SEISMOLOGY

Master Degree Programme in Physics - UNITS
Physics of the Earth and of the Environment

SEISMIC SURFACE WAVES

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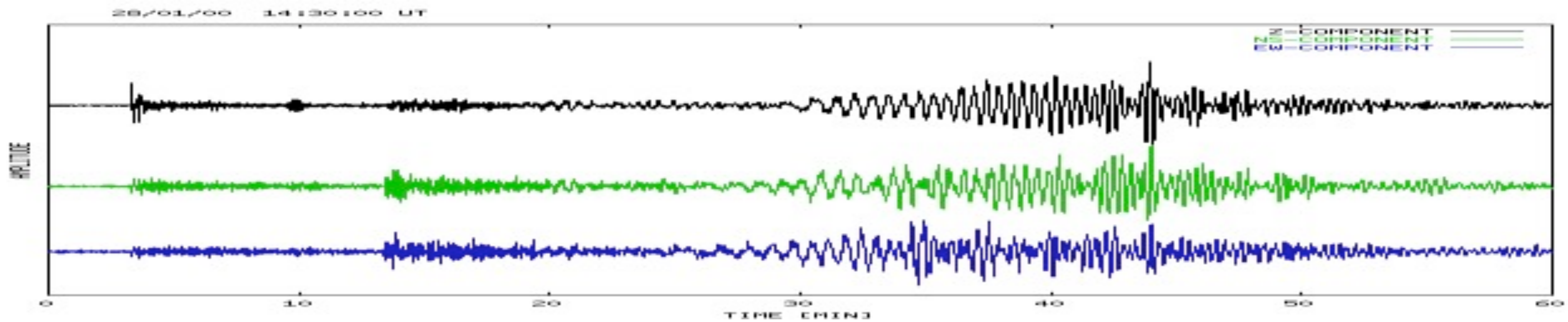
Surface Waves

Surface waves in an elastic half spaces: Rayleigh waves
- Potentials
- Free surface boundary conditions
- Solutions propagating along the surface, decaying with depth
- Lamb's problem

Surface waves in media with depth-dependent properties: Love waves
- Constructive interference in a low-velocity layer
- Dispersion curves
- Phase and Group velocity

Free Oscillations
- Spherical Harmonics
- Modes of the Earth
- Rotational Splitting

Data Example



Question:

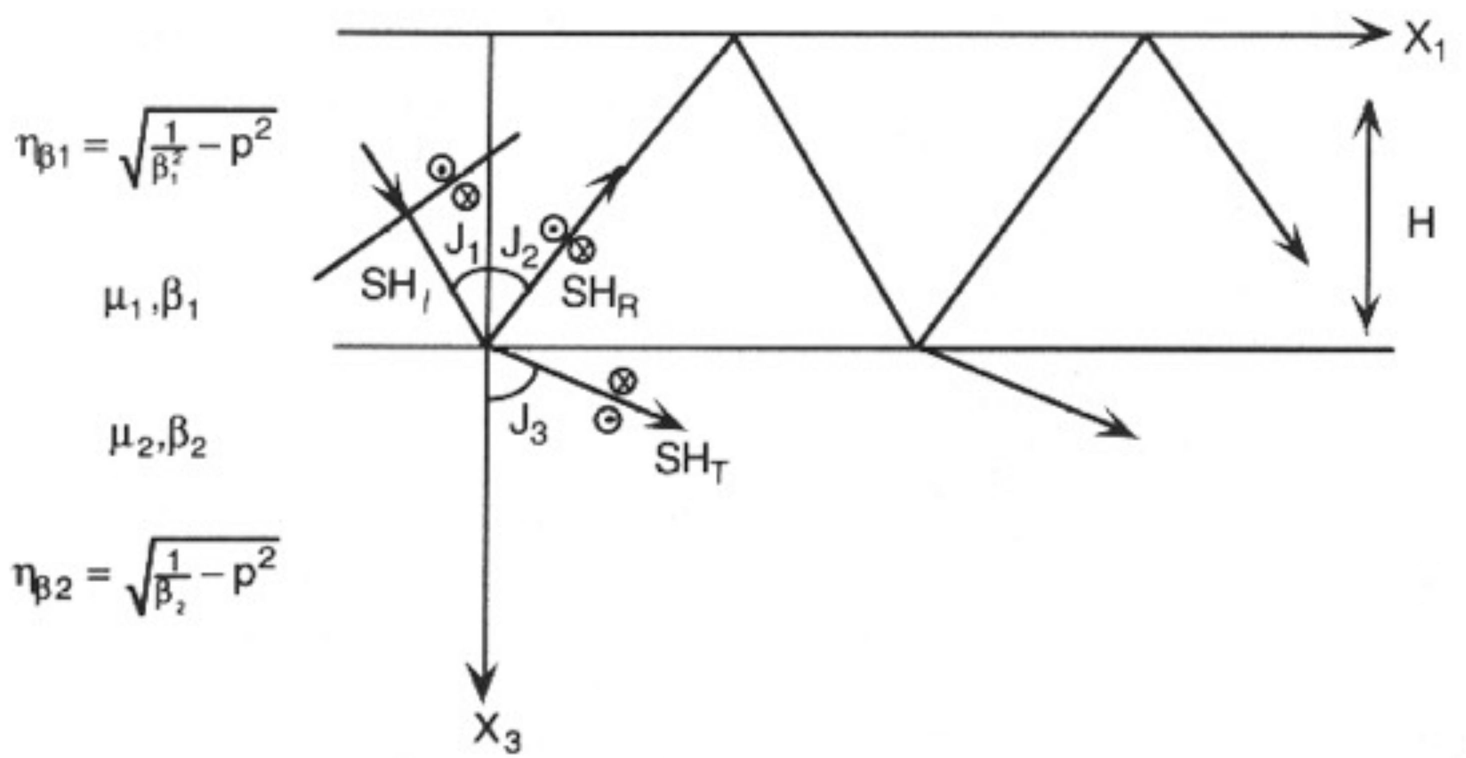
We derived that Rayleigh waves are non-dispersive!

But in the observed seismograms we clearly see a highly dispersed surface wave train?

We also see dispersive wave motion on both horizontal components!

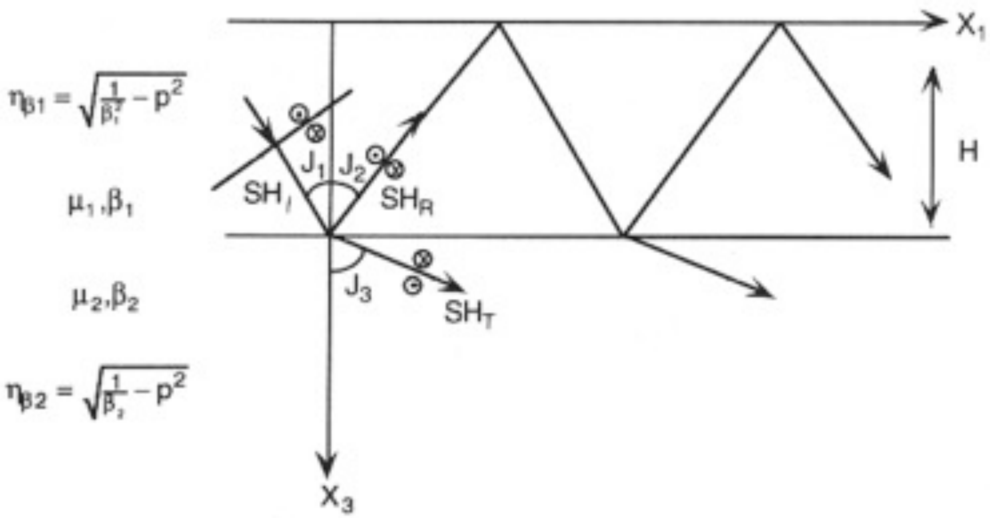
Do SH-type surface waves exist?
Why are the observed waves dispersive?

In an elastic half-space no SH type surface waves exist. Why?
Because there is total reflection and no interaction between an evanescent P wave and a phase shifted SV wave as in the case of Rayleigh waves. What happens if we have a layer over a half space (Love, 1911) ?



Repeated reflection in a layer over a half space.
Interference between incident, reflected and transmitted SH waves.
When the layer velocity is smaller than the halfspace velocity, then there is a critical angle beyond which SH reverberations will be totally trapped.

Love waves: trapping - 1



$$u_{y1} = A\exp[i(\omega t + \omega\eta_{\beta 1} z - kx)] + B\exp[i(\omega t - \omega\eta_{\beta 1} z - kx)]$$

$$u_{y2} = C\exp[i(\omega t - \omega\eta_{\beta 2} z - kx)]$$

$$k = k_x = \frac{\omega}{c}; \quad \omega\eta_\beta = k_z = \frac{\omega}{c}\sqrt{\frac{c^2}{\beta^2} - 1} = kr_\beta$$

$$u_{y1} = A\exp[i(\omega t + kr_{\beta 1} z - kx)] + B\exp[i(\omega t - kr_{\beta 1} z - kx)]$$

$$u_{y2} = C\exp[i(\omega t - kr_{\beta 2} z - kx)]$$

Love waves: trapping - 2

The formal derivation is very similar to the derivation of the Rayleigh waves. The conditions to be fulfilled are:

1. Free surface condition
2. Continuity of stress on the boundary
3. Continuity of displacement on the boundary
4. No radiation in the halfspace

$$1.\quad \sigma_{zy1}(0) = \mu_1 \left.\frac{\partial u_{y1}}{\partial z}\right|_0 = ikr_{\beta 1}\left\{A\exp[i(\omega t - kx)] - B\exp[i(\omega t - kx)]\right\} = 0$$

$$2.\quad \sigma_{zy1}(H) = \mu_1 \left.\frac{\partial u_{y1}}{\partial z}\right|_H = \sigma_{zy2}(H) = \mu_2 \left.\frac{\partial u_{y2}}{\partial z}\right|_H \qquad 3.\quad u_{y1}(H) = u_{y2}(H)$$

$$4.\quad \lim_{\infty} u_{y2}(z) = 0 \quad \text{i.e.} \quad c < \beta_2 \quad \text{i.e.} \quad r_{\beta 2} = -i\sqrt{1 - \frac{c^2}{\beta_2^2}}$$

We obtain a condition for which solutions exist. This time we obtain a frequency-dependent solution: a **dispersion** relation

$$\tan\left(H\omega\sqrt{1/\beta_1^2 - 1/c^2}\right) = \frac{\mu_2\sqrt{1/c^2 - 1/\beta_2^2}}{\mu_1\sqrt{1/\beta_1^2 - 1/c^2}}$$

... indicating that there are only solutions if ...

