 h_3}{h_1}\frac{\partial \Phi}{\partial u^1}\right) + \frac{\partial}{\partial u^2}\left(\frac{h_3 h_1}{h_2}\frac{\partial \Phi}{\partial u^2}\right) + \frac{\partial}{\partial u^3}\left(\frac{h_2 h_1}{h_3}\frac{\partial \Phi}{\partial u^3}\right)\right]}$$

Casi particolari

① Coordinate cartesiane ortogonali

In questo caso $h_1 = h_2 = h_3 \equiv 1$ e con le formule scritte sopra riotteniamo banalmente le formule classiche.

② Coordinate cilindriche (R, θ, z)



$$\begin{cases} x = R\cos\theta \\ y = R\,\mathrm{sen}\,\theta \\ z = z \end{cases} \rightarrow \bar{r} \equiv (R\cos\theta, R\,\mathrm{sen}\,\theta, z)$$

$$\overline{X}_R = \frac{\partial \bar r}{\partial R} = (\cos\theta, \mathrm{sen}\,\theta, 0)$$

$$\overline{X}_\theta = \frac{\partial \bar r}{\partial \theta} = (-R\,\mathrm{sen}\,\theta, R\cos\theta, 0)$$

$$\overline{X}_t = \frac{\partial \bar r}{\partial z} = (0, 0, 1)$$

$$g_{RR} = \overline{X}_R \cdot \overline{X}_R = \cos^2\theta + \mathrm{sen}^2\theta = 1 \rightarrow g_{RR} = h_R^2 \rightarrow \boxed{h_R \equiv 1}$$

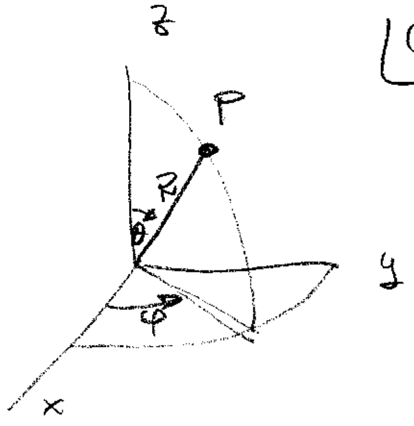
$$g_{\theta\theta} = \overline{X}_\theta \cdot \overline{X}_\theta = R^2\,\mathrm{sen}^2\theta + R^2\cos^2\theta = R^2 \rightarrow g_{\theta\theta} = h_\theta^2 \Rightarrow \boxed{h_\theta = R}$$

$$g_{zz} = \overline{X}_z \cdot \overline{X}_z = 1 \rightarrow g_{zz} = h_z^2 \rightarrow \boxed{h_z = 1}$$

- $dV = R\, dR\, d\theta\, d\varphi$
- $\overline{\nabla}\Phi(R,\theta,z) = \frac{\partial \Phi}{\partial R}\hat{e}_R + \frac{1}{R}\frac{\partial \Phi}{\partial \theta}\hat{e}_\theta + \frac{\partial \Phi}{\partial z}\hat{e}_z$
- $\overline{\nabla} \cdot \overline{F}(R,\theta,z) = \frac{1}{R}\left[\frac{\partial}{\partial R}(F_R \cdot R) + \frac{\partial F_\theta}{\partial \theta} + \frac{\partial}{\partial z}(F_z \cdot R)\right] =$
$$= \frac{1}{R}\frac{\partial (F_R \cdot R)}{\partial R} + \frac{1}{R}\frac{\partial F_\theta}{\partial \theta} + \frac{\partial F_z}{\partial z}$$
- $\nabla^2 \Phi(R,\theta,z) = \frac{1}{R}\left[\frac{\partial}{\partial R}\left(R\frac{\partial \Phi}{\partial R}\right) + \frac{\partial}{\partial \theta}\left(\frac{1}{R}\frac{\partial \Phi}{\partial \theta}\right) + \frac{\partial}{\partial z}\left(R\frac{\partial \Phi}{\partial z}\right)\right] =$
$$= \frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial \Phi}{\partial R}\right) + \frac{1}{R^2}\frac{\partial^2 \Phi}{\partial \theta^2} + \frac{\partial^2 \Phi}{\partial z^2}$$

③ Coordinate sferiche (R, θ, φ)

CC5



$$\begin{cases} x = R \operatorname{sen}\theta \cos\varphi \\ y = R \operatorname{sen}\theta \operatorname{sen}\varphi \\ z = R \cos\theta \end{cases}$$

$$\overline{r} = (R \operatorname{sen}\theta \cos\varphi, R \operatorname{sen}\theta \operatorname{sen}\varphi, R\cos\theta)$$

$$\overline{x}_R = \frac{\partial \overline{r}}{\partial R} = (\operatorname{sen}\theta\cos\varphi, \operatorname{sen}\theta\operatorname{sen}\varphi, \cos\theta)$$

$$\overline{x}_\theta = \frac{\partial \overline{r}}{\partial \theta} = (R\cos\theta\cos\varphi, R\cos\theta\operatorname{sen}\varphi, -R\operatorname{sen}\theta)$$

$$\overline{x}_\varphi = \frac{\partial \overline{r}}{\partial \varphi} = (-R\operatorname{sen}\theta\operatorname{sen}\varphi, R\operatorname{sen}\theta\cos\varphi, 0)$$

$$g_{RR} = \overline{x}_R \cdot \overline{x}_R = \operatorname{sen}^2\theta\cos^2\varphi + \operatorname{sen}^2\theta\operatorname{sen}^2\varphi + \cos^2\theta = 1 \rightarrow \boxed{h_R = \sqrt{g_{RR}} = 1}$$

$$g_{\theta\theta} = \overline{x}_\theta \cdot \overline{x}_\theta = R^2\cos^2\theta\cos^2\varphi + R^2\cos^2\theta\operatorname{sen}^2\varphi + R^2\operatorname{sen}^2\theta = R^2 \rightarrow \boxed{h_\theta = \sqrt{g_{\theta\theta}} = R}$$

$$g_{\varphi\varphi} = \overline{x}_\varphi \cdot \overline{x}_\varphi = R^2\operatorname{sen}^2\theta\operatorname{sen}^2\varphi + R^2\operatorname{sen}^2\theta\cos^2\varphi = R^2\operatorname{sen}^2\theta \rightarrow \boxed{h_\varphi = \sqrt{g_{\varphi\varphi}} = R\operatorname{sen}\theta}$$

- $dV = R^2 \operatorname{sen}\theta \, d\theta \, d\varphi$

- $$\overline{\nabla}\Phi(R,\theta,\varphi) = \frac{\partial\Phi}{\partial R}\hat{e}_R + \frac{1}{R}\frac{\partial\Phi}{\partial\theta}\hat{e}_\theta + \frac{1}{R\operatorname{sen}\theta}\frac{\partial\Phi}{\partial\varphi}\hat{e}_\varphi$$

- $$\overline{\nabla}\cdot\overline{F}(R,\theta,\varphi) = \frac{1}{R^2\operatorname{sen}\theta}\left[\frac{\partial}{\partial R}(R^2\operatorname{sen}\theta\, F_R) + \frac{\partial}{\partial\theta}(R\operatorname{sen}\theta\, F_\theta) + \frac{\partial}{\partial\varphi}(R\, F_\varphi)\right]$$

$$= \frac{1}{R^2}\frac{\partial}{\partial R}(R^2 F_R) + \frac{1}{R\operatorname{sen}\theta}\frac{\partial}{\partial\theta}(\operatorname{sen}\theta\cdot F_\theta) + \frac{1}{R\operatorname{sen}\theta}\frac{\partial F_\varphi}{\partial\varphi}$$

- $$\nabla^2\Phi(R,\theta,\varphi) = \frac{1}{R^2\operatorname{sen}\theta}\left[\frac{\partial}{\partial R}\left(R^2\operatorname{sen}\theta\frac{\partial\Phi}{\partial R}\right) + \frac{\partial}{\partial\theta}\left(\operatorname{sen}\theta\frac{\partial\Phi}{\partial\theta}\right) + \frac{\partial}{\partial\varphi}\left(\frac{1}{\operatorname{sen}\theta}\frac{\partial\Phi}{\partial\varphi}\right)\right]$$

$$= \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\cdot\frac{\partial\Phi}{\partial R}\right) + \frac{1}{R^2\operatorname{sen}\theta}\frac{\partial}{\partial\theta}\left(\operatorname{sen}\theta\frac{\partial\Phi}{\partial\theta}\right) + \frac{1}{R^2\operatorname{sen}^2\theta}\frac{\partial^2\Phi}{\partial\varphi^2}$$

