Lecture 1 – Unconstrained Optimization

Introduction to Optimisation

Many economic functions of interest (eg utility functions, production functions, profit functions, cost functions) are non linear

The idea behind optimisation is to choose the point where a function reaches a maximum or minimum value

Decision-makers are assumed to be "rational“ i.e.

1. each decision-maker is assumed to have a preference ordering over the outcomes to which her actions lead
2. Each decision makers chooses the action, among those feasible, that leads to the most preferred outcome (according to this ordering).

We usually make assumptions that guarantee that a decision-maker's preference ordering is represented by a *payoff function* (sometimes called *utility function*), so the decision-maker's problem is:

$$\max_a u(a) \text{ subject to } a \in S$$

$$\max_a u(a) \text{ subject to } a \in S$$

where: u is the decision-maker's payoff function over her actions

S is the set of her feasible actions.

classical consumer: ***a*** is a consumption bundle, ***u*** is the consumer's utility function, and ***S*** is the set of bundles of goods the consumer can afford.

Firm: ***a*** is an input-output vector, ***u*(*a*)** is the profit the action *a* generates, and ***S*** is the set of all feasible input-output vectors

In economic theory we sometimes need to solve a *minimization* problem of the form

$$\min_a u(a) \text{ subject to } a \in S$$

- we assume, for example, that firms choose input bundles to minimize the cost of producing any given output;
- an analysis of the problem of minimizing the cost of achieving a certain payoff greatly facilitates the study of a payoff-maximizing consumer.

Optimization: definitions

The optimization problems we study take the form

$$\max_x f(x) \text{ subject to } x \in S$$

where:

- f is a function,
- x is an n-vector (which we can also write as $(x_1, ..., x_n)$),
- S is a set of n-vectors.

We call:

- f the **objective function**,
- x the **choice variable**, and
- S the **constraint set** or **opportunity set**.

Definition

The value x^* of the variable x solves the problem

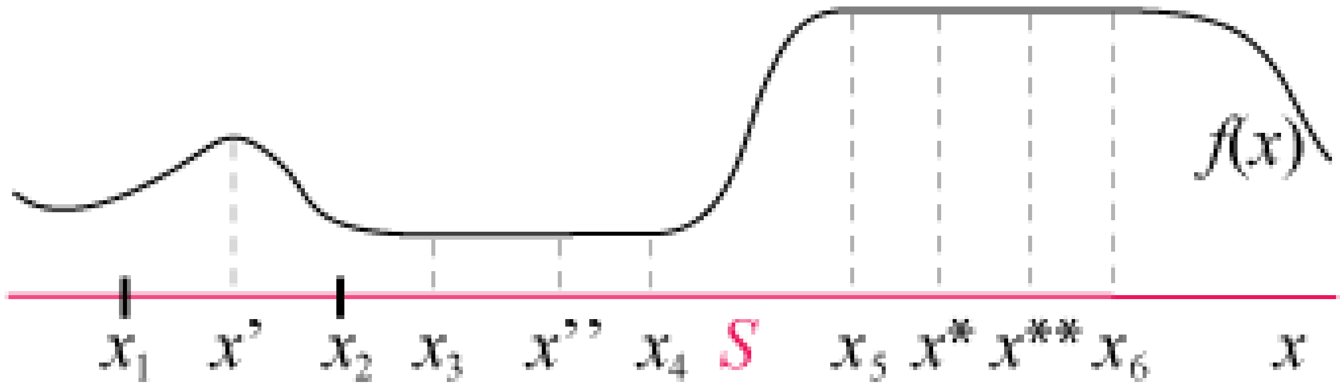
$$\max_x f(x) \text{ subject to } x \in S$$

if $f(x) \leq f(x^*)$ for all $x \in S$.

In this case we say that:

- x^* is a **maximizer** of the function f subject to the constraint $x \in S$
- $f(x^*)$ is the **maximum** (or **maximum value**) of the function f subject to the constraint $x \in S$.

A **minimizer** is defined analogously



x* and ***x***** are **maximizers** of *f* subject to the constraint $x \in S$
x'' is a **minimizer**

What is ***x'*** ?