$$\beta_1 < c < \beta_2$$

Love Waves: Solutions

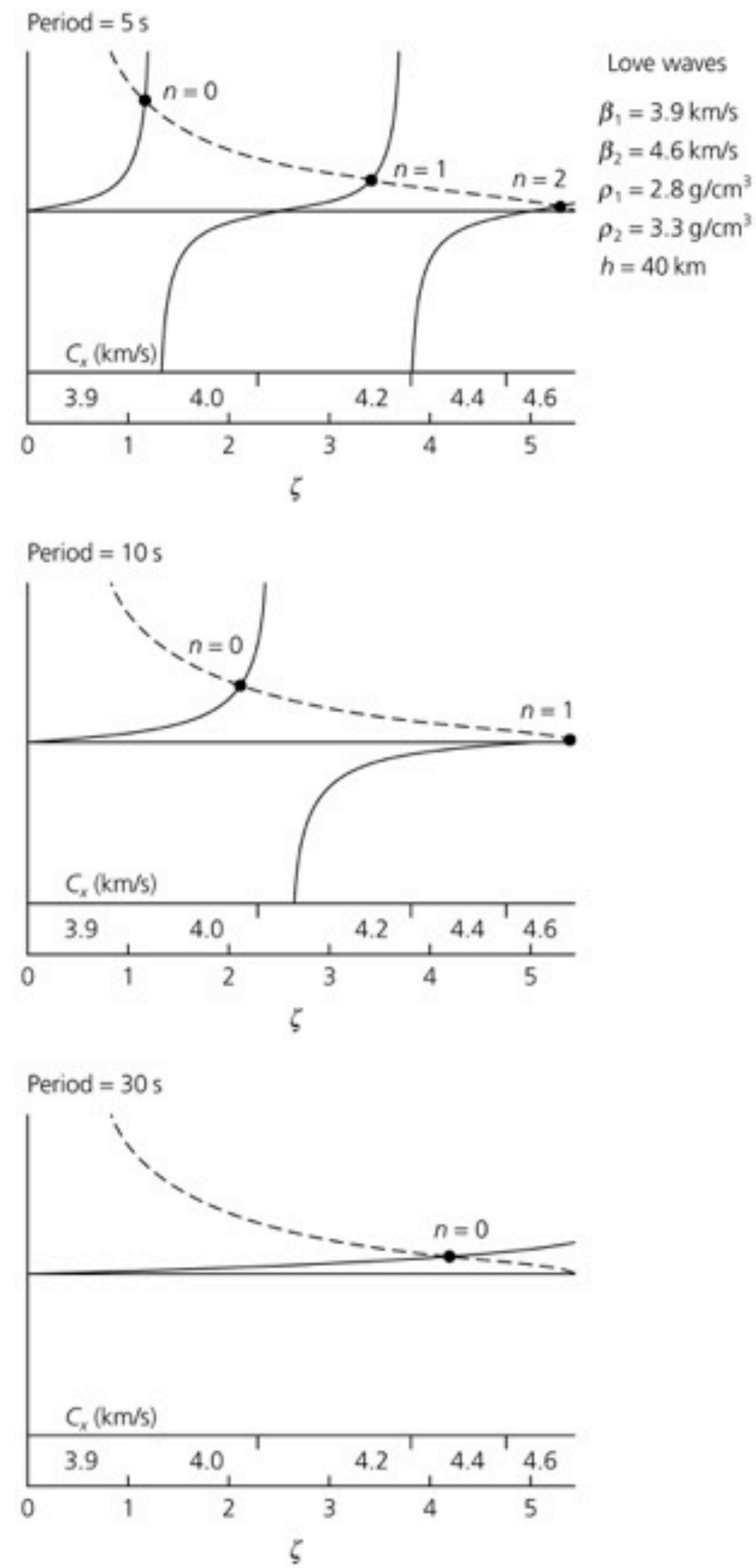
Graphical solution of the previous equation. Intersection of dashed and solid lines yield discrete **modes**.

$$\tan\left(H\omega\sqrt{1/\beta_1^2 - 1/c^2}\right) = \tan(\omega\zeta)$$

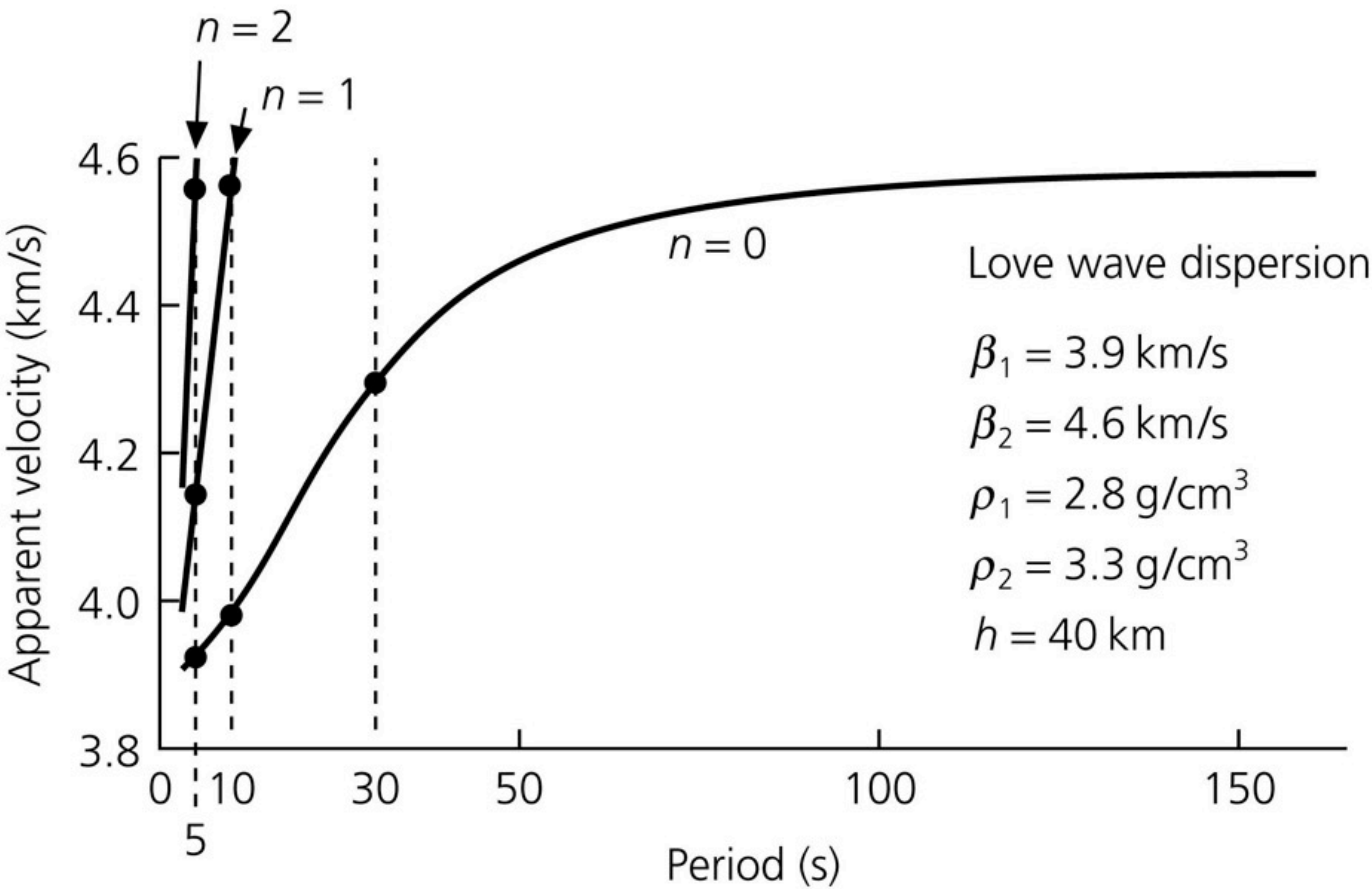
$$\text{that vanishes when } \zeta = n\frac{\pi}{\omega}$$

New modes appear at **cut-off frequencies**

$$\omega_n = \frac{n\pi}{H\left(\frac{1}{\beta_1^2} - \frac{1}{\beta_2^2}\right)^{1/2}}$$



Love Waves: Solutions

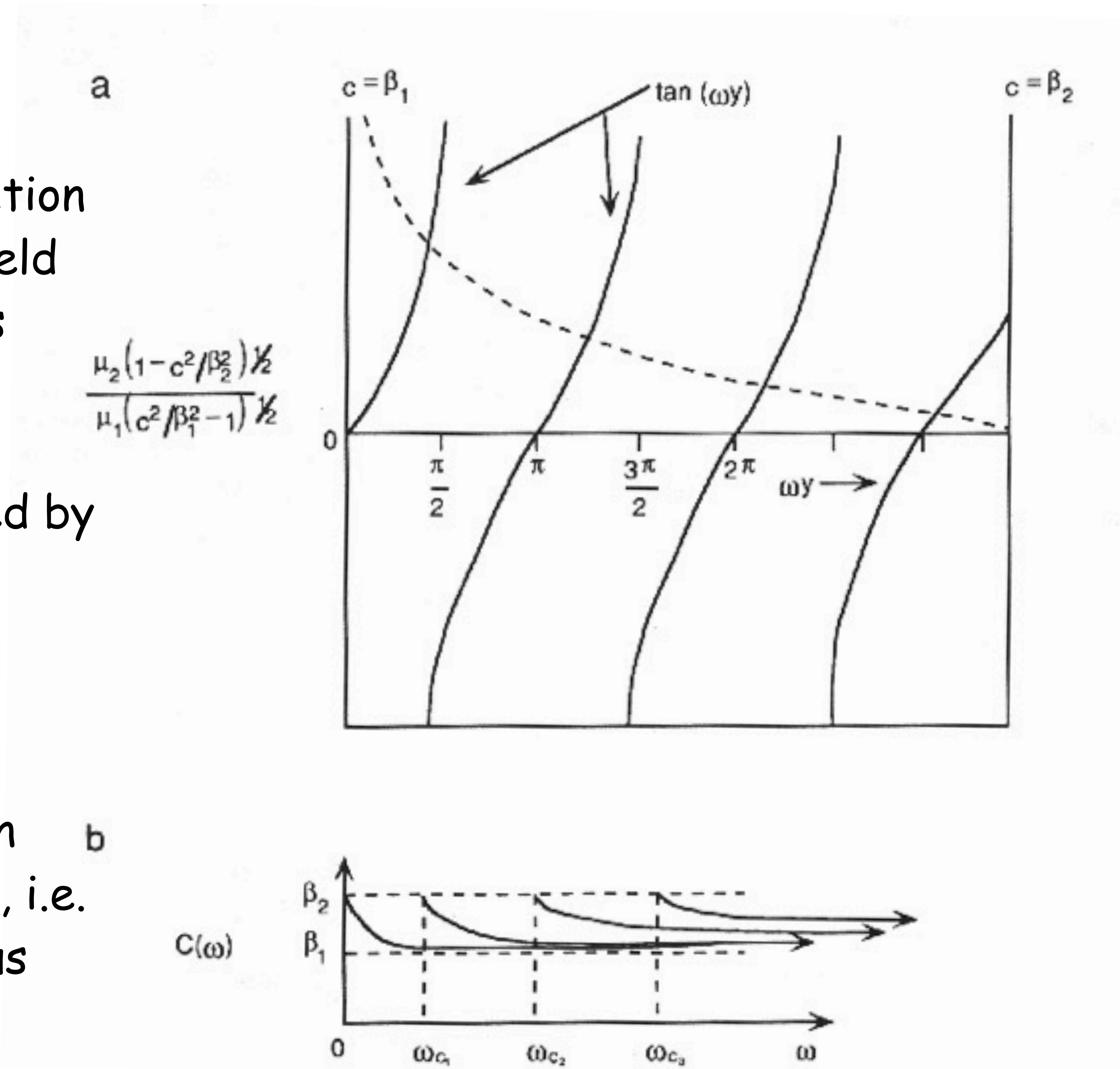


Love Waves: Solutions

Graphical solution of the previous equation. Intersection of dashed and solid lines yield solutions while frequency is varying: discrete modes.

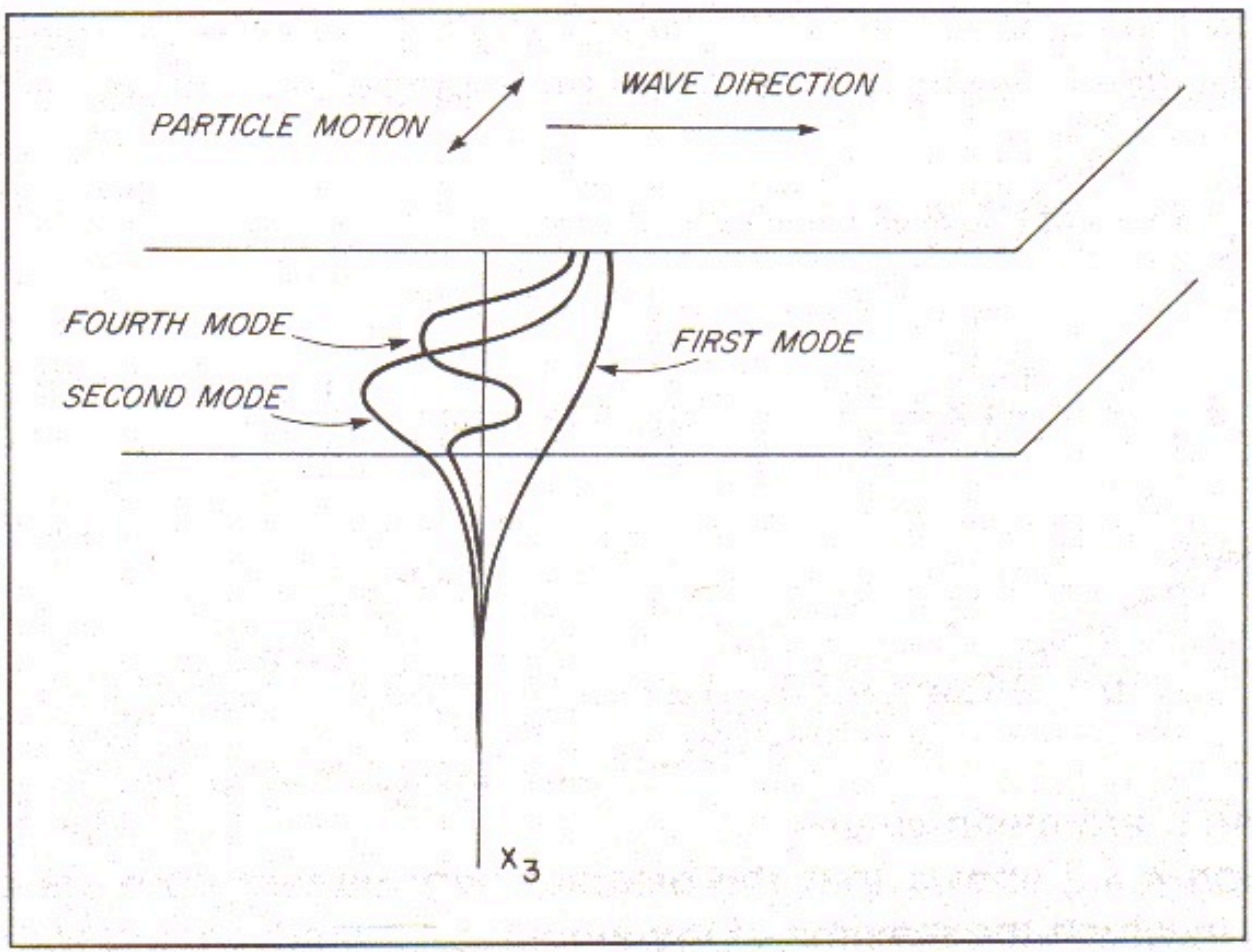
Every mode is characterized by a dispersion curve **c=c(ω)**, showing the solution to the **eigenvalue** problem.