Lavoriamo in coordinate cartesiane ortogonali

$$\rightarrow \boxed{\overline{\nabla} f^\alpha = \sum_i \bar{e}_i \frac{\partial f^\alpha}{\partial x^i} = \sum_i \bar{e}_i \alpha f^{\alpha-1} \frac{\partial f}{\partial x^i} = \alpha \cdot f^{\alpha-1} \overline{\nabla} f}$$

$$\bullet\ \overline{\nabla}_x \left(\frac{1}{|\bar{x}'-\bar{x}|} \right) = \overline{\nabla}_x \left\{ \left[(\bar{x}'-\bar{x})^2 \right]^{-1/2} \right\} = -\frac{1}{2} \frac{1}{\left[(\bar{x}'-\bar{x})^2\right]^{3/2}} \cdot \overline{\nabla}_x (\bar{x}'-\bar{x})^2$$

ma

$$\overline{\nabla}_x (\bar{x}'-\bar{x})^2 = \sum_i \bar{e}_i \frac{\partial}{\partial x^i} (\bar{x}'-\bar{x})^2 = \sum_i \bar{e}_i \frac{\partial}{\partial x^i} (x^{i\prime}-x^i)^2 =$$

$$= \sum_i \bar{e}_i \cdot 2 (x^{i\prime}-x^i) \cdot -1 = -2 \sum_i \bar{e}_i (x^{i\prime}-x^i) = -2(\bar{x}'-\bar{x})$$

da cui

$$\boxed{\overline{\nabla}_x \left(\frac{1}{|\bar{x}'-\bar{x}|} \right) = -\frac{1}{2} \cdot \frac{1}{|\bar{x}'-\bar{x}|^3} \cdot -2 (\bar{x}'-\bar{x}) = \frac{\bar{x}'-\bar{x}}{|\bar{x}'-\bar{x}|^3}} \quad \text{cvd}$$

$$\rightarrow \boxed{\overline{\nabla} \cdot (\lambda \bar{u}) = \sum_i \frac{\partial}{\partial x^i} (\lambda u^i) = \sum_i \lambda \frac{\partial u^i}{\partial x^i} + \sum_i u^i \frac{\partial \lambda}{\partial x^i} = \lambda \overline{\nabla} \cdot \bar{u} + \bar{u} \cdot \overline{\nabla} \lambda}$$

$$\bullet\ \overline{\nabla}_x \cdot \frac{\bar{x}'-\bar{x}}{|\bar{x}'-\bar{x}|^3} = \frac{1}{|\bar{x}'-\bar{x}|^3} \cdot \overline{\nabla}_x \cdot (\bar{x}'-\bar{x}) + (\bar{x}'-\bar{x}) \cdot \overline{\nabla}_x \left(\frac{1}{|\bar{x}'-\bar{x}|^3} \right)$$

Ma $\overline{\nabla}_x \cdot \bar{x} = \sum_i \frac{\partial}{\partial x^i} (x^i) = 1+1+1 = 3 \rightarrow \boxed{\overline{\nabla}_x \cdot (\bar{x}'-\bar{x}) = -3}$

Inoltre

$$\overline{\nabla}_x \left(\frac{1}{|\bar{x}'-\bar{x}|^3} \right) = \overline{\nabla}_x \left\{ \left[(\bar{x}'-\bar{x})^2 \right]^{-3/2} \right\} = -\frac{3}{2} \frac{1}{\left[(\bar{x}'-\bar{x})^2\right]^{5/2}} \cdot \overline{\nabla} (\bar{x}'-\bar{x})^2 =$$

$$= -\frac{3}{2} \frac{1}{|\bar{x}'-\bar{x}|^5} \cdot -2 (\bar{x}'-\bar{x}) = +3 \frac{\bar{x}'-\bar{x}}{|\bar{x}'-\bar{x}|^5}$$

Allora

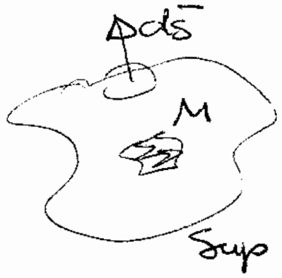
$$\boxed{\overline{\nabla}_x \cdot \left(\frac{\bar{x}'-\bar{x}}{|\bar{x}'-\bar{x}|^3} \right) = -3 \cdot \frac{1}{|\bar{x}'-\bar{x}|^3} + 3 \frac{(\bar{x}'-\bar{x}) \cdot (\bar{x}'-\bar{x})}{|\bar{x}'-\bar{x}|^5}}$$

Se $\bar{x}'-\bar{x} \neq 0$ posso semplificare un fattore $|\bar{x}'-\bar{x}|^2$ sopra e sotto nel 2° termine del 2° membro, per cui ho

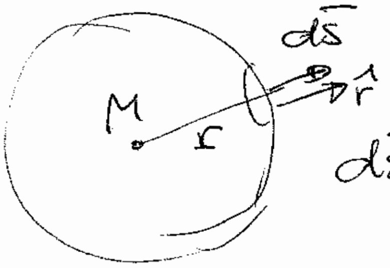
$$\boxed{\overline{\nabla}_x \cdot \left(\frac{\bar{x}'-\bar{x}}{|\bar{x}'-\bar{x}|^3} \right) = \emptyset \qquad \text{se } \bar{x}' \neq \bar{x}}$$

Applicazioni teorema di Gauss

$$\boxed{4\pi G M = -\int_{Sup} d\bar{S}\cdot\bar{g}}$$



① Masse puntiformi



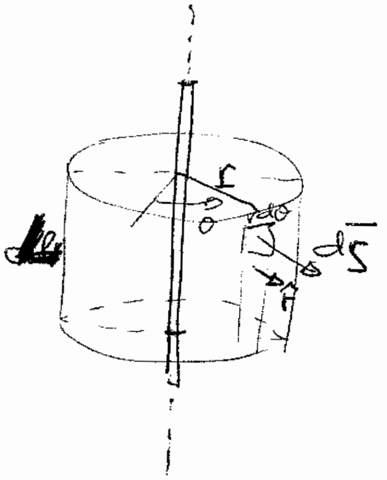
$d\bar{S} \parallel \hat{r}$

$$4\pi G M = -\int r^2 d\Omega\, g_r = -4\pi r^2 g_r$$

$$\Downarrow$$

$$\boxed{g_r = -\frac{GM}{r^2}}$$

② Filo infinito con densità lineare $\lambda = \frac{dm}{dl}$

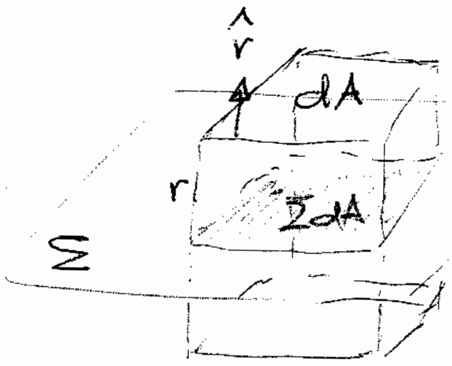


In porzione di filo di lunghezza L:

$$\overset{2}{\cancel{4\pi}} G\cdot\lambda\cdot\cancel{L} = -\int_0^{2\pi} r\,d\theta\cdot\cancel{L}\cdot g_r = -\cancel{2\pi}\cancel{L}\, r\, g_r$$

$$\boxed{g_r = -\frac{2G\lambda}{r}}$$

③ Piano infinito di densità superficiale Σ e spessore nullo



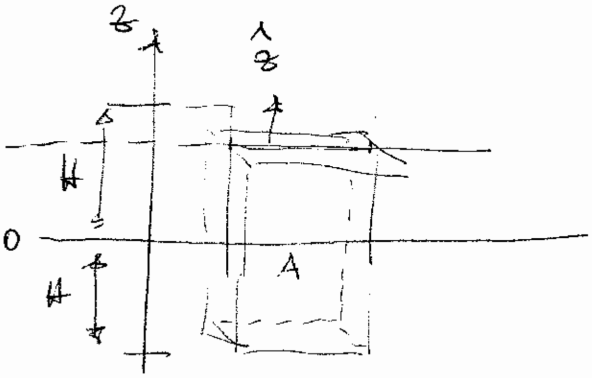
Per un elemento di superficie dA

$$\overset{2}{\cancel{4}}\pi G\cdot\Sigma\, dA = -g_r\cdot dA\cdot\cancel{2}$$

$$\boxed{g_r = -2\pi G\Sigma}$$

costante: x invarianza di scala

(4) Piano infinito di spessore $2H$ e distribuzione di densità lungo la verticale $\rho(z)=\rho_0 e^{-\frac{|z|}{H}}$ con $\rho_0 = cost.$