It is not a maximizer, because $f(x^*) > f(x')$,
It is not a minimizer, because $f(x'') < f(x')$

But it is a maximum *among the points close to it*. We call such a point a <u>**local maximizer**</u>

Definition

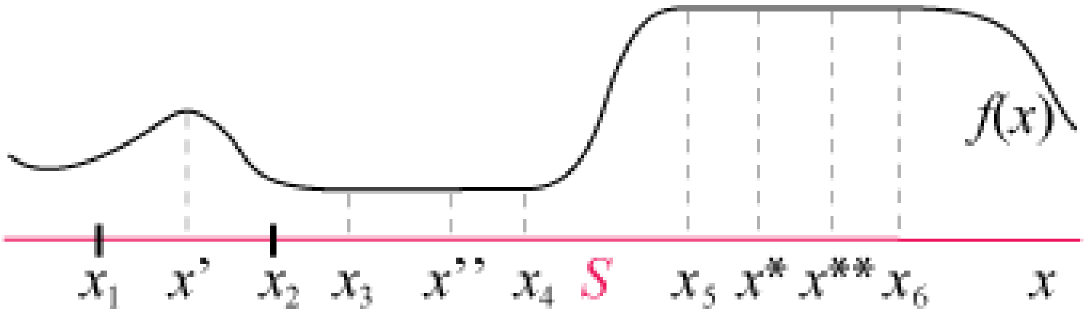
The variable x^* is a ***local maximizer*** *of the function f subject to the constraint $x \in S$ if there is a number $\varepsilon > 0$ such that $f(x) \leq f(x^*)$ for all $x \in S$ for which the distance between x and x^* is at most ε.*

Note: suppose that x and x' are vectors, then the *distance* between two points x and x' is the square root of $\sum_{i=1}^{n}(xi - xi')2$

A **local minimizer** is defined analogously.

Sometimes we refer to a maximizer as a **global maximizer** to emphasize that it is not only a local maximizer.

Every **global maximizer** is, in particular, **a local maximizer** (ε can take any value), and every minimizer is a local minimizer.



$f(x) \leq f(x')$ for all x between x_1 and x_2, where $|x_1 - x'| = |x_2 - x'|$

then point x' is a local maximizer of f (set $\varepsilon = |x_1 - x'|$).

But note that the point x'' is also a local maximizer of f , even though it is a global *minimizer*.

The function is constant between x_3 and x_4. The point x_4 is closer to x'' than is the point x_3, so we can take the ε in the definition of a local maximizer to be $x_4 - x''$. For every point x within the distance ε of x'', we have $f(x) = f(x'')$, so that in particular $f(x) \leq f(x'')$.

Transforming the objective function

Let g be a strictly increasing function of a single variable.

i.e. if $z' > z$ then $g(z') > g(z)$

Then the set of solutions to the problem

$$\max_x f(x) \text{ subject to } x \in S \qquad (1)$$

is identical to the set of solutions to the problem

$$\max_x g(f(x)) \text{ subject to } x \in S. \qquad (2)$$

Proof:

If x^* is a solution to the first problem then by definition $f(x) \leq f(x^*)$ for all $x \in S$.

But if $f(x) \leq f(x^*)$ then $g(f(x)) \leq g(f(x^*))$, so that $g(f(x) \leq g(f(x^*))$ for all $x \in S$.

Hence x^* is a solution of the second problem.

Minimization problems

We concentrate on maximization problems. What about minimization problems?

Any minimization problem can be turned into a maximization problem by taking the negative of the objective function.

That is, the problem

$\min_x f(x)$ subject to $x \in S$

is equivalent (i.e. has the same set of solutions) to

$\max_x -f(x)$ subject to $x \in S$.

Thus we can solve any minimization problem by taking the negative of the objective function and apply the results for maximization problems.

Existence of an optimum

Let f be a function of n variables defined on the set S. The problems we consider take the form

$$\max_x f(x) \text{ subject to } x \in S \qquad \text{where } x = (x_1, ..., x_n).$$

Before we start to think about how to find the solution to a problem, we need to think about **whether the problem *has* a solution**

Some problems that do **not** have any solution.

1. $f(x) = x$, $S = [0, \infty)$

In this case, f increases without bound, and never attains a maximum.

2. $f(x) = 1 - 1/x$, $S = [1, \infty)$.

In this case, f converges to the value 1, but never attains this value.

3. $f(x) = x$, $S = (0, 1)$.

In this case, the points 0 and 1 are excluded from S.

As x approaches 1, the value of the function approaches 1, but this value is never attained for values of x in S, because S excludes $x = 1$.

4. $f(x) = x$ if $x < 1/2$ and $f(x) = x - 1$ if $x \geq 1/2$; $S = [0, 1]$.