For every value of c one can calculate the **eigenfunction**, i.e. the displacement , u_y, versus depth.



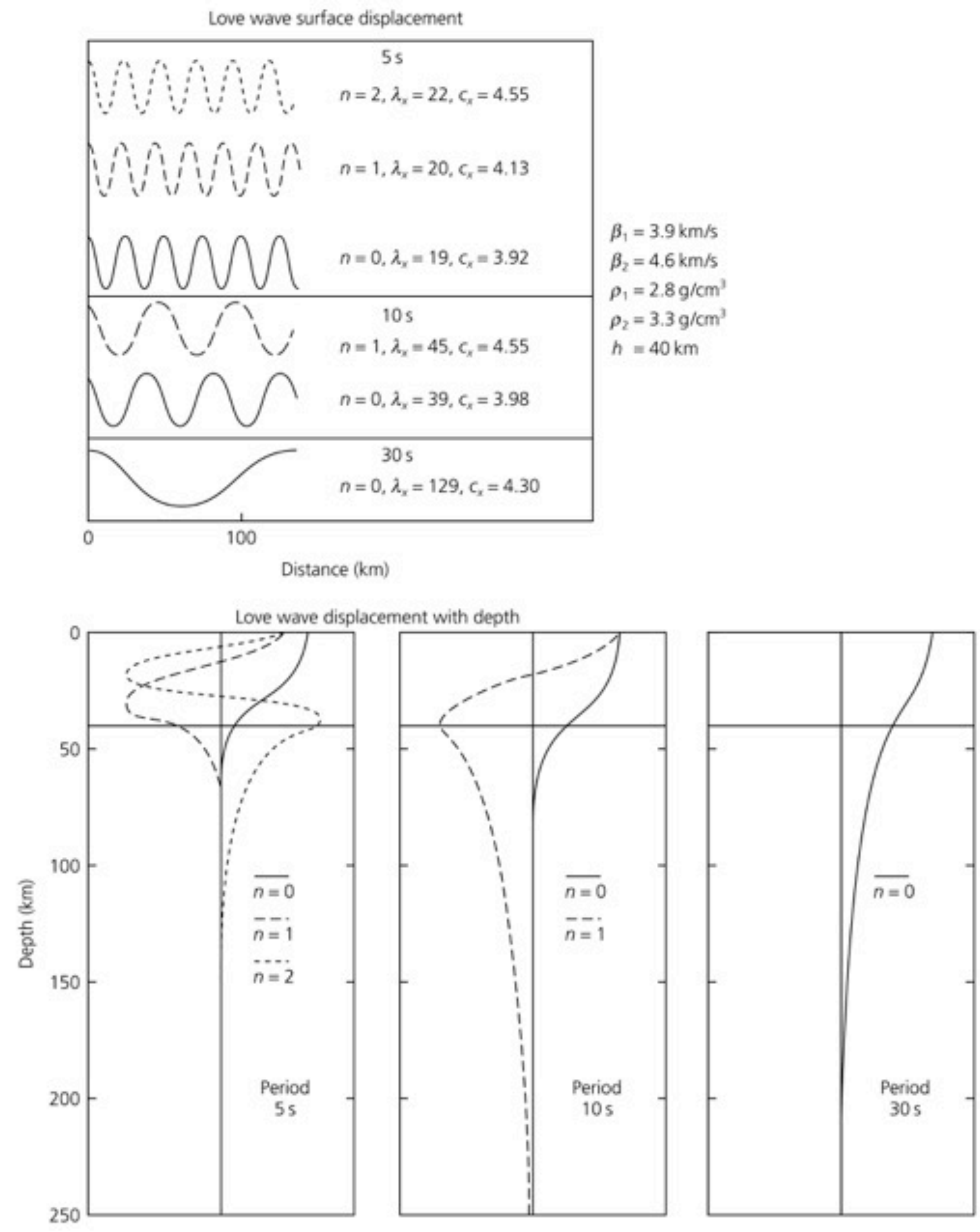
Love Waves: modes

Some modes for Love waves



Love Waves: modes

Some eigenvectors (displacement) for Love waves



Liquid layer over a half space

The conditions to be fulfilled are:

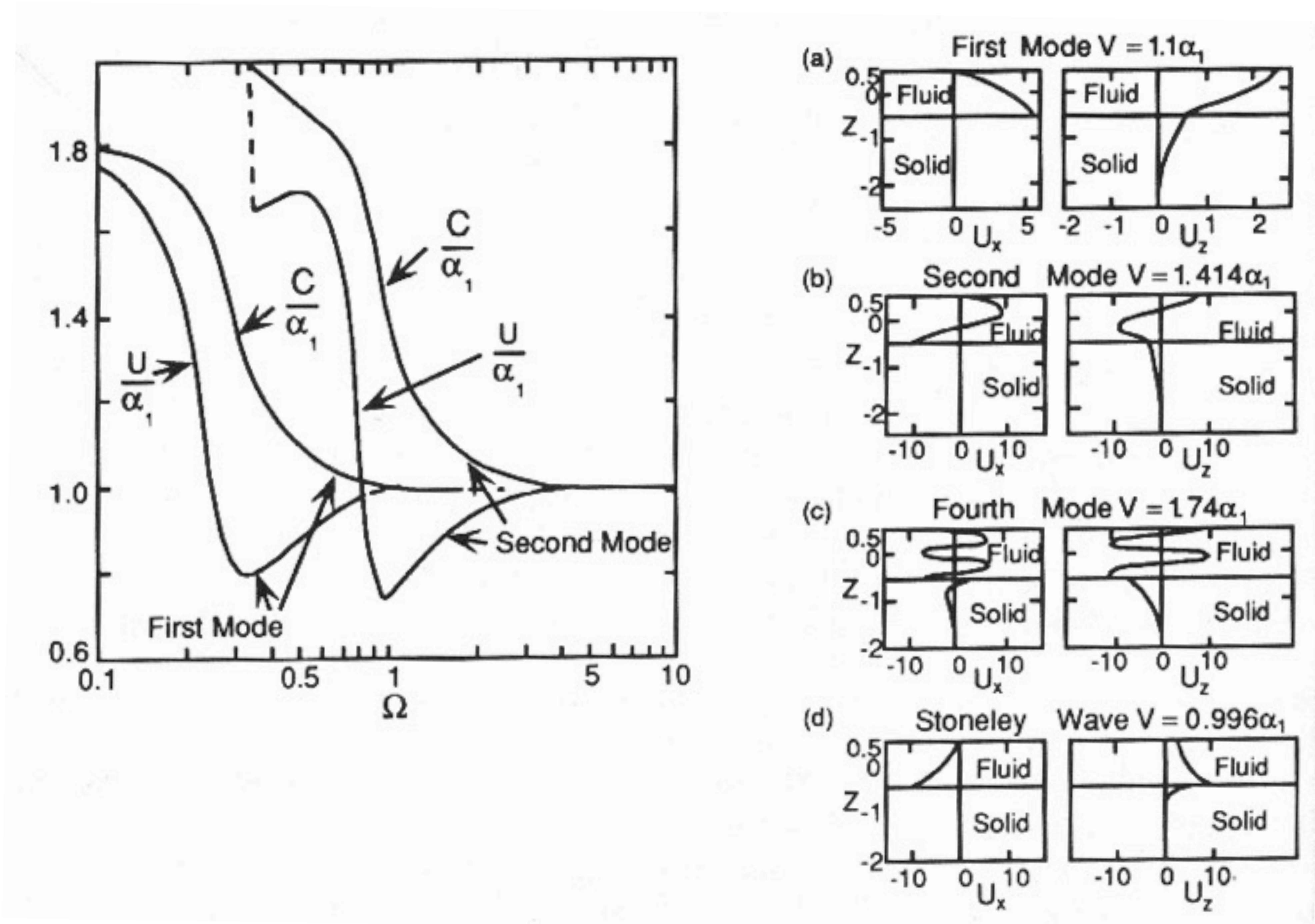
1. Free surface condition
2. No S-wave potential and shear stress in the liquid layer
3. Continuity of stress at the liquid-layer interface
4. Continuity of vertical component of displacement at the liquid layer interface (horizontal is free due to no viscosity in perfect liquid)

$$\tan\left(H\omega\sqrt{1/\alpha_w^2 - 1/c^2}\right) = \frac{\rho\beta^4\sqrt{c^2/\alpha_w{}^2 - 1}}{\rho_w c^4\sqrt{1 - c^2/\alpha^2}}$$

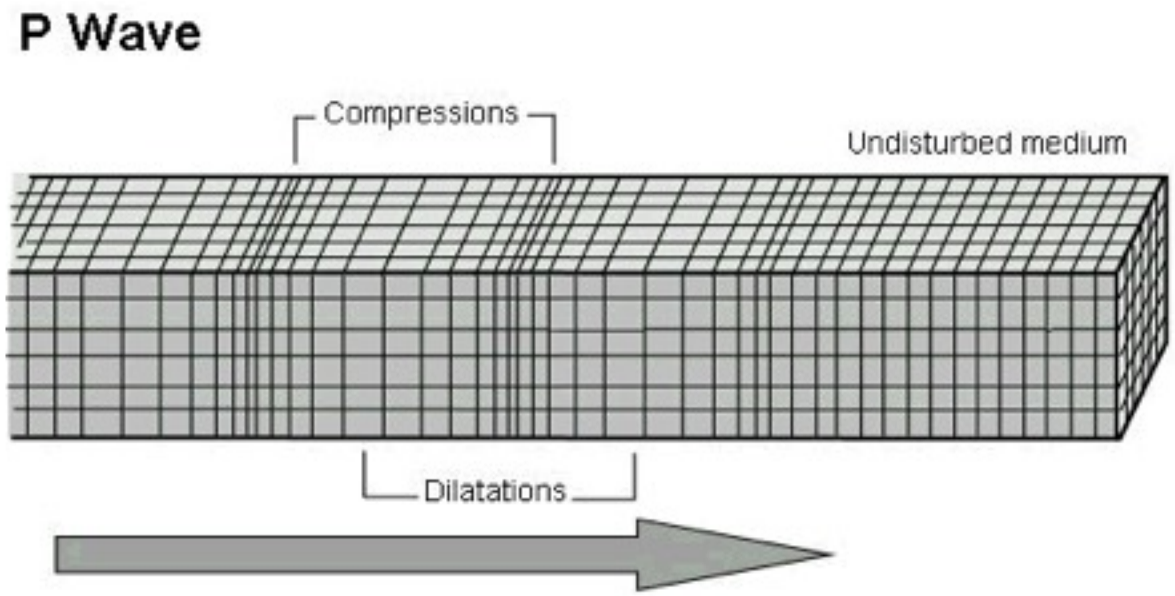
$$\left[-(2 - c^2/\beta^2)^2 + 4(1 - c^2/\alpha^2)^{1/2}(1 - c^2/\beta^2)^{1/2}\right]$$