Massa entro parallelepipedo di base A e altezza z

$$dV = A \cdot dz$$

$$dM = \rho(z)\, A\, dz$$

$$M(z) = \int_{-z}^{+z} A \rho_0 e^{-|z|/H} dz \quad (\text{per } |z| \le H)$$

$$= 2A\rho_0 \int_0^z e^{-z/H} \frac{dz}{H} \cdot H =$$

$$= 2H\rho_0 A \int_0^{z/H} e^{-x} dx$$

$$M(z) = 2H\rho_0 A \left| -e^{-x} \right|_0^{z/H} = 2H\rho_0 A\left(1 - e^{-z/H}\right) \quad (z>0)$$

altrimenti $z \to |z|$

Dal teor di Gauss

$$4\pi G \cdot \cancel{2} H\rho_0 A\left(1-e^{-z/H}\right) = -g_z \cdot \cancel{A} \cdot \cancel{2}$$

$$\boxed{g_z(z) = -4\pi H G\rho_0\left(1 - e^{-|z|/H}\right)} \quad |z|<H$$

Se $|z|>H$, la massa diventa costante e

$$\boxed{g_z(z) = -4\pi H G\rho_0\left(1-\frac{1}{e}\right)} \quad z>H$$

Quanto vale $\Sigma \equiv \int_{-\infty}^{+\infty} \rho(z)dz$?

$$\Sigma = \int_{-\infty}^{+\infty} \rho_0 e^{-|z|/H} dz = 2\int_0^{H} \rho_0 e^{-\frac{z}{H}} \frac{dz}{H} H = 2\rho_0 H \int_0^H e^{-x} dx = 2\rho_0 H\left(1-\frac{1}{e}\right)$$

quindi

$$\Rightarrow g_z = -4\pi H G\rho_0\left(1-\frac{1}{e}\right) = -2\pi G \cdot 2H\rho_0\left(1-\frac{1}{e}\right) = -2\pi G\,\Sigma$$

$z>H$

(2.13) $\delta W = \int d^3\bar{x}\, \delta\rho(\bar{x})\, \phi(\bar{x})$

Potenziale additivo | $\nabla^2(\delta\phi) = 4\pi G\, \delta\rho(\bar{x})$

(2.14) $\delta W = \frac{1}{4\pi G} \int d^3x\, \phi(\bar{x})\, \nabla^2(\delta\phi)$

(*) [Ricordiamo (56 bis): $\bar{\nabla}\cdot(\lambda\bar{u}) = \lambda\bar{\nabla}\cdot\bar{u} + \bar{u}\cdot\bar{\nabla}\lambda$]

se $\lambda \to \phi$, $\bar{u} \to \bar{\nabla}(\delta\phi)$ $\to$ $\bar{\nabla}\cdot(\phi\bar{\nabla}(\delta\phi)) = \phi\underbrace{\bar{\nabla}\cdot\bar{\nabla}(\delta\phi)}_{\nabla^2(\delta\phi)} + \bar{\nabla}(\delta\phi)\cdot\bar{\nabla}\phi$

da cui

$$\phi\nabla^2(\delta\phi) = \bar{\nabla}\cdot[\phi\bar{\nabla}(\delta\phi)] - \bar{\nabla}(\delta\phi)\cdot\bar{\nabla}\phi$$

da questa $\to$ (2.15) usando il teorema della divergenza.

Inoltre $\delta[\bar{\nabla}\phi\cdot\bar{\nabla}\phi] = 2\bar{\nabla}\phi\cdot\delta\bar{\nabla}\phi$

ma $\delta\bar{\nabla}\phi = \bar{\nabla}(\phi+\delta\phi) - \bar{\nabla}\phi = \bar{\nabla}\phi + \bar{\nabla}(\delta\phi) - \bar{\nabla}\phi = \bar{\nabla}(\delta\phi)$

da cui $\bar{\nabla}\phi\cdot\bar{\nabla}(\delta\phi) = \frac{1}{2}\delta[|\bar{\nabla}\phi|^2]$

e

$$\delta W = -\frac{1}{4\pi G}\int d^3\bar{x}\cdot\frac{1}{2}\delta[|\bar{\nabla}\phi|^2]$$

Ma la variazione della somma è la somma delle variazioni, per cui

(2.16) $\delta W = -\frac{1}{8\pi G}\delta\int d^3\bar{x}\,|\bar{\nabla}\phi|^2$

Uso di nuovo la (*) con $\lambda \to \phi$ e $\bar{u} \to \bar{\nabla}\phi$:

$$\bar{\nabla}\cdot(\phi\bar{\nabla}\phi) = \phi\bar{\nabla}\cdot\bar{\nabla}\phi + \underbrace{\bar{\nabla}\phi\cdot\bar{\nabla}\phi}_{|\bar{\nabla}\phi|^2} \to |\bar{\nabla}\phi|^2 = \bar{\nabla}\cdot(\phi\bar{\nabla}\phi) - \phi\nabla^2\phi$$

$$W = -\frac{1}{8\pi G}\int_{Vol} d^3\bar{x}\,|\bar{\nabla}\phi|^2 = -\frac{1}{8\pi G}\Big[\underbrace{\int_{Vol} d^3\bar{x}\,\bar{\nabla}\cdot(\phi\bar{\nabla}\phi)}_{\underbrace{\int_{sup} d\bar{S}\cdot\phi\bar{\nabla}\phi}_{\to 0 \text{ se } r\to\infty}} - \int_{Vol} d^3\bar{x}\,\phi\,\underbrace{\nabla^2\phi}_{4\pi G\rho(\bar{x})}\Big]$$

$d\bar{S} \propto r^2$, $\phi \sim \frac{1}{r}$, $\bar{\nabla}\phi \sim \frac{1}{r^2}$

da cui

$$W = \frac{1}{2}\int d^3\bar{x}\,\rho(\bar{x})\phi(\bar{x}) \quad (2.18)$$

Ricorda: integrazione per parti

$$\int f' g \, ds = fg - \int f g' \, ds$$

con

$g \to GM(s) \qquad f \to \frac{1}{s}$

$$\int -\frac{1}{s^2} GM(s)\, ds = \frac{GM(s)}{s} - \int \frac{1}{s} \cdot G \frac{dM}{\cancel{ds}} \cancel{ds}$$

$$-\int_r^\infty \frac{GM(s)}{s^2}\, ds = \underbrace{\frac{GM(s)}{s}\Big|_r^\infty}_{-\frac{GM(r)}{r} \,=\, -\frac{G}{r}\int_0^r dM(s)} - \int_r^\infty \frac{G}{s}\, dM(s)$$

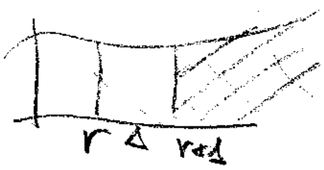
cioè

$$\phi(r) = -\frac{G}{r}\int_0^r dM - G\int_r^\infty \frac{dM}{s} = \qquad (2.28)$$

$$= -\int_r^\infty \frac{GM(s)}{s^2}\, ds \quad //$$

$\bar{g} = -\bar{\nabla}\phi$ lungo la direzione radiale: $g_r = -\frac{d\phi}{dr}$

$$g_r = \frac{d}{dr}\int_r^\infty \frac{GM(s)}{s^2}\, ds$$



cioè

$$g_r = \lim_{\Delta \to 0} \frac{1}{\Delta}\left[\int_{r+\Delta}^\infty - \int_r^\infty\right] = -\frac{1}{\Delta}\int_r^{r+\Delta} \frac{GM(s)}{s^2}\, ds =$$

$$= -\frac{1}{\Delta}\cdot \Delta \cdot \frac{GM(r)}{r^2} \to g_r = -\frac{GM(r)}{r^2} \quad // \text{ cvd}$$

$$\boxed{M(r,\alpha,\beta) = 4\pi\rho_0 a^3 \int_0^{r/a} \frac{s^{2-\alpha}}{(1+s)^{\beta-\alpha}}\, ds}$$