In this case, as x approaches 1/2 the value of the function approaches 1/2, but this value is never attained, because at $x = 1/2$ the function jumps down to $-1/2$.

in the first two cases are that the set S is unbounded;

in the third case is that the interval S is open (does not contain its endpoints);

in the last case is that the function f is discontinuous.

Reminder: Definition of Bounded set

For functions of many variables, we need to define the concept of a *bounded* set.

The set S is **bounded** if there exists a number k such that the distance of every point in S from the origin is at most k.

Example

The set $[-1, 100]$ is bounded, because the distance of any point in the set from 0 is at most 100. The set $[0, \infty)$ is not bounded, because for any number k, the number $2k$ is in the set, and the distance of $2k$ to 0 is $2k$ which exceeds k.

Example

The set $\{(x, y): x^2 + y^2 \leq 4\}$ is bounded, because the distance of any point in the set from $(0, 0)$ is at most 2.

Example

The set $\{(x, y): xy \leq 1\}$ is not bounded, because for any number k the point $(2k, 0)$ is in the set, and the distance of this point from $(0, 0)$ is $2k$, which exceeds k.

We say that a set that is closed and bounded is **compact.**

Proposition (Extreme value theorem) :

A continuous function on a compact set attains both a maximum and a minimum on the set

Note that the requirement of boundedness is on the *set*, not the *function*.

Note also that the result gives only a **sufficient** condition for a function to have a maximum.

If a function is continuous and is defined on a compact set **then** it definitely has a maximum and a minimum.

The result does **not** rule out the possibility that a function has a maximum and/or minimum if it is not continuous or is not defined on a compact set.

UNCONSTRAINED OPTIMIZATION WITH MANY VARIABLES

Consider the problem:

$$max_x\ f(x)\ subject\ to\ x\ \in S$$

where x is a vector

Proposition (First Order Conditions, FOC)

Let f be a differentiable function of n variables defined on the set S. If the point x in the interior of S is a local or global maximizer or minimizer of f then

$$f_i'(x) = 0\ for\ i = 1, \ldots, n.$$

Then the condition that all partial derivatives are equal to zero is a **necessary condition** for an interior optimum (and therefore for an optimum in an unconstrained optimization where each element of x could be any of the real numbers.

Conditions under which a stationary point is a local optimum

(Second Order Conditions, SOC)

Let f be a function of n variables with continuous partial derivatives of first and second order, defined on the set S.
Suppose that x^* is a stationary point of f in the interior of S (so that $f_i'(x^*) = 0$ for all i).

If $H(x^*)$ is negative definite then x^* is a local maximizer.
If x^* is a local maximizer then $H(x^*)$ is negative semidefinite.
If $H(x^*)$ is positive definite then x^* is a local minimizer.
If x^* is a local minimizer then $H(x^*)$ is positive semidefinite.

where $H(x)$ denotes the Hessian of f at x.

When these conditions are satisfied, FOCs are necessary and sufficient conditions

Conditions under which a stationary point is a global optimum

Suppose that the function f has continuous partial derivatives in a convex set S and x^* is a stationary point of f in the interior of S (so that $f_i'(x^*) = 0$ for all i).

1. if f is concave then $\boldsymbol{x^*}$ is a global maximizer of f in S **if and only if** it is a stationary point of f

2. if f is convex then $\boldsymbol{x^*}$ is a global minimizer of f in S **if and only if** it is a stationary point of f .

$H(z)$ is negative semidefinite for all $z \in S$

$$\Rightarrow$$

x is a global maximizer of f in S **if and only if** x is a stationary point of f $\quad (f_i'(x) = 0)$

$H(z)$ is positive semidefinite for all $z \in S$

$$\Rightarrow$$

x is a global minimizer of f in S if and only if x is a stationary point of f ,

where $H(x)$ denotes the Hessian of f at x.

Given that conditions for definiteness are easier to check we apply the following procedure:

1. Check concavity of f to see if the conditions represent a maximum.

   a. We compute the Hessian

   b. We check if it is negative definite

If these conditions hold, H is negative definite, f is strictly concave and the stationary point is a maximum

2. If these conditions are violated by equality, i.e. are equal to zero, check the conditions for semi definiteness

3. If these conditions hold, H is negative semidefinite, f is concave and the stationary point is a maximum

4. If these conditions are violated, we need further investigation

Example 1: Unconstrained Maximization with two variables

For example Utility = U(x, y) or Output = F(K, L)

Now try to find the values of x and y which maximize a function $f(x, y)$

Three steps:

1. Set **both** 1st order conditions equal to zero $f_x = 0$ and $f_y = 0$

(the slope of the function with respect to both variables must be simultaneously zero)

2. Solve the equations simultaneously for x and y

However this is a necessary but not sufficient condition (saddle points, minimum points,….)