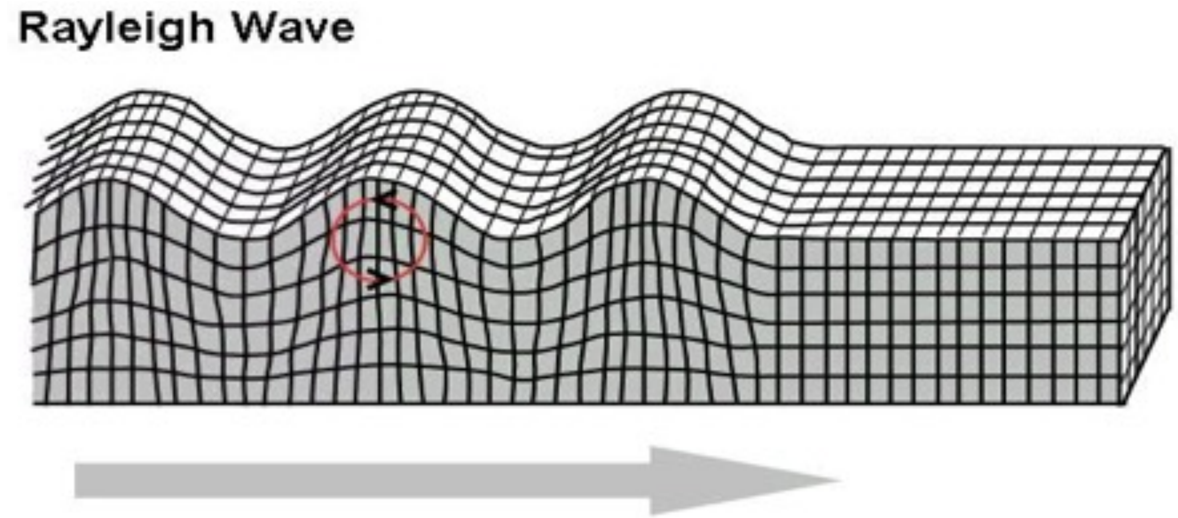
Liquid layer over a half space

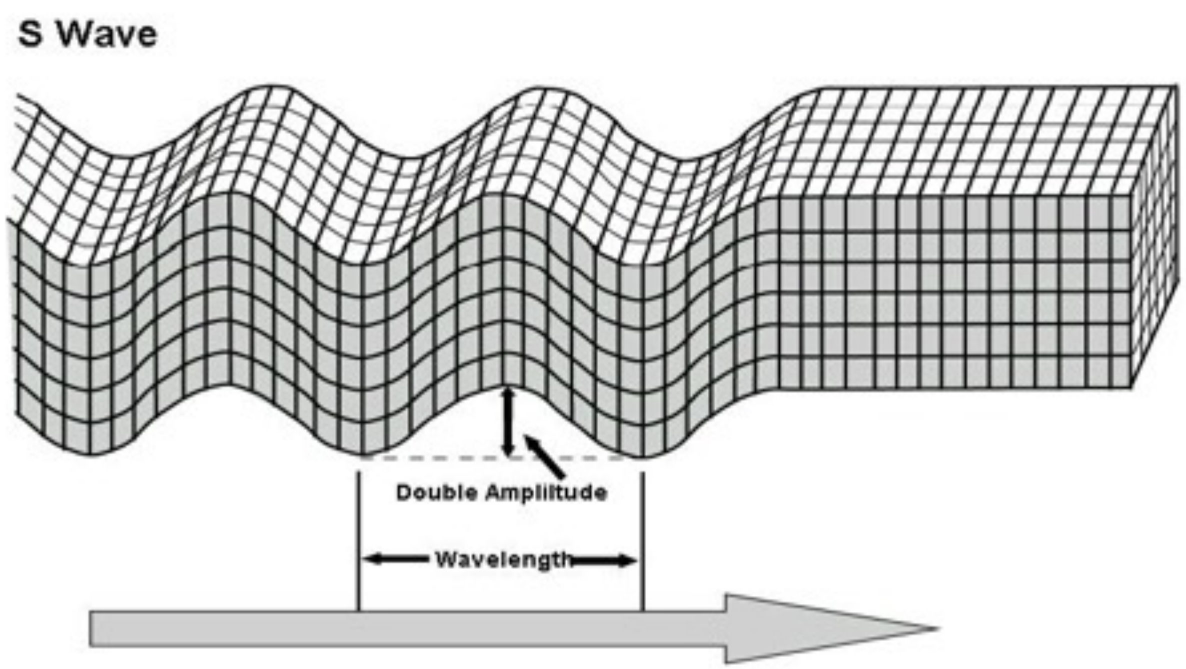
Similar derivation for Rayleigh type motion leads to dispersive behavior