Jaffe: $\beta=4$, $\alpha=2$

$$\int \frac{dx}{(1+x)^2} = -\frac{1}{1+x}$$

$$M_J(r) = 4\pi\rho_0 a^3 \int_0^{r/a} \frac{ds}{(1+s)^2} = 4\pi\rho_0 a^3 \left[-\frac{1}{1+r/a} + 1\right] = 4\pi\rho_0 a^3 \frac{\cancel{-1}+\cancel{1}+r/a}{1+r/a}$$

$$\boxed{M_J(r) = 4\pi\rho_0 a^3 \cdot \frac{r/a}{1+r/a}}$$

se $r\to\infty$, $M_J(r)\to 4\pi\rho_0 a^3$ finite

Herquist: $\beta=4$, $\alpha=1$

$$\int \frac{x}{(1+x)^3}dx = -\frac{1}{1+x} + \frac{1}{2(1+x)^2}$$

$$M_H(r) = 4\pi\rho_0 a^3 \int_0^{r/a} \frac{s}{(1+s)^3}ds = 4\pi\rho_0 a^3 \left[\underbrace{-\frac{1}{1+r/a} + \frac{1}{2(1+r/a)^2} + 1 - \frac{1}{2}}\right]$$

$$\frac{1}{2} - \frac{1}{1+r/a} + \frac{1}{2(1+r/a)^2} = \frac{\cancel{1}+(\frac{r}{a})^2+\cancel{2\frac{r}{a}}-\cancel{2}-\cancel{2\frac{r}{a}}+\cancel{1}}{2(1+r/a)^2}$$

$$\boxed{M_H(r) = 4\pi\rho_0 a^3 \cdot \frac{(r/a)^2}{2(1+r/a)^2}}$$

se $r\to\infty$, $M_H \to 2\pi\rho_0 a^3$

NFW: $\beta=3$, $\alpha=1$

$$\int \frac{x}{(1+x)^2}dx = \ln(1+x) + \frac{1}{1+x}$$

$$M_{NFW}(r) = 4\pi\rho_0 a^3 \int_0^{r/a} \frac{s}{(1+s)^2} = 4\pi\rho_0 a^3 \left[\ln\left(1+\frac{r}{a}\right) + \underbrace{\frac{1}{1+r/a} - 1}_{\frac{\cancel{1}-\cancel{1}-r/a}{1+r/a}}\right]$$

$$\boxed{M_{NFW}(r) = 4\pi\rho_0 a^3 \left[\ln\left(1+\frac{r}{a}\right) - \frac{r/a}{1+r/a}\right]}$$

(logaritmicamente) $M_{NFW}\to\infty$ se $r\to\infty$!

Non va bene a $r \gg a$

70 ter

$$\boxed{v_c(r) = \sqrt{\frac{G M(r)}{r}}}$$

$$v_c^2(r)_J = \frac{G}{\cancel{r}} \cdot 4\pi \rho_0 a^{\cancel{3}\,2} \frac{\cancel{r/a}}{1+r/a} = 4\pi G \rho_0 a^2 \cdot \frac{1}{1+r/a}$$

$$\boxed{v_c(r)_J = \sqrt{4\pi G \rho_0 a^2} \cdot \frac{1}{(1+r/a)^{1/2}}}$$

$$v_c^2(r)_H = \frac{G}{\cancel{r}_1} 4\pi \rho_0 a^{\cancel{3}\,2} \cdot \frac{(r/a)^{\cancel{2}\,1}}{2(1+r/a)^2} = 4\pi G \rho_0 a^2 \cdot \frac{r/a}{2(1+r/a)^2}$$

$$\boxed{v_c(r)_H = \sqrt{4\pi G \rho_0 a^2} \cdot \frac{\sqrt{r/a}}{\sqrt{2}(1+r/a)}}$$

$$v_c^2(r)_{NFW} = \frac{G}{r} \cdot 4\pi \rho_0 a^3 \cdot \left[\ln(1+\frac{r}{a}) - \frac{r/a}{1+r/a} \right] = 4\pi G \rho_0 a^2 \left[\frac{\ln(1+r/a)}{r/a} - \frac{1}{1+r/a} \right]$$

$$\boxed{v_c(r)_{NFW} = \sqrt{4\pi G \rho_0 a^2} \cdot \left[\frac{\ln(1+r/a)}{r/a} - \frac{1}{1+r/a} \right]^{1/2}}$$

$$\boxed{\Phi_{(r)} = -G \int_r^\infty \frac{M(s)}{s^2} ds}$$

$$\int \frac{dx}{x(1+x)} = -\ln \left| \frac{1+x}{x} \right|$$

$$\Phi_J(r) = -G \int_r^\infty 4\pi \rho_0 a^3 \frac{s/a}{1+s/a} \cdot \frac{1}{s^2} \frac{ds}{a} \cdot a = -4\pi G \rho_0 a^2 \int_{r/a}^\infty \frac{1}{x(1+x)} dx$$

$$\boxed{= -4\pi G \rho_0 a^2 \cdot \ln\left[1+\frac{a}{r}\right]}$$

$$\int \frac{dx}{(1+x)^2} = -\frac{1}{1+x} \;?$$

$$\Phi_H(r) = -4\pi G \rho_0 \int_r^\infty a^3 \cdot \frac{1}{s^2} \cdot \frac{(s/a)^2}{2(1+s/a)^2} \frac{ds}{a} \cdot a = -4\pi G \rho_0 a^2 \underbrace{\int_{r/a}^\infty \frac{dx}{2(1+x)^2}}_{\frac{1}{2} \cdot \frac{1}{1+r/a}}$$

$$\boxed{\Phi_H(r) = -4\pi G \rho_0 a^2 \cdot \frac{1}{2(1+r/a)}}$$

$$\Phi_{NFW}(r) = -4\pi G \rho_0 \int_r^\infty a^3 \cdot \frac{1}{s^2} \cdot \left[\ln(1+\frac{s}{a}) - \frac{s/a}{1+s/a} \right] \frac{ds}{a} \cdot a =$$

$$= -4\pi G \rho_0 a^2 \left[\int_{r/a}^\infty \frac{\ln(1+x)}{x^2} dx - \int_{r/a}^\infty \frac{dx}{x(1+x)} \right]$$

$$\int \frac{\ln(1+x)}{x^2} dx =$$

$$= \ln x - \left(\frac{1}{x}+1\right) \ln(1+x)$$

$$\Phi_{NFW}(r) = -4\pi G\rho_0 a^2 \left[\cancel{\ln x} - \underbrace{\frac{1+x}{x}\ln(1+x) + \ln(1+x)}_{\ln(1+x)\left[1-\frac{1+x}{x}\right]} - \cancel{\ln x} \right]_{r/a}^{\infty}$$

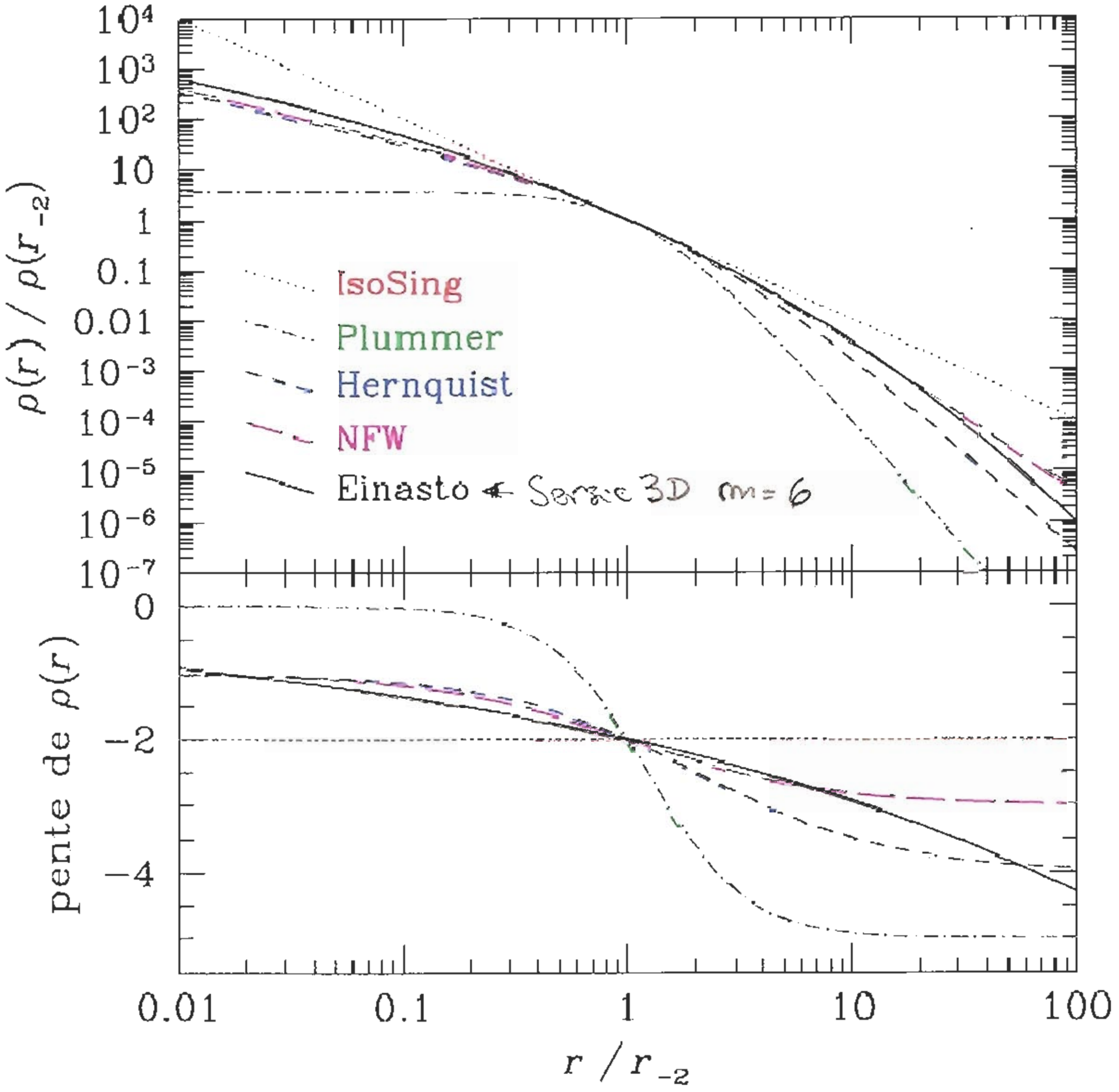
70 queker

$$\ln(1+x)\left[1 - \underbrace{\frac{1+x}{x}}\right]$$

$$\underbrace{\frac{\cancel{x} - 1 - \cancel{x}}{x}} = -\frac{\ln(1+x)}{x}$$

$$\boxed{\Phi_{NFW}(r) = -4\pi G\rho_0 a^2 \cdot \frac{\ln(1+r/a)}{r/a}}$$

71 bis



Modello di Sersic 3D: (detto anche di Einasto)