3. Second order conditions (for maximization)

$$H = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{pmatrix}$$

$$f_{xx} \leq 0, \quad f_{yy} \leq 0 \quad \text{and} \quad f_{xx}f_{yy} - f^2_{xy} \geq 0$$

Note: Second order conditions (for minimization) are

$f_{xx} \geq 0$, $f_{yy} \geq 0$ and $f_{xx}f_{yy} - f^2_{xy} \geq 0$

$f(x,y) = 4x - 2x^2 + 2xy - y^2$

1. (i). $f_x = 4 - 4x + 2y = 0$

   (ii). $f_y = 2x - 2y = 0$

2. Solve: from (ii) we have $x = y$

   insert into (i) to get $4 - 4x + 2x = 0$ or

   $4 = 2x$ or $x = 2$

   so $y = x = 2$

3. $H = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{pmatrix} = \begin{pmatrix} -4 & 2 \\ 2 & -2 \end{pmatrix}$

The first order leading principal minor is $f_{xx} = -4 < 0$

The second order leading principal minor is

$$f_{xx}f_{yy} - f^2_{xy} = (-4)(-2) - (2)^2 = 4 > 0$$

Then the matrix H is negative definite

f is (strictly) concave, so we have a maximum point where $x = 2$ and $y = 2$

Example 2

Maximize $f(x) = -x_1^2 - 2x_2^2$

The first order conditions are:

$$\begin{pmatrix} \dfrac{\partial f}{\partial x_1} \\ \dfrac{\partial f}{\partial x_2} \end{pmatrix} = \begin{pmatrix} -2x_1 \\ -4x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Is this a maximum? – it will be if function is concave

1. H is,

$$H = \begin{pmatrix} \dfrac{\partial^2 f}{\partial x_1^2} & \dfrac{\partial^2 f}{\partial x_1 \partial x_2} \\ \dfrac{\partial^2 f}{\partial x_2 \partial x_1} & \dfrac{\partial^2 f}{\partial x_2^2} \end{pmatrix} = \begin{pmatrix} -2 & 0 \\ 0 & -4 \end{pmatrix}$$

From H find the leading principal matrices by eliminating:

1. The last n-1 rows and columns – written as $D_1 = (-2)$
2. The last n-2 rows and columns – written as $D_2 = H$

Compute the determinants of these leading principal matrices.

1. $|D_1| = -2$
2. $|H| = 8$

Then the matrix H is negative definite

f is (strictly) concave

the values that satisfy FOC ($x = 0$ and $y = 0$) give a maximum.

Example 3

Total revenue R = $12q_1 + 18q_2$

Total Cost = $2q_1^2 + q_1q_2 + 2q_2^2$

Find the values of q_1 and q_2 that maximise profit

Profit = revenue – cost = $12q_1 + 18q_2 - (2q_1^2 + q_1q_2 + 2q_2^2)$

The first order conditions are:

$$\begin{pmatrix} \dfrac{\partial \pi}{\partial q_1} \\ \dfrac{\partial \pi}{\partial q_2} \end{pmatrix} = \begin{pmatrix} 12 - 4q_1 - q_2 \\ 18 - q_1 - 4q_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Solving for q_1 and q_2 gives $q_1 = 2$ and $q_2 = 4$

Is this a maximum? –it will be if function is concave

The Hessian is

$$H = \begin{pmatrix} \dfrac{\partial^2 \pi}{\partial q_1^2} & \dfrac{\partial^2 \pi}{\partial q_1 \partial q_2} \\[2ex] \dfrac{\partial^2 \pi}{\partial q_2 \partial q_1} & \dfrac{\partial^2 \pi}{\partial q_2^2} \end{pmatrix} = \begin{pmatrix} -4 & -1 \\ -1 & -4 \end{pmatrix}$$

From H find the leading principal matrices by eliminating:

1. The last n-1 rows and columns – written as $D_1 = (-4)$
2. The last n-2 rows and columns – written as $D_2 = H$

Compute the determinants of these leading principal matrices.