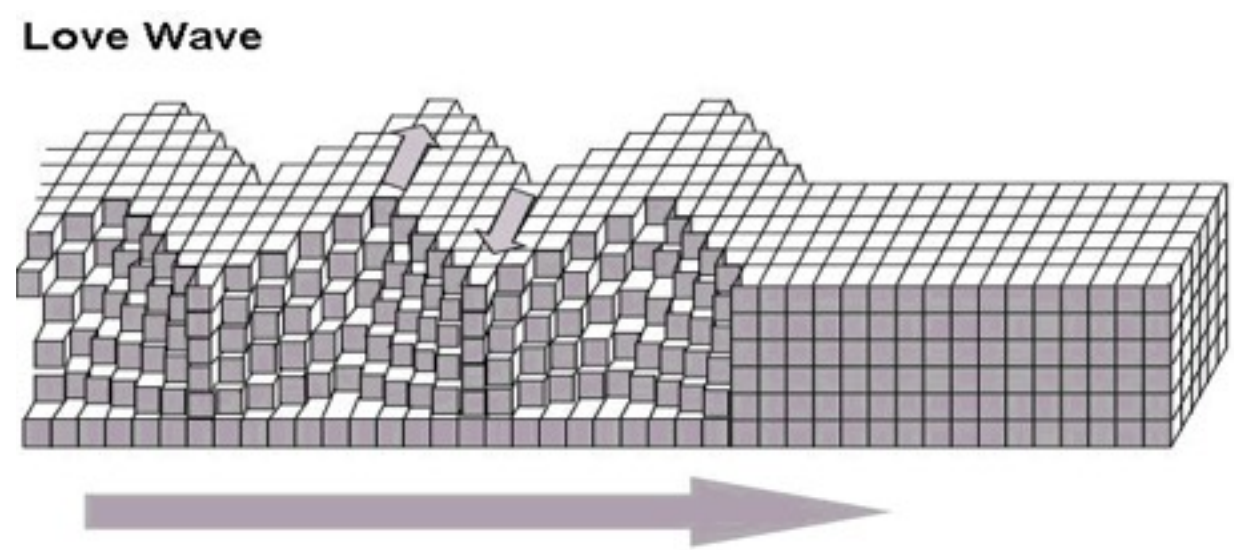


Wavefields visualization

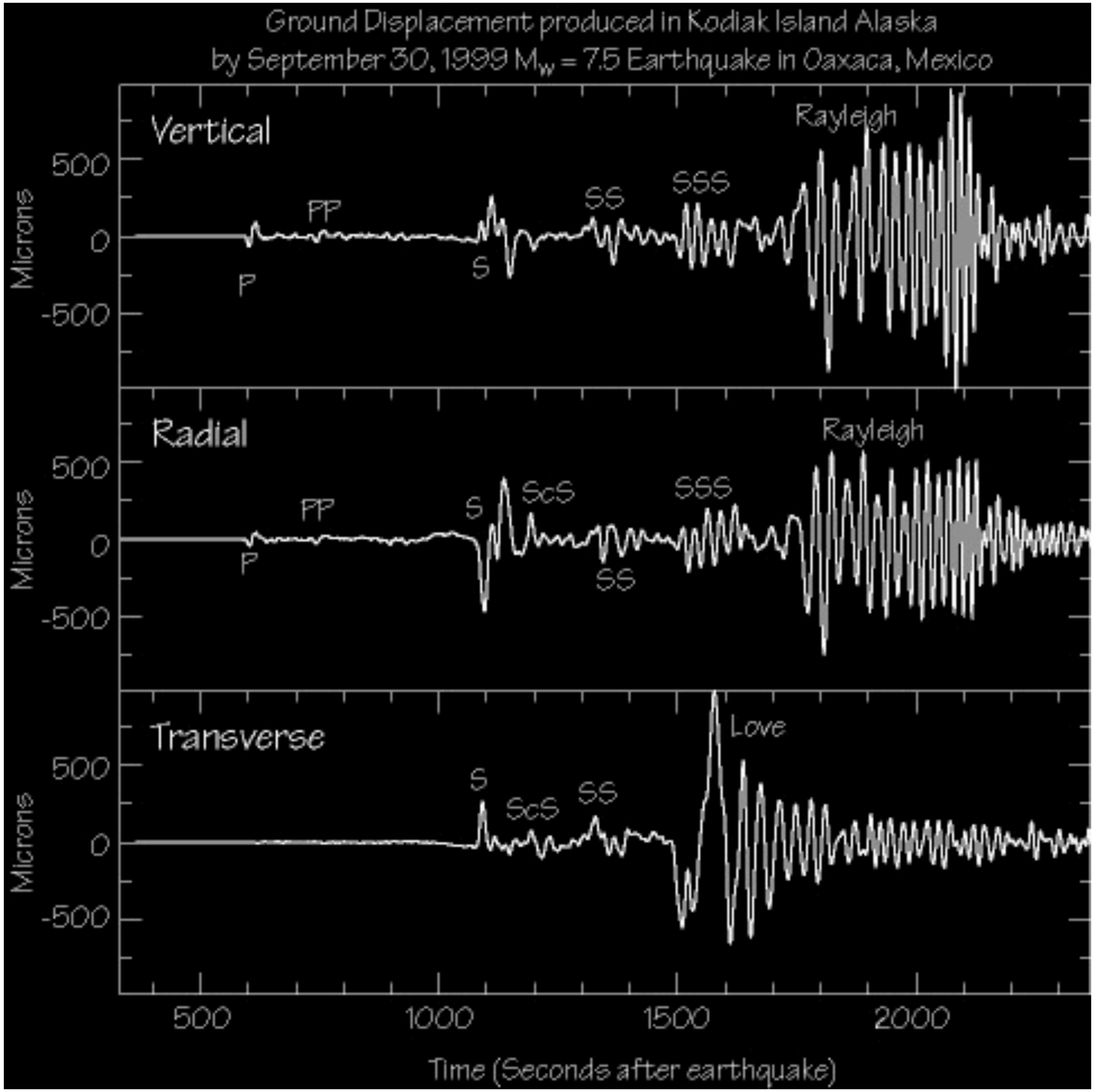




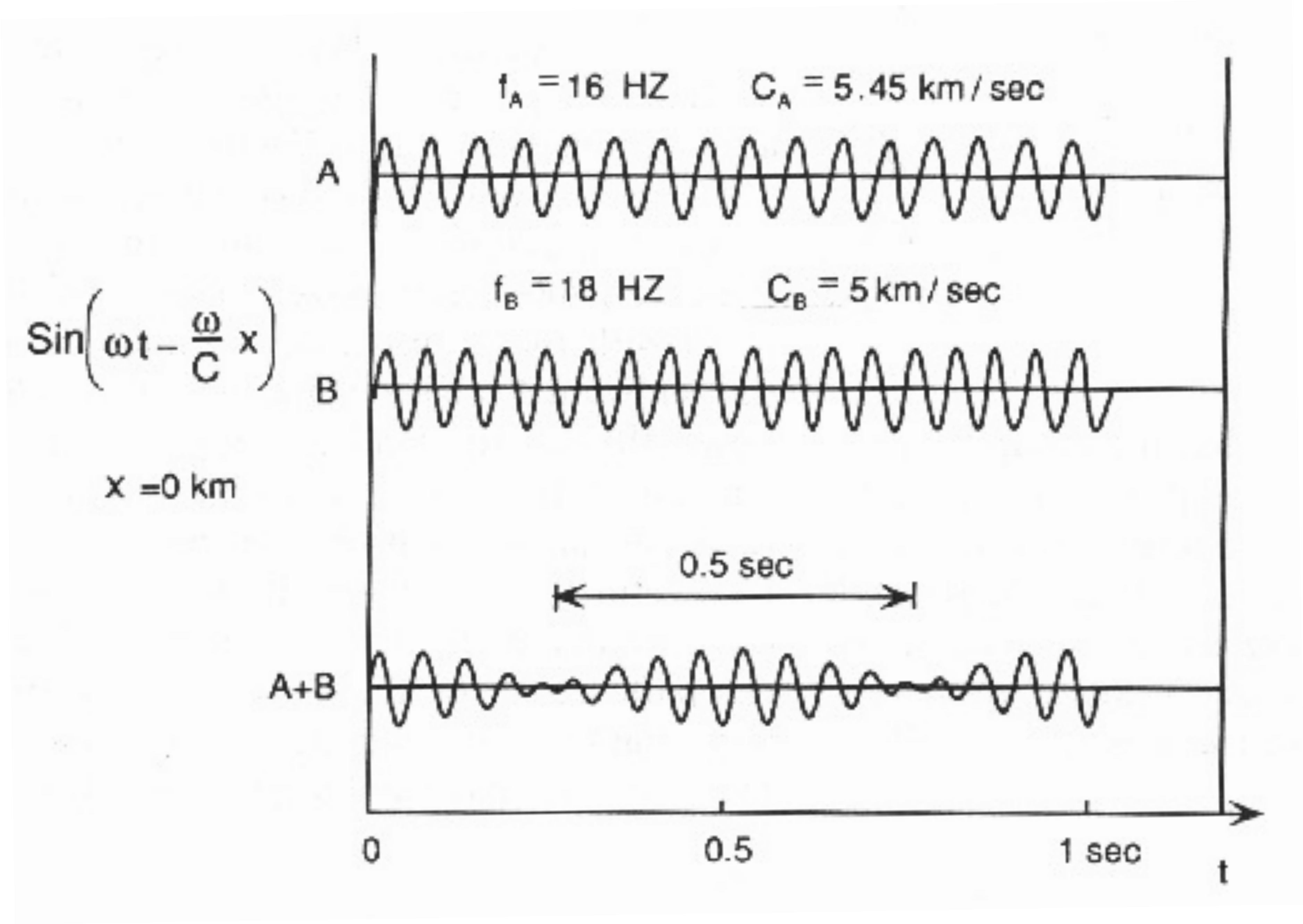




Data example - 2

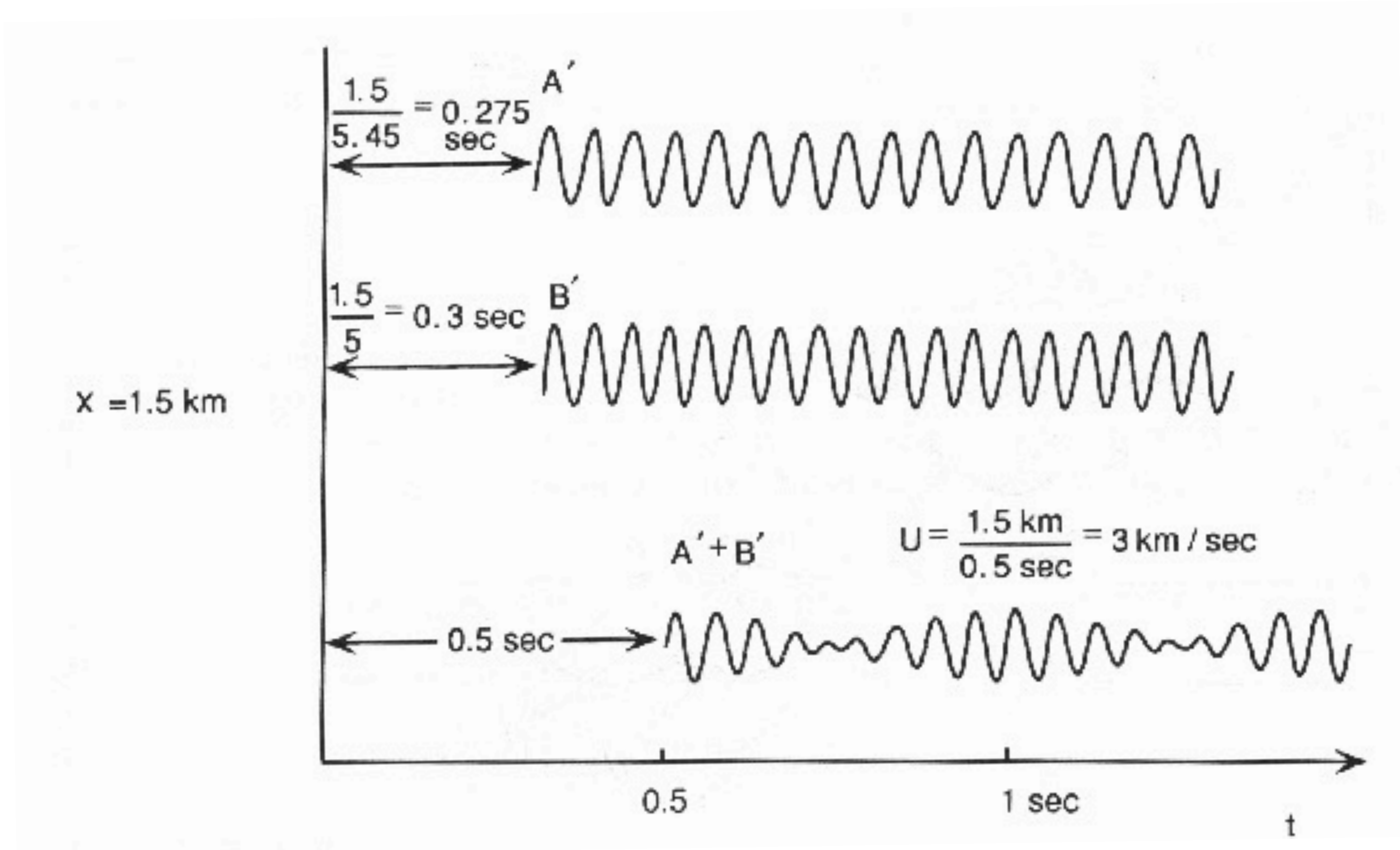


Interference of two waves at two positions (1)



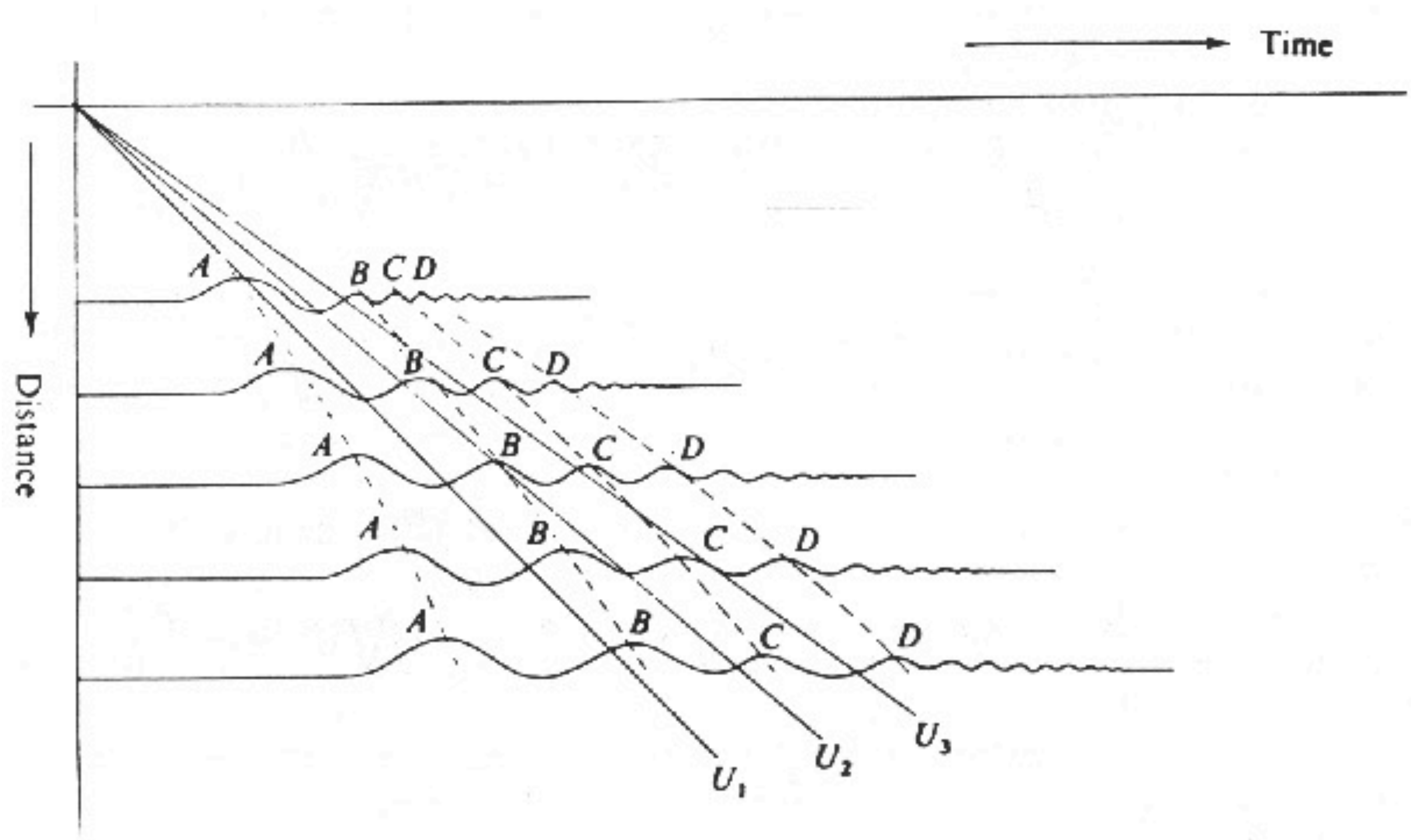
Velocity

Interference of two waves at two positions (2)



Dispersion

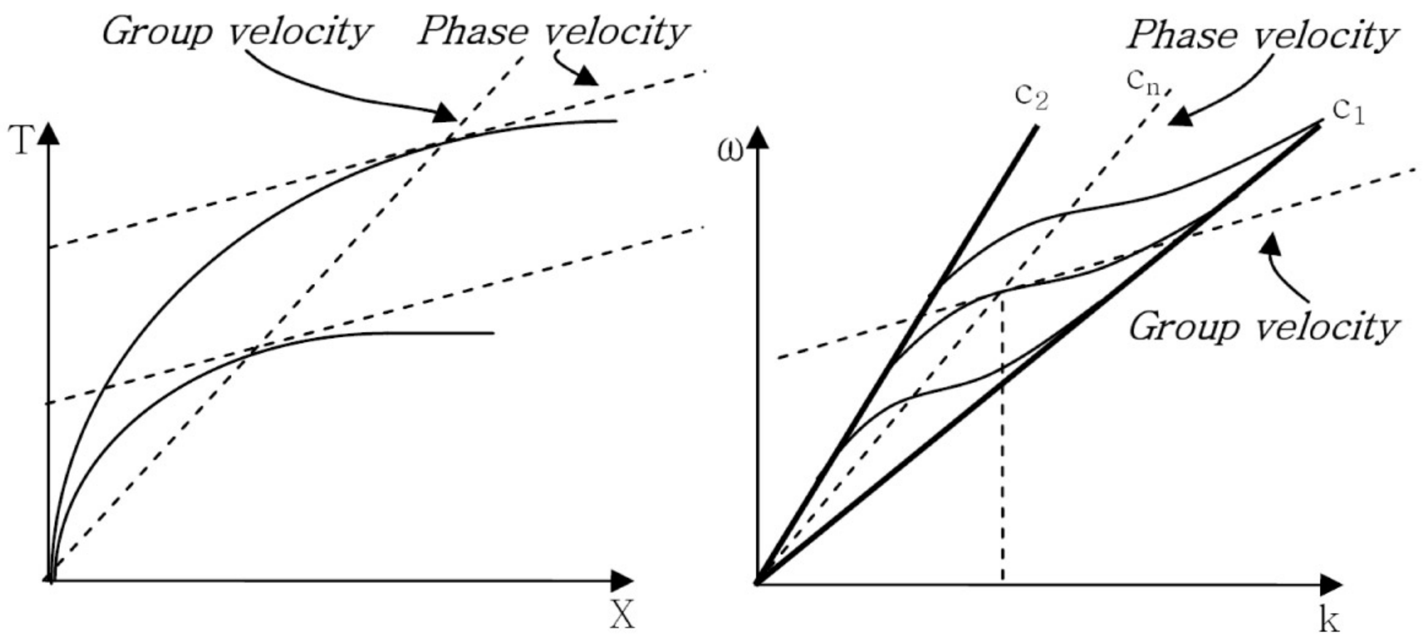
The typical dispersive behavior of surface waves
solid – group velocities; dashed – phase velocities