$$\rho_m(r) = \rho_0 \, e^{-(r/a)^{1/m}}$$

Hanno il vantaggio che se $r \to 0$ $\rho(r) \to \rho_0$ finito ed hanno anche una massa che converge:

$$M = \int_0^\infty 4\pi \frac{r^2}{a^2} \rho_0 \, e^{-(r/a)^{1/m}} \, d\frac{r}{a} \cdot a^3 = 4\pi a^3 \rho_0 \int_0^\infty x^2 e^{-x^{1/m}} dx$$

funzione Gamma

$$\int_0^\infty x^{m-1} e^{-x} dx = \Gamma(m)$$

$$x^{1/m} \equiv u \to x = u^m, \quad x^2 = u^{2m}, \quad dx = m\, u^{m-1} du$$

$$\boxed{M = 4\pi a^3 \rho_0 \int_0^\infty u^{3m-1} e^{-u} du = 4\pi a^3 \rho_0 \cdot m\,\Gamma(3m)}$$

$$\ln\rho = \ln\rho_0 - (r/a)^{1/m} \to \frac{d\ln\rho}{dr} = -\frac{1}{m}\left(\frac{r}{a}\right)^{1/m - 1} / a \to \frac{d\ln\rho}{d\ln r} = -\frac{1}{m}\, r \cdot \left(\frac{r}{a}\right)^{1/m - 1} \cdot \frac{a}{a}$$

$$\boxed{\frac{d\ln\rho}{d\ln r} = -\frac{a}{m}\left(r/a\right)^{1/m}}$$

Arter

Modelli N-body di Dark Matter

Navarro et al 2004 :

Mon. Not. R. Astron. Soc. 349, 1039–1051 (2004)

After some experimentation, we have found that a density profile where $\beta(r)$ is a power law of radius is a reasonable compromise that satisfies these constraints whilst retaining simplicity, i.e.

$$\beta_\alpha(r) = -\mathrm{d}\ln\rho/\mathrm{d}\ln r = 2(r/r_{-2})^\alpha, \qquad (4)$$

$\alpha \equiv 1/m$

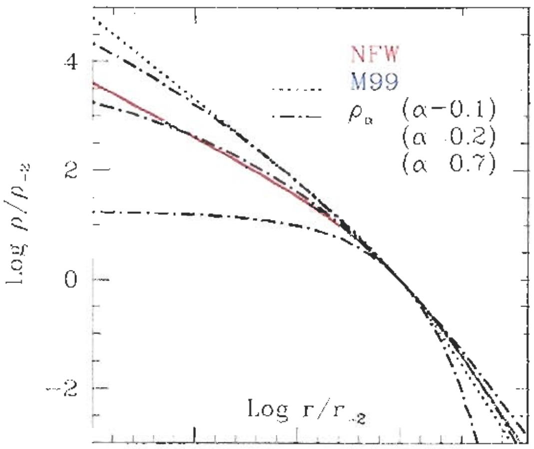
which corresponds to a density profile of the form

$$\ln(\rho_\alpha/\rho_{-2}) = (-2/\alpha)[(r/r_{-2})^\alpha - 1]. \qquad (5)$$

ρ_{-2} : densità a $r = r_{-2}$, r a cui la pendenza $= -2$

This profile has finite total mass (the density cuts off exponentially at large radius) and has a logarithmic slope that decreases inward more gradually than the NFW or M99 profiles. The thick dot-dashed curves in Figs 3 and 4 show that equation (5) (with $\alpha \sim 0.17$) $= \frac{1}{6}$ does indeed reproduce fairly well the radial dependence of $\beta(r)$ and $\beta_{\max}(r)$ in simulated haloes.

È modello di Sersic 3D con $m=6$



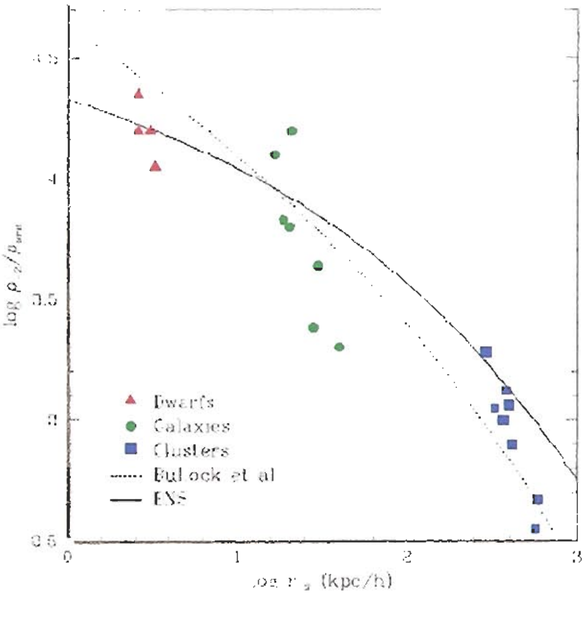


Figure 8. The radius, r_{-2}, where the logarithmic slope of the density profile takes the 'isothermal' value, $\beta(r_{-2}) = 2$, plotted versus the local density at that radius, $\rho_{-2} = \rho(r_{-2})$, for all simulated haloes in our series. This figure illustrates the mass dependence of the central concentration of dark matter haloes: low-mass haloes are systematically denser than their more massive counterparts. Solid and dotted lines indicate the scale radius-characteristic density correlation predicted by the formalisms presented by Eke et al. (2001) and Bullock et al. (2001). These parameters may be used, in conjunction with equation (5), to predict the mass profile of ΛCDM haloes.

$$\to \frac{d\ln\rho}{d\ln r} = -2\left(\frac{r}{r_{-2}}\right)^\alpha$$

$$\int d\ln\rho = -2\int\left(\frac{r}{r_{-2}}\right)^\alpha \frac{dr/r_{-2}}{r/r_{-2}} = -2\int x^{\alpha-1}dx$$

$$\downarrow$$

$$\ln\rho + \mathrm{cost} = -\frac{2}{\alpha}\left(\frac{r}{r_{-2}}\right)^\alpha$$

Se $r = r_{-2}$ $\rho = \rho_{-2}$ $\to$ $\ln\rho_{-2} + \mathrm{cost} = -\frac{2}{\alpha}$

$$\boxed{\ln(\rho/\rho_{-2}) = \left(-\frac{2}{\alpha}\right)\left[\left(\frac{r}{r_{-2}}\right)^\alpha - 1\right] \quad \text{cvd}}$$

$$\rho_{cr} = \frac{3H^2}{8\pi G} \sim 2\times10^{-29}h^2\ g/cm^3$$

densità critica dello Universo (vedi Cosmologia)

Eq. di Poisson in sistemi molto schiacciati

Coor. cilindriche: eq. di Poisson

$$\nabla^2\phi = \frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial\phi}{\partial R}\right) + \underbrace{\frac{1}{R^2}\frac{\partial^2\phi}{\partial\theta^2}}_{\substack{\equiv 0 \\ \times \text{ simm. cilindrica}}} + \frac{\partial^2\phi}{\partial z^2} = 4\pi G\rho(R,z)$$

detto $F_R \equiv -\frac{\partial\phi}{\partial R}$ si ha

$$\boxed{\frac{\partial^2\phi}{\partial z^2} = 4\pi G\rho(R,z) + \frac{1}{R}\frac{\partial}{\partial R}(R F_R)}$$