1. $|D_1| = -4$
2. $|H| = (-4) * (-4) - 1 = 15$

So H is negative definite, then f is (strictly) concave and the values for q_1 and q_2 maximise profits

Example with three variables

Maximize $f(x) = -x_1{}^2 - 2\,x_2{}^2 - x_3{}^2$

The first order conditions are:

$$\begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \\ \frac{\partial f}{\partial x_3} \end{pmatrix} = \begin{pmatrix} -2x_1 \\ -4x_2 \\ -2x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The Hessian is:

$$H = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_1 \partial x_3} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \frac{\partial^2 f}{\partial x_2 \partial x_3} \\ \frac{\partial^2 f}{\partial x_3 \partial x_1} & \frac{\partial^2 f}{\partial x_3 \partial x_2} & \frac{\partial^2 f}{\partial x_3^2} \end{pmatrix} = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

From H find the leading principal matrices by eliminating:

1. The last n-1 rows and columns – $D_1 = (-2)$
2. The last n-2 rows and columns – $D_2 = \begin{pmatrix} -2 & 0 \\ 0 & -4 \end{pmatrix}$
3. The last 0 rows and columns – $D_3 = H$

1. Compute the determinants of these leading principal matrices.
   1. $|D_1| = -2,$
   2. $|D_2| = 8$
   3. $|H| = -16$

H is negative definite, then f is (strictly) concave

Summing up – two variable maximization

1. Differentiate f(x) and solve the first order conditions are:

$$\begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

2. Check concavity of f to see if the conditions represent a maximum.

   a. We compute the Hessian

   b. We check if it is negative definite

   c. i.e. check if, for all x_1 and x_2,

$$\frac{\partial^2 f}{\partial x_1^2} < 0 \quad \text{and} \quad \begin{vmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} \end{vmatrix} \; or \; \begin{vmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{vmatrix} = f_{11} f_{22} - f_{21} f_{12} > 0$$

3. If these conditions hold, H is negative definite, f is strictly concave and the stationary point is a maximum

4. If these conditions are violated by equality, i.e. are equal to zero, check the conditions for semi definiteness

$$\frac{\partial^2 f}{\partial x_1^2} \leq 0 \qquad \frac{\partial^2 f}{\partial x_2^2} \leq 0 \qquad \begin{vmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{vmatrix} = f_{11} f_{22} - f_{21} f_{12} \geq 0$$

5. If these conditions hold, H is negative semidefinite, f is concave and the stationary point is a maximum

6. If these conditions are violated, we need further investigation

Summing up – 3 variable maximization

1. Differentiate f(x) and solve the the first order conditions are:

$$\begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \\ \frac{\partial f}{\partial x_3} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

2. Check concavity of f to see if the conditions represent a maximum.

    a. We compute the Hessian

    b. We check if it is negative definite

b. We check if it is negative definite

$$\frac{\partial^2 f}{\partial x_1^2} < 0$$

$$\begin{vmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} \end{vmatrix} \quad or \quad \begin{vmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{vmatrix} = f_{11} f_{22} - f_{21} f_{12} > 0$$

$$\begin{vmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{vmatrix} = f_{11} \begin{vmatrix} f_{22} & f_{23} \\ f_{32} & f_{33} \end{vmatrix} - f_{12} \begin{vmatrix} f_{21} & f_{23} \\ f_{31} & f_{33} \end{vmatrix} + f_{13} \begin{vmatrix} f_{21} & f_{22} \\ f_{31} & f_{32} \end{vmatrix} < 0$$

3. If these conditions hold, H is negative definite, f is strictly concave and the stationary point is a maximum

4. If these conditions are violated by equality, i.e. are equal to zero, check the conditions for semi definiteness

$$f_{11} \le 0, f_{22} \le 0 \; f_{33} \le 0$$

$$\begin{vmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{vmatrix} \ge 0 \begin{vmatrix} f_{11} & f_{13} \\ f_{31} & f_{33} \end{vmatrix} \ge 0 \begin{vmatrix} f_{22} & f_{23} \\ f_{32} & f_{33} \end{vmatrix} \ge 0$$

$$\begin{vmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{vmatrix} \le 0$$

5. If these conditions hold, H is negative semidefinite, f is concave and the stationary point is a maximum

6. If these conditions are violated, we need further investigation

Economic applications

From chapter 11.6 of the textbook

- Multiproduct firm
- Price discrimination
- Input decisions of afirm