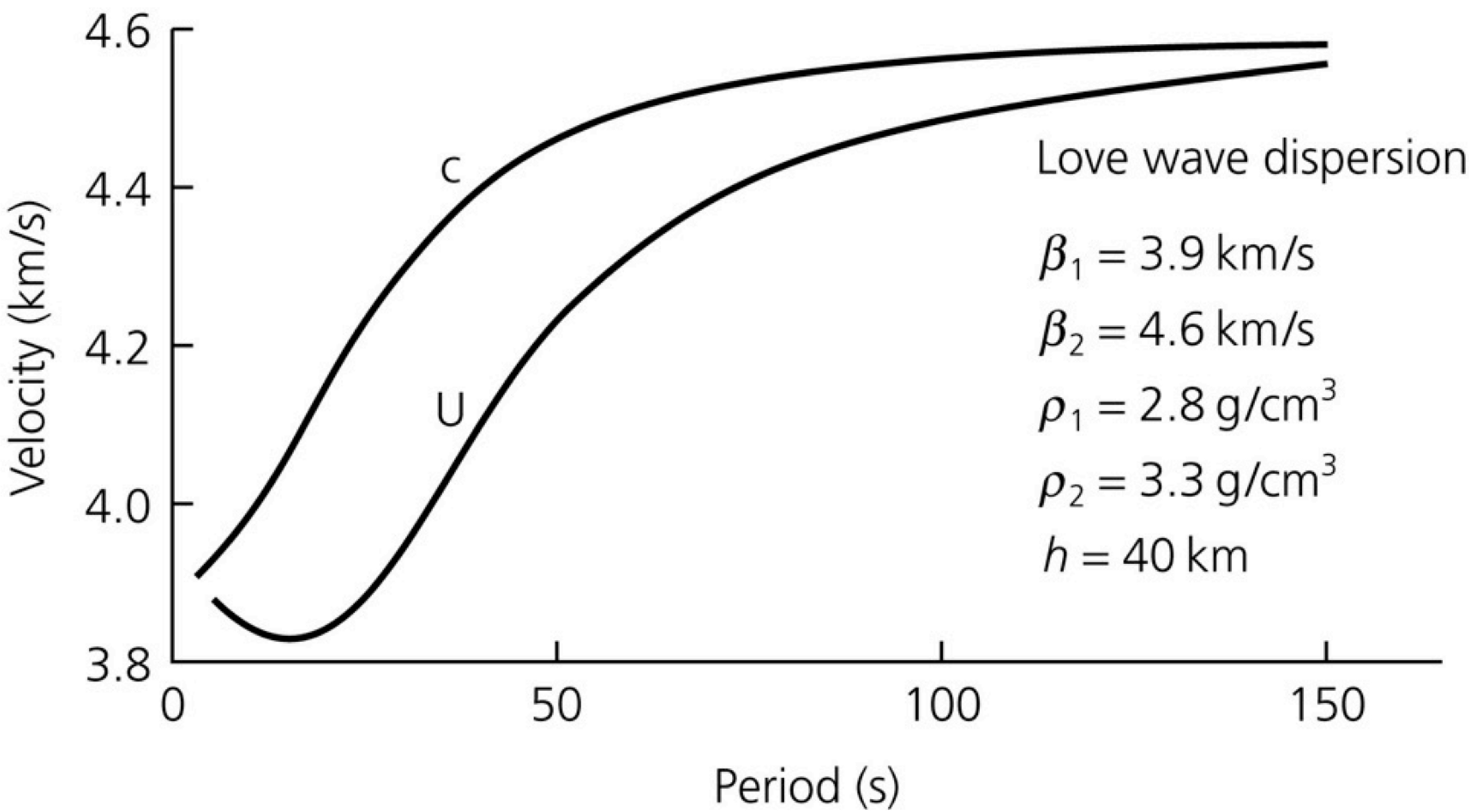
Dispersion...

The group velocity itself is usually a function of the wave's frequency. This results in group velocity dispersion (GVD),that is often quantified as the group delay dispersion parameter : If D is < 0, the medium is said to have **positive dispersion**. If D is > 0, the medium has **negative dispersion**.

$$D = \frac{dv_g}{d\omega}$$



Airy Phase –
wave that arises if the phase and the change in group velocity are stationary and gives the highest amplitude in terms of group velocity and are prominent on the seismogram.

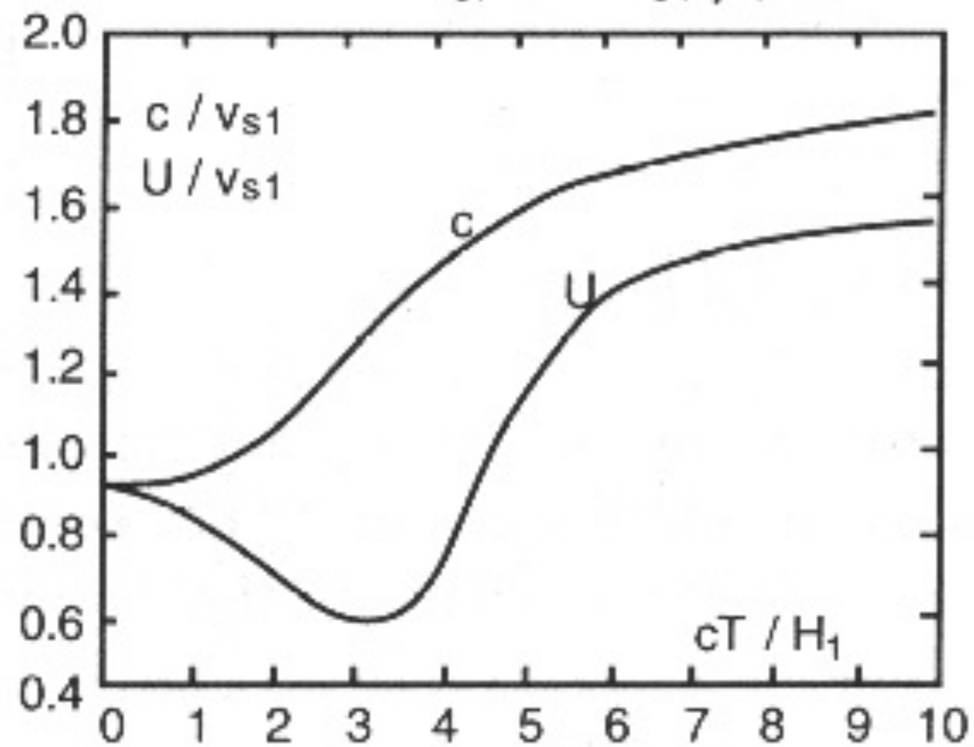


$$U = \frac{d\omega}{dk} = \frac{d(ck)}{dk} = c + k\,\frac{dc}{dk} = c - \lambda\,\frac{dc}{d\lambda}$$

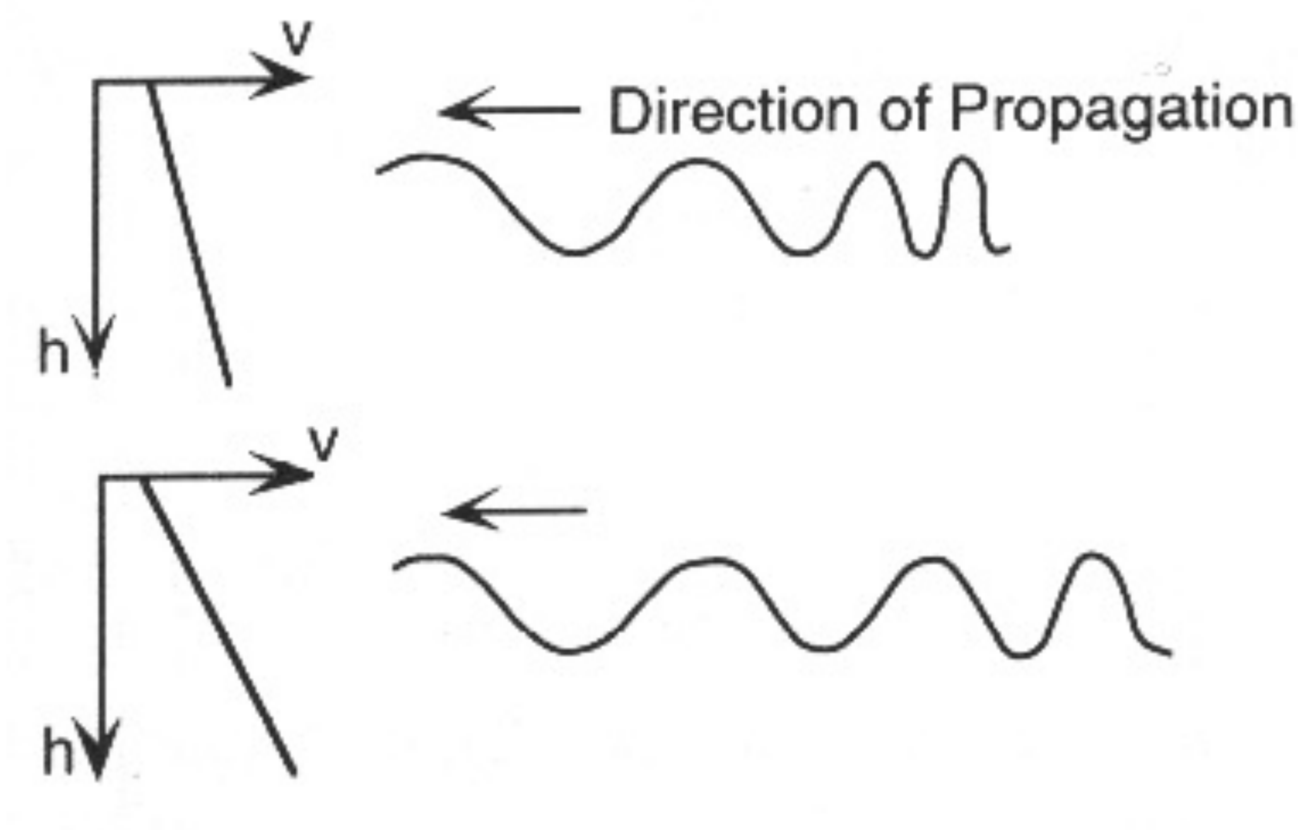
Dispersion

Fundamental Mode Rayleigh dispersion curve for
a layer over a half space.

Layer	v_p	v_s	ρ	H
1	$1.732 v_{s1}$	v_{s1}	ρ_1	H_1
2	$3.873 v_{s1}$	$2.236 v_{s1}$	ρ_1	∞