Consideriamo disco di Miyamoto-Nagai con

$$\rho_M(R,z) = \frac{1}{4\pi}\frac{M}{a^3}\left(\frac{b}{a}\right)^2 \frac{\left(\frac{R}{a}\right)^2 + \left(1+3\sqrt{\left(\frac{z}{a}\right)^2+\left(\frac{b}{a}\right)^2}\right)\left(1+\sqrt{\left(\frac{z}{a}\right)^2+\left(\frac{b}{a}\right)^2}\right)^2}{\left[\left(\frac{R}{a}\right)^2+\left(1+\sqrt{\left(\frac{z}{a}\right)^2+\left(\frac{b}{a}\right)^2}\right)^2\right]^{5/2}\left[\left(\frac{z}{a}\right)^2+\left(\frac{b}{a}\right)^2\right]^{3/2}}$$

Se $b \to 0$ $(b/a) \to 0$ distribuzione di densità sempre più schiacciata e, a R fisso, $\rho(R, z=0)$ cresce come $(b/a)^{-1}$ e diverge per $b=0$.
Ma F_R resta finita: se $b=0$ $\phi_M \to \phi_K$ (di Kuzmin)

$$\phi_M = -\frac{GM}{\sqrt{R^2+(a+\sqrt{z^2+b^2})^2}} \to \phi_K = -\frac{GM}{\sqrt{R^2+(a+|z|)^2}}$$

e $F_R = -\frac{\partial\phi_K}{\partial R} = -(-1)\cdot\frac{1}{\cancel{2}}\cdot GM\,(R^2+(a+z)^2)^{-3/2}\cdot\cancel{2}R = -\frac{GMR}{(R^2+(a+z)^2)^{3/2}}$

$$F_R(R, z=0) \equiv -\frac{GMR}{(R^2+a^2)^{3/2}} \quad \underline{\underline{\text{finito}}}$$

Quindi, mentre il termine con F_R rimane finito, il termine con $\rho(R,z) \to \infty$, per cui domina e si può scrivere

$$\boxed{\frac{\partial^2\phi(R,z)}{\partial z^2} \simeq 4\pi G\rho(R,z)}$$

La variazione verticale del potenziale ad un dato raggio R dipende solo dalla distribuzione di densità a quel raggio.

Il potenziale della Galassia

Vediamo delle rappresentazioni più semplici di quelle proposte da B&T, II ed. Rappresentiamo le 3 componenti principali: il Bulge, il Disco e l'Alone.

Bulge Si può usare un modello di Hernquist

$$\rho_B(r) = \frac{\rho_0}{(r/a_B)(1+\frac{r}{a_B})^3} \qquad M_B = 2\pi\rho_0 a_B^3 \rightarrow \rho_0 = \frac{M_B}{2\pi a_B^3}$$

$$\boxed{\rho_B(r) = \frac{M_B}{2\pi a_B^3 (\frac{r}{a_B})(1+\frac{r}{a_B})^3}}$$

$$\phi_B = -\frac{\cancel{4}^{2}\pi G \rho_0 a_B^2}{\cancel{2}(1+r/a_B)}$$

$$\boxed{\phi_B = -\frac{G M_B}{r+a_B}}$$

$a_B \sim 0.7-0.8$ kpc
$M_B \sim 3.5\times 10^{10} M_\odot$

coord. sferiche → cilindriche: $\boxed{r \rightarrow \sqrt{R^2+z^2}}$

Disco Modello di Miyamoto & Nagai

$$\phi_D(R,z) = -\frac{G M_D}{\sqrt{R^2+(a_D+\sqrt{z^2+b_D^2})^2}}$$

$$\rho_D(R,z) = \frac{1}{4\pi}\frac{M_D}{a_D^3}\cdot\left(\frac{b_D}{a_D}\right)^2 \cdot \frac{\left(\frac{R}{a_D}\right)^2+\left(1+3\sqrt{\left(\frac{z}{a_D}\right)^2+\left(\frac{b_D}{a_D}\right)^2}\right)\left(1+\sqrt{\left(\frac{z}{a_D}\right)^2+\left(\frac{b_D}{a_D}\right)^2}\right)^2}{\left[\left(\frac{R}{a_D}\right)^2+\left(1+\sqrt{\left(\frac{z}{a_D}\right)^2+\left(\frac{b_D}{a_D}\right)^2}\right)^2\right]^{5/2}\left[\left(\frac{z}{a_D}\right)^2+\left(\frac{b_D}{a_D}\right)^2\right]^{3/2}}$$

$a_D \sim 6.5-8.5$ kpc
$b_D \sim 0.26$ kpc
$M_D \sim 1-2\times 10^{11} M_\odot$

Alone Potenziale logaritmico (per avere un possibile schiacciamento)

$$\boxed{\Phi_A \simeq \frac{1}{2} v_0^2 \ln\left(R_c^2 + R^2 + \frac{z^2}{q_\phi^2}\right) + \text{cost.}}$$

$$\rho(R,z) = \frac{1}{4\pi G}\left\{\frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial \Phi_A}{\partial R}\right) + \frac{\partial^2 \Phi_A}{\partial z^2}\right\}$$

$$\rho(R,z) = \frac{v_0^2}{4\pi G q_\phi^2} \frac{R^2 + R_c^2(2q_\phi^2+1) + z^2(2 - 1/q_\phi^2)}{\left(R_c^2 + R^2 + \frac{z^2}{q_\phi^2}\right)^2}$$

$v_0 \simeq 170$ km/s
$R_c \sim 12$ kpc
$q_\phi > 0.95$ 90% $q_\phi > 0.85$ 99%

NB: lo schiacciamento delle curve isopotenziali è $\sim 1/3$ dello schiacciamento delle curve di isodensità

Se $q_\phi < \frac{1}{\sqrt{2}} \sim 0.707$ $\rho(R,z)$ diventa negativa attorno all'asse z!

Modello sferico NFW:

$$\boxed{\Phi_A \simeq -4\pi G \rho_{0H} a_H^2 \frac{\ln(1+r/a_H)}{r/a_H}}$$

$a_H \sim 30$ kpc
$\rho_{0H} \sim 0.3 \times 10^{-24}$ g/cm^3

troncato a ~ 350 kpc, dopo di che $\phi \sim 1/r$

$$\boxed{\rho_A(r) \simeq \frac{\rho_{0H}}{(r/a_H)(1+r/a_H)^2}}$$

Ricordiamo che la massa di NFW diverge!

$$M_{NFW}(r) = 4\pi\rho_{0H} a_H^3 \left[\ln\left(1+\frac{r}{a_H}\right) - \frac{r/a_H}{1+r/a_H}\right]$$

1 g/cm^3 $\equiv 1.48 \times 10^{31}$ M$_\odot$/kpc^3 $\rightarrow 0.3 \times 10^{-24}$ g/cm^3 $= 4.4 \times 10^6$ M$_\odot$/kpc^3

NB: Dimensionalmente, ϕ è il quadrato di una velocità (vedi il pot. logaritmico) e si può misurare in (km/s)^2.
Se usiamo le masse in M$_\odot$, le distanze in kpc, allora per avere ϕ in (km/s)^2 si prende $G \cong 4.3 \times 10^{-6}$

$$\frac{6.67\times10^{-8}\ \text{cgs} \cdot M_\odot \cdot 1.99\times10^{33}\ \text{g}/M_\odot}{R_{kpc} \cdot 3.086\times10^{21}\ \text{cm/kpc}} = (\text{cm/s})^2 / \cdot \frac{1}{(10^5)^2} \rightarrow (\text{km/s})^2$$

Velocità $\quad \bar v = \dot{\bar r}$

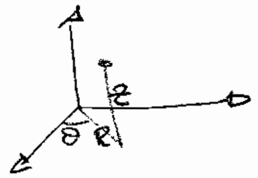
$$\boxed{\dot{\bar r} = \frac{d\bar r}{dt} = \frac{\partial \bar r}{\partial u^i}\frac{du^i}{dt} = \underbrace{\left|\frac{\partial \bar r}{\partial u^i}\right|}_{h_i}\hat e_i \cdot \frac{du^i}{dt} = h_i \frac{du^i}{dt}\hat e_i}$$

Coord. cilindriche

$u^1 = R \quad u^2 = \theta \quad u^3 = z$

$h_R \equiv 1 \qquad h_\theta \equiv R \qquad h_z \equiv 1$

$$\dot{\bar r} = \bar v = \underbrace{\dot R}_{v_R}\hat e_R + \underbrace{R\dot\theta}_{v_\theta}\hat e_\theta + \underbrace{\dot z}_{v_z}\hat e_z$$



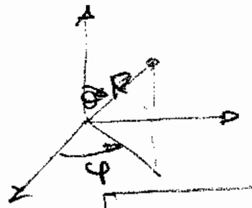
$\boxed{\dot R = v_R} \qquad R\dot\theta = v_\theta \to \boxed{\dot\theta = v_\theta/R} \qquad \boxed{\dot z = v_z}$

Coord. sferiche

$u^1 = R \quad u^2 = \theta \quad u^3 = \varphi$

$h_R = 1 \qquad h_\theta = R \qquad h_\varphi = R\,\mathrm{sen}\,\theta$

$$\dot{\bar r} = \bar v = \underbrace{\dot R}_{v_R}\hat e_R + \underbrace{R\dot\theta}_{v_\theta}\hat e_\theta + \underbrace{R\,\mathrm{sen}\,\theta\,\dot\varphi}_{v_\varphi}\hat e_\varphi$$



$\boxed{\dot R = v_R} \qquad R\dot\theta = v_\theta \to \boxed{\dot\theta = v_\theta/R} \qquad R\,\mathrm{sen}\,\theta\,\dot\varphi = v_\varphi \to \boxed{\dot\varphi = \frac{v_\varphi}{R\,\mathrm{sen}\,\theta}}$

Da queste relazioni otteniamo le derivate delle velocità:

Coord. cilindriche

$$\begin{cases} v_R = \dot R \\ v_\theta = R\dot\theta \\ v_z = \dot z \end{cases} \Longrightarrow \begin{cases} \dot v_R = \ddot R \\ \dot v_\theta = \dot R\dot\theta + R\ddot\theta \\ \dot v_z = \ddot z \end{cases}$$

Coord. sferiche

$$\begin{cases} v_R = \dot R \\ v_\theta = R\dot\theta \\ v_\varphi = R\,\mathrm{sen}\,\theta\,\dot\varphi \end{cases} \Longrightarrow \begin{cases} \dot v_R = \ddot R \\ \dot v_\theta = \dot R\dot\theta + R\ddot\theta \\ \dot v_\varphi = \dot R\,\mathrm{sen}\,\theta\,\dot\varphi + R\cos\theta\,\dot\theta\dot\varphi + \\ \qquad + R\,\mathrm{sen}\,\theta\,\ddot\varphi \end{cases}$$

Accelerazione

CC7

$$\dot{\bar r} = h_i \frac{du^i}{dt}\hat e_i \Rightarrow \ddot{\bar r} = \frac{d}{dt}\left[h_i \frac{du^i}{dt}\hat e_i\right] =$$

$$\ddot{\bar r} = \dot h_i \frac{du^i}{dt}\hat e_i + h_i\left[\frac{d^2u^i}{dt^2}\hat e_i + \frac{du^i}{dt}\dot{\hat e}_i\right] =$$

$$\boxed{\ddot{\bar r} = \left[\dot h_i \frac{du^i}{dt} + h_i\frac{d^2u^i}{dt^2}\right]\hat e_i + h_i \frac{du^i}{dt}\dot{\hat e}_i}$$

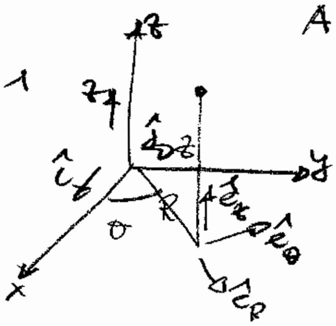
Gli $\dot{\hat e}_i$ dipendono dal sistema di riferimento

Coord. cilindriche

$u^1=R \quad u^2=\theta \quad u^3=z$

$h_R=1 \quad h_\theta=R \quad h_z=1$

$$\bar A\times\bar B = \begin{vmatrix} \hat i & \hat j & \hat k\\ A_x & A_y & A_z\\ B_x & B_y & B_z\end{vmatrix}$$



$$\hat e_\theta = \hat e_z \times \hat e_R$$

$\hat e_R = \hat i\cos\theta + \hat j\,\mathrm{sen}\,\theta$ (NB: modulo = 1)

$\hat e_z = \hat k$

$$\hat e_\theta = \begin{vmatrix} \hat i & \hat j & \hat k\\ 0 & 0 & 1\\ \cos\theta & \mathrm{sen}\,\theta & 0\end{vmatrix} = -\hat i\,\mathrm{sen}\,\theta + \hat j\cos\theta$$

$\hat e_R$ e $\hat e_\theta$ dipendono dall'angolo θ, $\hat e_z$ nemmeno da quello, per cui tra le derivate $\frac{\partial \hat e_i}{\partial u^j}$ sono $\neq 0$ solo

$$\begin{cases} \dfrac{\partial \hat e_R}{\partial \theta} = -\hat i\,\mathrm{sen}\,\theta + \hat j\cos\theta \equiv \hat e_\theta \\ \dfrac{\partial \hat e_\theta}{\partial \theta} = -\hat i\cos\theta - \hat j\,\mathrm{sen}\,\theta \equiv -\hat e_R \end{cases}$$

Perciò $\dot{\hat e}_i \equiv \frac{\partial \hat e_i}{\partial u^j}\cdot \dot u^j \Rightarrow \begin{cases} \dot{\hat e}_R = \dot\theta\,\hat e_\theta \\ \dot{\hat e}_\theta = -\dot\theta\,\hat e_R \end{cases}$

$+R\dot\theta\dot{\hat e}_\theta = $
$= R\dot\theta(-\dot\theta\,\hat e_R)$

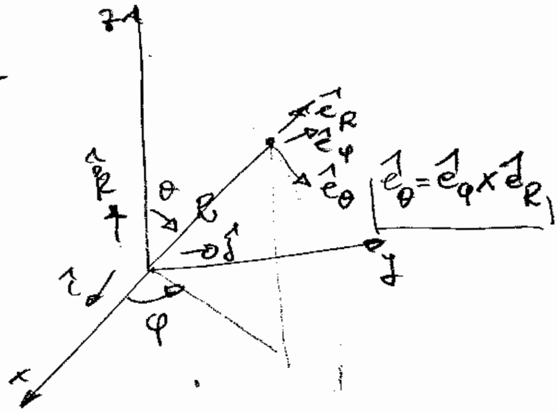
A questo punto possiamo scrivere:

$$\ddot{\bar r} = \ddot R\hat e_R + \dot R\underbrace{\dot\theta\hat e_\theta}_{\dot{\hat e}_R} + [\dot R\dot\theta + R\ddot\theta]\hat e_\theta - R\dot\theta^2\hat e_R + \ddot z\hat e_z$$

$$\boxed{\ddot{\bar r} = (\ddot R - R\dot\theta^2)\hat e_R + (2\dot R\dot\theta + R\ddot\theta)\hat e_\theta + \ddot z\,\hat e_z}$$

$\boxed{\text{Coord. sferiche}}$ $u^1 = R \quad u^2 = \theta \quad u^3 = \varphi$

$h_R = 1 \quad h_\theta = R \quad h_\varphi = R \operatorname{sen}\theta$



$\vec{e}_\theta = \vec{e}_\varphi \times \vec{e}_R$

$$\begin{cases} \hat{e}_R = \hat{\imath} \operatorname{sen}\theta\cos\varphi + \hat{j}\operatorname{sen}\theta\operatorname{sen}\varphi + \hat{k}\cos\theta \\ \hat{e}_\varphi = \dfrac{\hat{k}\times\hat{e}_R}{\operatorname{sen}\theta} = \dfrac{1}{\operatorname{sen}\theta}\cdot\left[-\hat{\imath}\operatorname{sen}\theta\operatorname{sen}\varphi + \hat{j}\operatorname{sen}\theta\cos\varphi\right] \\ \quad = -\hat{\imath}\operatorname{sen}\varphi + \hat{j}\cos\varphi \\ \vec{e}_\theta = \hat{\imath}\cos\varphi\cos\theta + \hat{j}\operatorname{sen}\varphi\cos\theta - \hat{k}\operatorname{sen}\theta \end{cases}$$

$$\hat{k}\times\hat{e}_R \equiv \begin{vmatrix} \hat{\imath} & \hat{j} & \hat{k} \\ 0 & 0 & 1 \\ \operatorname{sen}\theta\cos\varphi & \operatorname{sen}\theta\operatorname{sen}\varphi & \cos\theta \end{vmatrix}$$

$$\vec{e}_\varphi\times\vec{e}_R = \begin{vmatrix} \hat{\imath} & \hat{j} & \hat{k} \\ -\operatorname{sen}\varphi & \cos\varphi & 0 \\ \operatorname{sen}\theta\cos\varphi & \operatorname{sen}\theta\operatorname{sen}\varphi & \cos\theta \end{vmatrix}$$