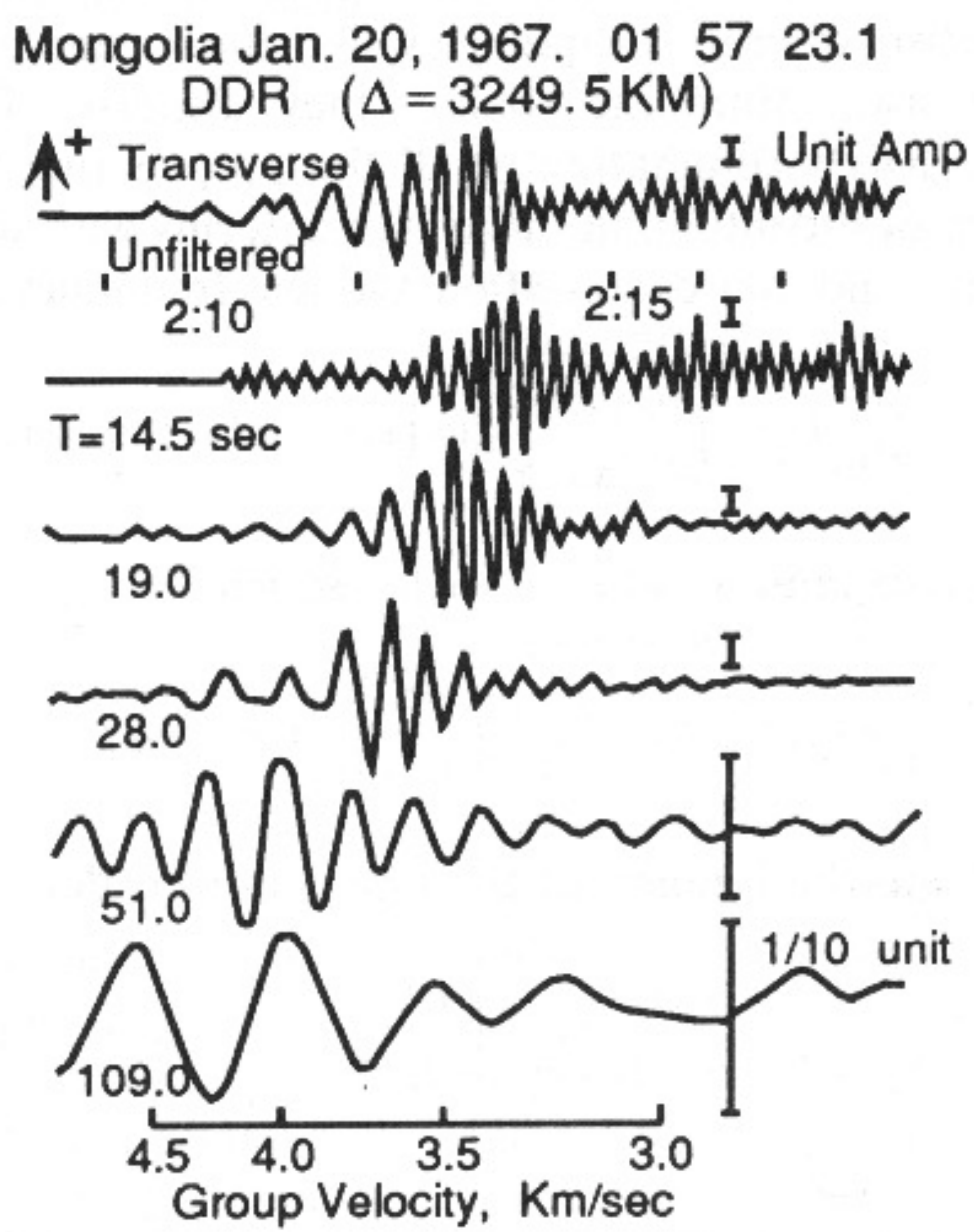
Dispersion



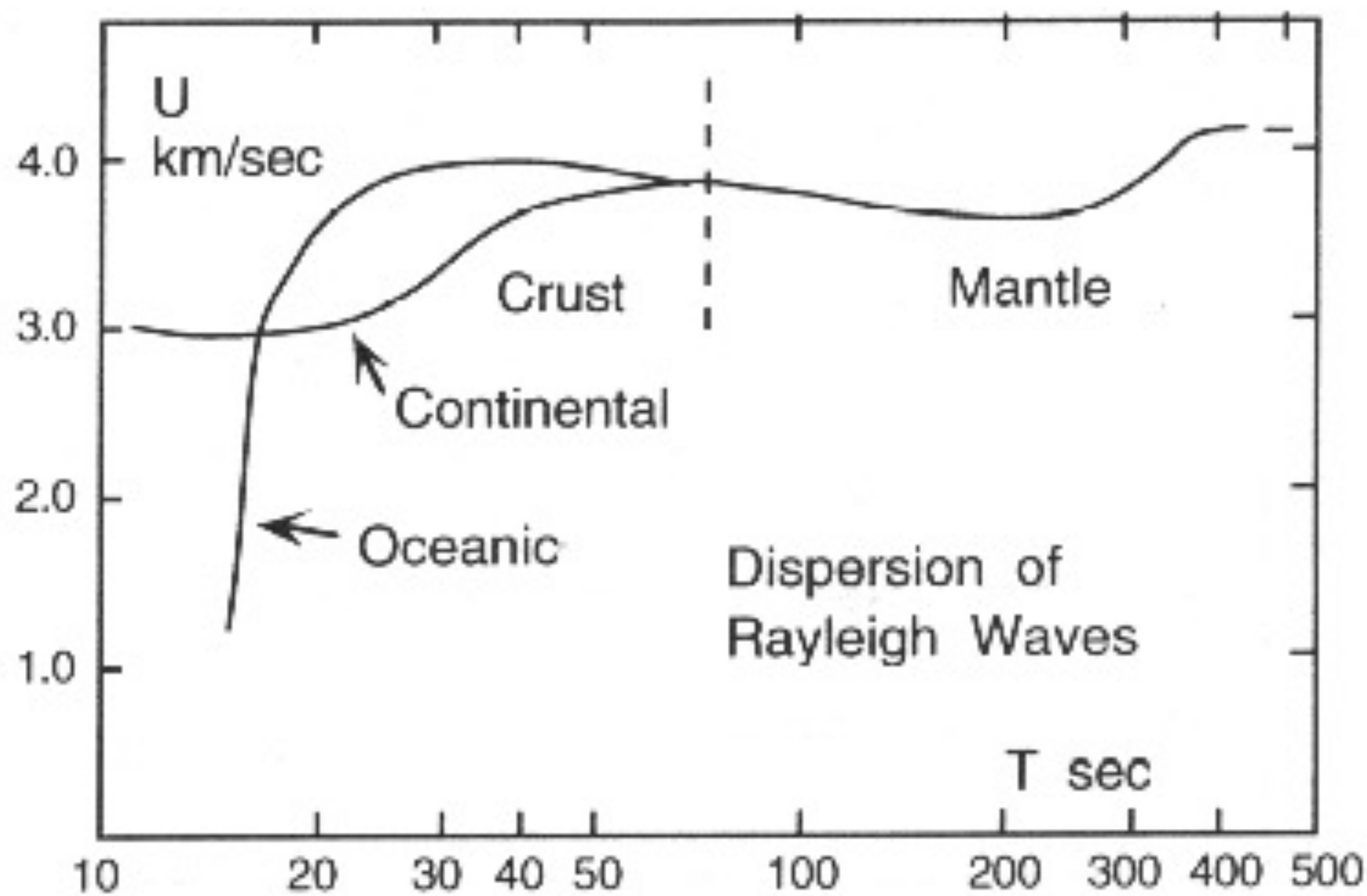
Stronger gradients cause greater dispersion

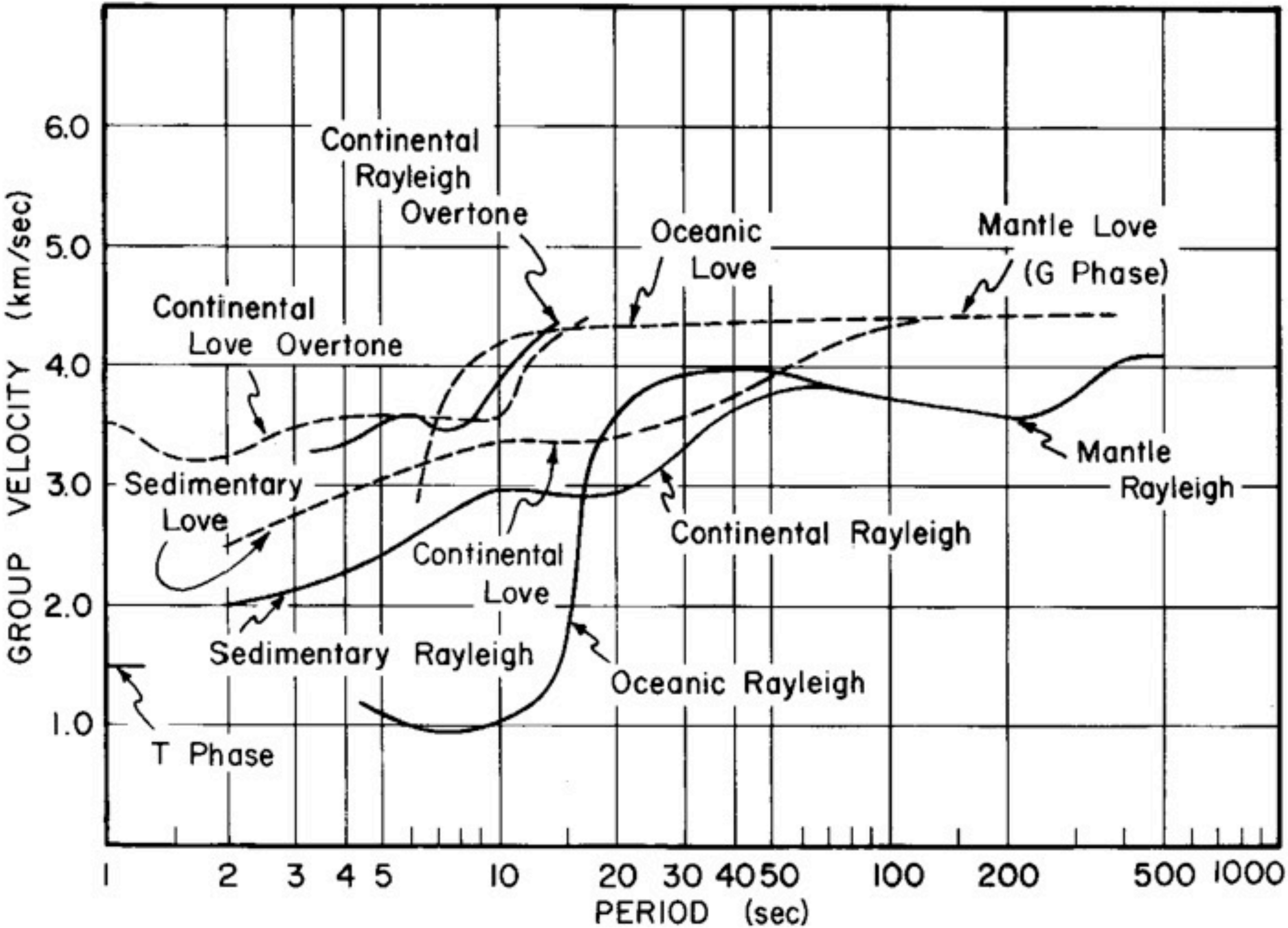
Wave Packets

Seismograms of a Love wave train filtered with different central periods. Each narrowband trace has the appearance of a wave packet arriving at different times.



Observed Group Velocities (T<80s)




GROUP VELOCITY (km/sec)
PERIOD (sec)
6.0
5.0
4.0
3.0
2.0
1.0
1
2
3
4
5
10
20
30
40
50
100
200
500
1000
Continental Rayleigh Overtone
Oceanic Love
Mantle Love (G Phase)
Continental Love Overtone
Sedimentary Love
Continental Love
Continental Rayleigh
Mantle Rayleigh
Sedimentary Rayleigh
Oceanic Rayleigh
T Phase

Measuring group velocity

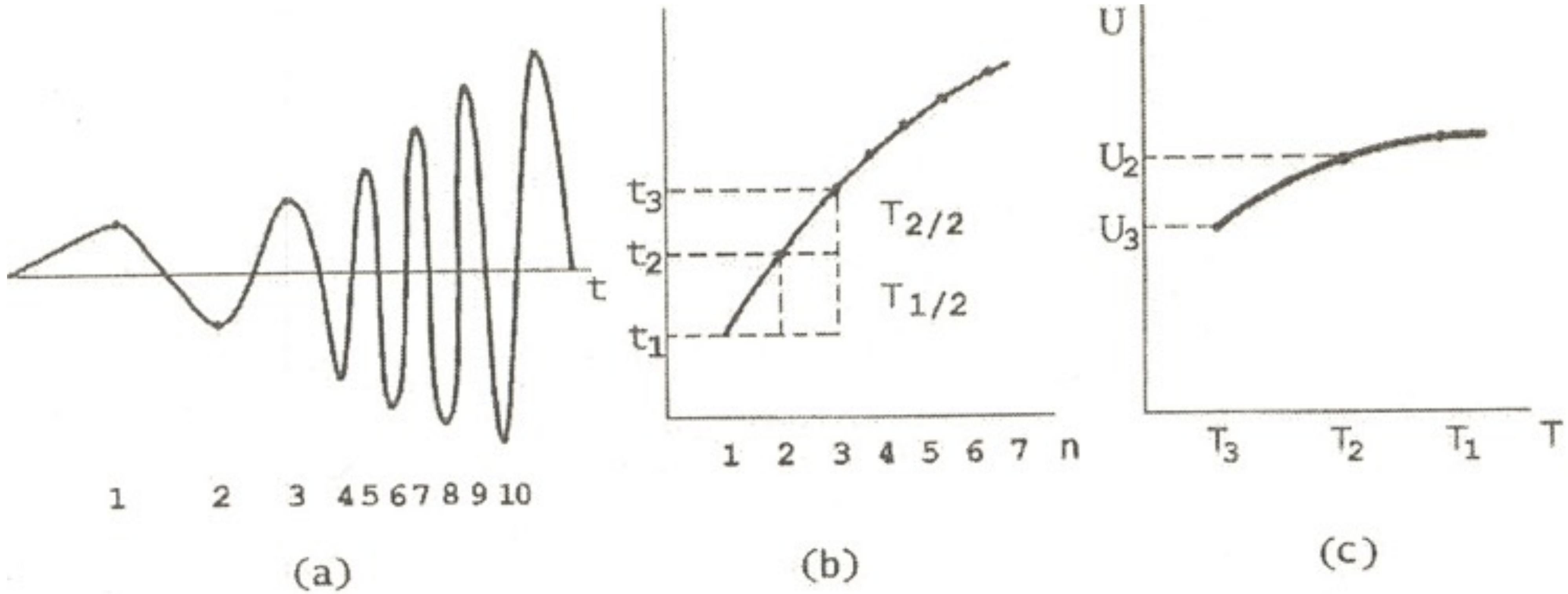
One station method

1. Directly measure the arrival time of the peaks and troughs on one seismogram
2. Narrow filtered the seismogram, measure the arrival of the peak of the wave packet (more accurate)

$$U(\omega) = \frac{x}{t}$$

Need know the origin time and the location of the earthquake source

Determination of group velocities at one station



Two stations method

If two stations are located on the same great circle path, group velocity can be obtained by measuring the difference in arrival times of filtered wave packets.