$\hat{e}_R$ e $\hat{e}_\theta$ dipendono da θ e φ, $\hat{e}_\varphi$ solo da φ

Saranno quindi non nulle solo le seguenti derivate:

$$\begin{cases} \dfrac{\partial\vec{e}_R}{\partial\theta} = \hat{\imath}\cos\theta\cos\varphi + \hat{j}\cos\theta\operatorname{sen}\varphi - \hat{k}\operatorname{sen}\theta \equiv \hat{e}_\theta \\ \dfrac{\partial\hat{e}_R}{\partial\varphi} = -\hat{\imath}\operatorname{sen}\theta\operatorname{sen}\varphi + \hat{j}\operatorname{sen}\theta\cos\varphi = \operatorname{sen}\theta\cdot\hat{e}_\varphi \\ \dfrac{\partial\vec{e}_\varphi}{\partial\varphi} = -\hat{\imath}\cos\varphi - \hat{j}\operatorname{sen}\varphi = -(\vec{e}_R\operatorname{sen}\theta + \vec{e}_\theta\cos\theta) \\ \dfrac{\partial\hat{e}_\theta}{\partial\theta} = -\hat{\imath}\cos\varphi\operatorname{sen}\theta - \hat{j}\operatorname{sen}\varphi\operatorname{sen}\theta - \hat{k}\cos\theta = -\vec{e}_R \\ \dfrac{\partial\hat{e}_\theta}{\partial\varphi} = -\hat{\imath}\operatorname{sen}\varphi\cos\theta + \hat{j}\cos\varphi\cos\theta = \cos\theta\cdot\vec{e}_\varphi \end{cases}$$

Perciò sarà

$$\dot{\vec{e}}_i \equiv \frac{\partial\vec{e}_i}{\partial u^j}\dot{u}^j \Longrightarrow \begin{cases} \dot{\hat{e}}_R = \dot{\theta}\vec{e}_\theta + \operatorname{sen}\theta\,\dot{\varphi}\,\vec{e}_\varphi \\ \dot{\hat{e}}_\varphi = -\dot{\varphi}\left(\vec{e}_R\operatorname{sen}\theta + \vec{e}_\theta\cos\theta\right) \\ \dot{\hat{e}}_\theta = -\dot{\theta}\vec{e}_R + \cos\theta\,\dot{\varphi}\,\vec{e}_\varphi \end{cases}$$

Scriveremo:

$$\ddot{\vec{r}} = \ddot{R}\hat{e}_R + \dot{R}\dot{\theta}\vec{e}_\theta + \dot{R}\operatorname{sen}\theta\,\dot{\varphi}\,\vec{e}_\varphi + \left[\dot{R}\dot{\theta} + R\ddot{\theta}\right]\vec{e}_\theta + R\dot{\theta}\left[-\dot{\theta}\vec{e}_R + \cos\theta\,\dot{\varphi}\,\vec{e}_\varphi\right] +$$
$$+ \left[\dot{R}\operatorname{sen}\theta\,\dot{\varphi} + R\cos\theta\,\dot{\theta}\,\dot{\varphi} + R\operatorname{sen}\theta\,\ddot{\varphi}\right]\vec{e}_\varphi + R\operatorname{sen}\theta\,\dot{\varphi}\left[-\dot{\varphi}\operatorname{sen}\theta\,\vec{e}_R - \dot{\varphi}\cos\theta\,\vec{e}_\theta\right] =$$
$$= \hat{e}_R\left(\ddot{R} - R\dot{\theta}^2 - R\operatorname{sen}^2\theta\,\dot{\varphi}^2\right] + \vec{e}_\theta\left[2\dot{R}\dot{\theta} + R\ddot{\theta} - R\operatorname{sen}\theta\cos\theta\,\dot{\varphi}^2\right] +$$
$$+ \vec{e}_\varphi\left[2\dot{R}\operatorname{sen}\theta\,\dot{\varphi} + 2R\dot{\theta}\cos\theta\,\dot{\varphi} + R\operatorname{sen}\theta\,\ddot{\varphi}\right]$$

Se ci troviamo in un sistema a simmetria sferica la forza agirà solamente lungo $\hat{e}_R$ e sarà

$$\bar{F} = -m \frac{d\phi}{dR} \hat{e}_R = m\bar{a} \rightarrow \boxed{\bar{a} = \ddot{\bar{r}} = -\frac{d\phi}{dR} \cdot \hat{e}_R}$$

mentre saranno nulle le componenti di $\bar{a}$ lungo $\hat{e}_\theta$ ed $\hat{e}_\varphi$.

Dalla relazione appena ricavata per $\ddot{\bar{r}}$ avremo allora:

- componente $\hat{e}_R$?

$$\ddot{R} - R\dot{\theta}^2 - R \operatorname{sen}^2\theta \, \dot{\varphi}^2 \equiv -\frac{d\phi}{dR}$$

dalla $v_R = \dot{R} \rightarrow \dot{v}_R = \ddot{R}$ cioè

$$\dot{v}_R = \ddot{R} = -\frac{d\phi}{dR} + R\dot{\theta}^2 + R \operatorname{sen}^2\theta \, \dot{\varphi}^2$$

$$= -\frac{d\phi}{dR} + \cancel{R} \cdot \frac{v_\theta^2}{R^{\cancel{2}}} + \cancel{R} \, \cancel{\operatorname{sen}^2\theta} \, \frac{v_\varphi^2}{R^{\cancel{2}} \cancel{\operatorname{sen}^2\theta}}$$

$$\boxed{\dot{v}_R = -\frac{d\phi}{dR} + \frac{v_\theta^2 + v_\varphi^2}{R}}$$

Ricordiamo [cc6]

$\dot{R} = v_R \quad \dot{\theta} = v_\theta / R \quad \dot{\varphi} = \frac{v_\varphi}{R \operatorname{sen}\theta}$

$\dot{v}_R = \ddot{R}$

$\dot{v}_\theta = \dot{R}\dot{\theta} + R\ddot{\theta}$

$\dot{v}_\varphi = \dot{R} \operatorname{sen}\theta \, \dot{\varphi} + R\cos\theta \, \dot{\theta} \dot{\varphi} + R \operatorname{sen}\theta \, \ddot{\varphi}$

- componente $\hat{e}_\theta = 0$

$$2\dot{R}\dot{\theta} + R\ddot{\theta} - R \operatorname{sen}\theta \cos\theta \, \dot{\varphi}^2 \equiv 0$$

$$\boxed{\dot{v}_\theta = \dot{R}\dot{\theta} + R\ddot{\theta} = \cancel{R} \, \cancel{\operatorname{sen}\theta} \cos\theta \cdot \frac{v_\varphi^2}{R^{\cancel{2}} \operatorname{sen}^{\cancel{2}}\theta} - v_R \cdot \frac{v_\theta}{R} = \cot\theta \, \frac{v_\varphi^2}{R} - \frac{v_R v_\theta}{R}}$$

- componente $\hat{e}_\varphi = 0$

$$2\dot{R} \operatorname{sen}\theta \, \dot{\varphi} + 2R\dot{\theta} \cos\theta \, \dot{\varphi} + R \operatorname{sen}\theta \, \ddot{\varphi} \equiv 0$$

$$\dot{v}_\varphi = \dot{R} \operatorname{sen}\theta \, \dot{\varphi} + R\cos\theta \, \dot{\theta}\dot{\varphi} + R \operatorname{sen}\theta \, \ddot{\varphi} = -\dot{R} \operatorname{sen}\theta \, \dot{\varphi} - R\cos\theta \, \dot{\theta}\dot{\varphi} =$$

$$= -v_R \, \cancel{\operatorname{sen}\theta} \, \frac{v_\varphi}{R \, \cancel{\operatorname{sen}\theta}} - \cancel{R} \cos\theta \, \frac{v_\theta}{\cancel{R}} \cdot \frac{v_\varphi}{R \operatorname{sen}\theta} =$$

$$\boxed{\dot{v}_\varphi = -\frac{v_R v_\varphi + \cot\theta \, v_\theta v_\varphi}{R}}$$

Se ho una funzione $f = f(R, \theta, \varphi, v_R, v_\theta, v_\varphi)$ e voglio fare la sua $\frac{df}{dt}$, sarà

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \dot{R}\frac{\partial f}{\partial R} + \dot{\theta}\frac{\partial f}{\partial \theta} + \dot{\varphi}\frac{\partial f}{\partial \varphi} + \dot{v}_R \frac{\partial f}{\partial v_R} + \dot{v}_\theta \frac{\partial f}{\partial v_\theta} + \dot{v}_\varphi \frac{\partial f}{\partial v_\varphi} =$$

$$= \frac{\partial f}{\partial t} + v_R \frac{\partial f}{\partial R} + \frac{v_\theta}{R}\frac{\partial f}{\partial \theta} + \frac{v_\varphi}{R \operatorname{sen}\theta}\frac{\partial f}{\partial \varphi} + \left(\frac{v_\theta^2 + v_\varphi^2}{R} - \frac{d\phi}{dR}\right)\frac{\partial f}{\partial v_R} + \frac{\cot\theta \, v_\varphi^2 - v_R v_\theta}{R}\frac{\partial f}{\partial v_\theta} +$$

$$- \frac{v_R v_\varphi + \cot\theta \, v_\theta v_\varphi}{R}\frac{\partial f}{\partial v_\varphi}$$

Come si vedrà, $\frac{df}{dt} = 0$ è l'eq. di Boltzmann non collisionale.