Distance between two stations

$$U(\omega) = \frac{\Delta x}{\Delta t}$$

Different of arrival times at the two stations

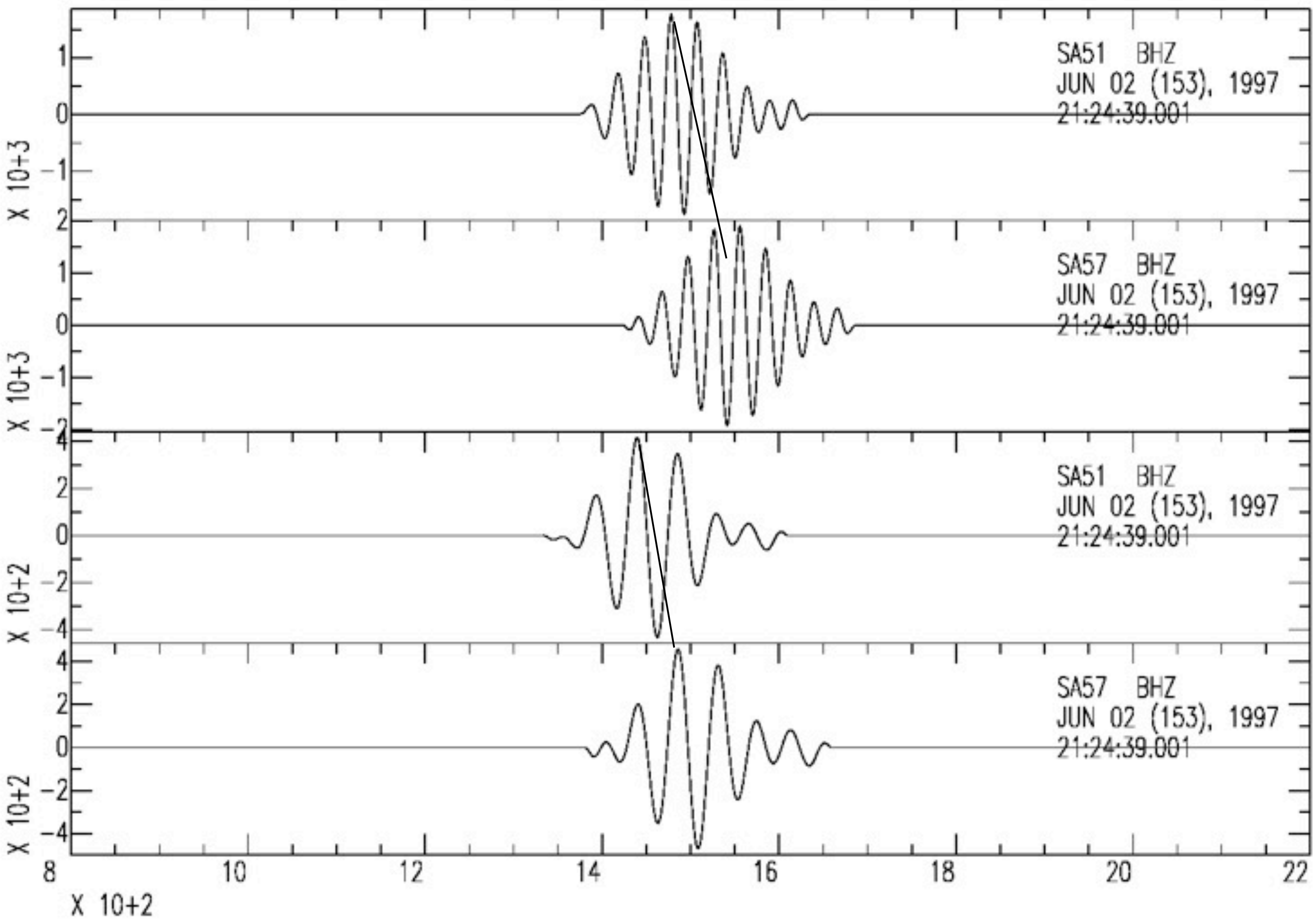
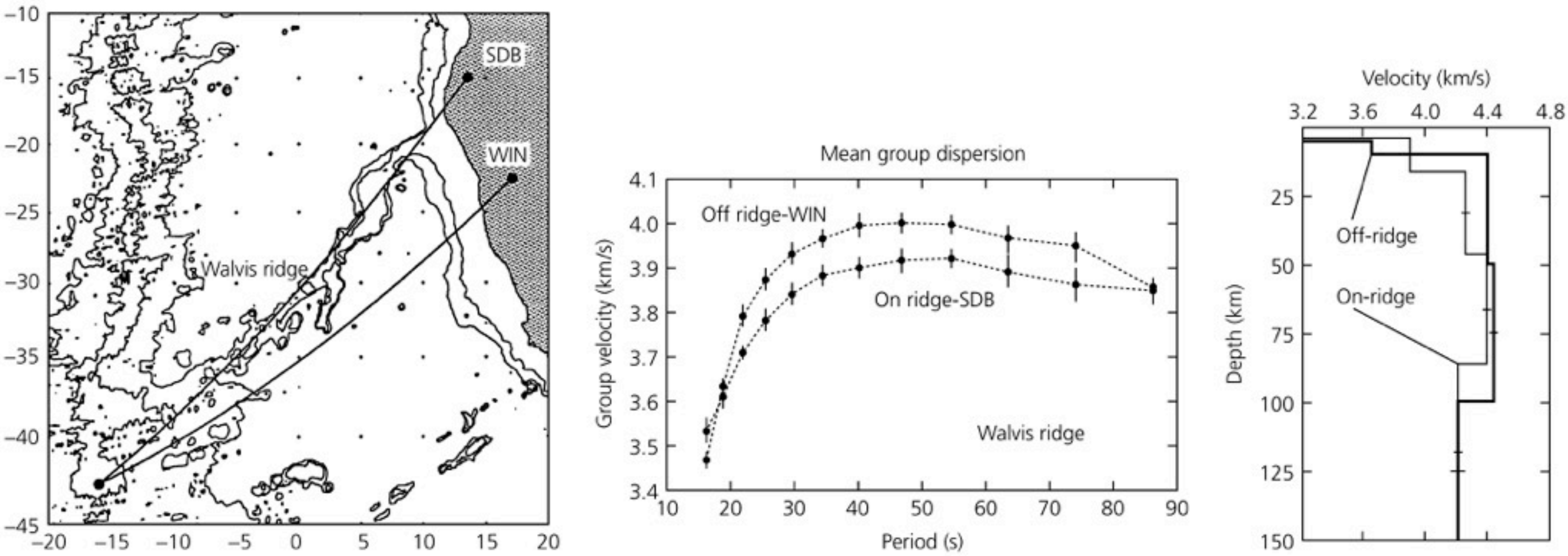
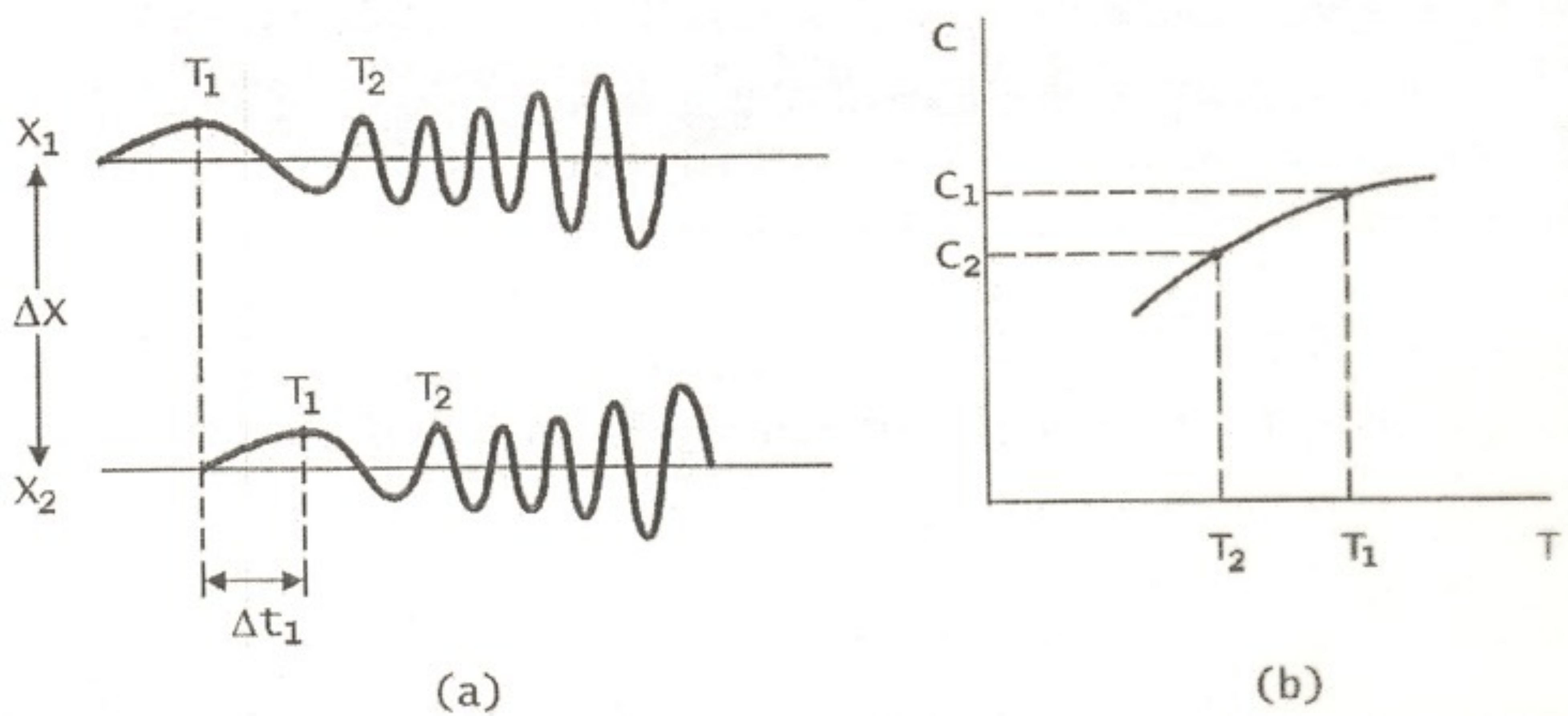

SA51 BHZ
JUN 02 (153), 1997
21:24:39.001
SA57 BHZ
JUN 02 (153), 1997
21:24:39.001
SA51 BHZ
JUN 02 (153), 1997
21:24:39.001
SA57 BHZ
JUN 02 (153), 1997
21:24:39.001
X 10+3
X 10+3
X 10+2
X 10+2
8
10
12
14
16
18
20
22
X 10+2

Figure 2.8-5: Rayleigh wave group velocity study of the Walvis ridge.



Directly measured at two stations



Measuring phase velocity

Measured by taking Fourier transform and obtaining phase spectrum

A surface wave can be represented:

$$u(x,t) = \frac{1}{\pi}\int_0^\infty A(\omega, x)\cos(\omega t - \frac{\omega}{c(\omega)}x + \phi_0(x))d\omega$$

$$\phi(\omega) = \omega t - \frac{\omega}{c(\omega)}x + \phi_0(\omega) + 2\pi N$$

One-station method

$$\phi_1(\omega) = \omega t_1 - \frac{\omega}{c(\omega)} x_1 + \phi_0(\omega) + 2\pi N$$

Need know the initial phase ϕ_0

N can be determined by by allowing c(ω) for the longest period converge to the global average

c measures

On a seismogram recorded at a distance x from the earthquake at time t after the earthquake, the phase has three terms:

$$\Phi(\omega) = [\omega t - k(\omega)x] + \phi_i(\omega) + 2n\pi$$

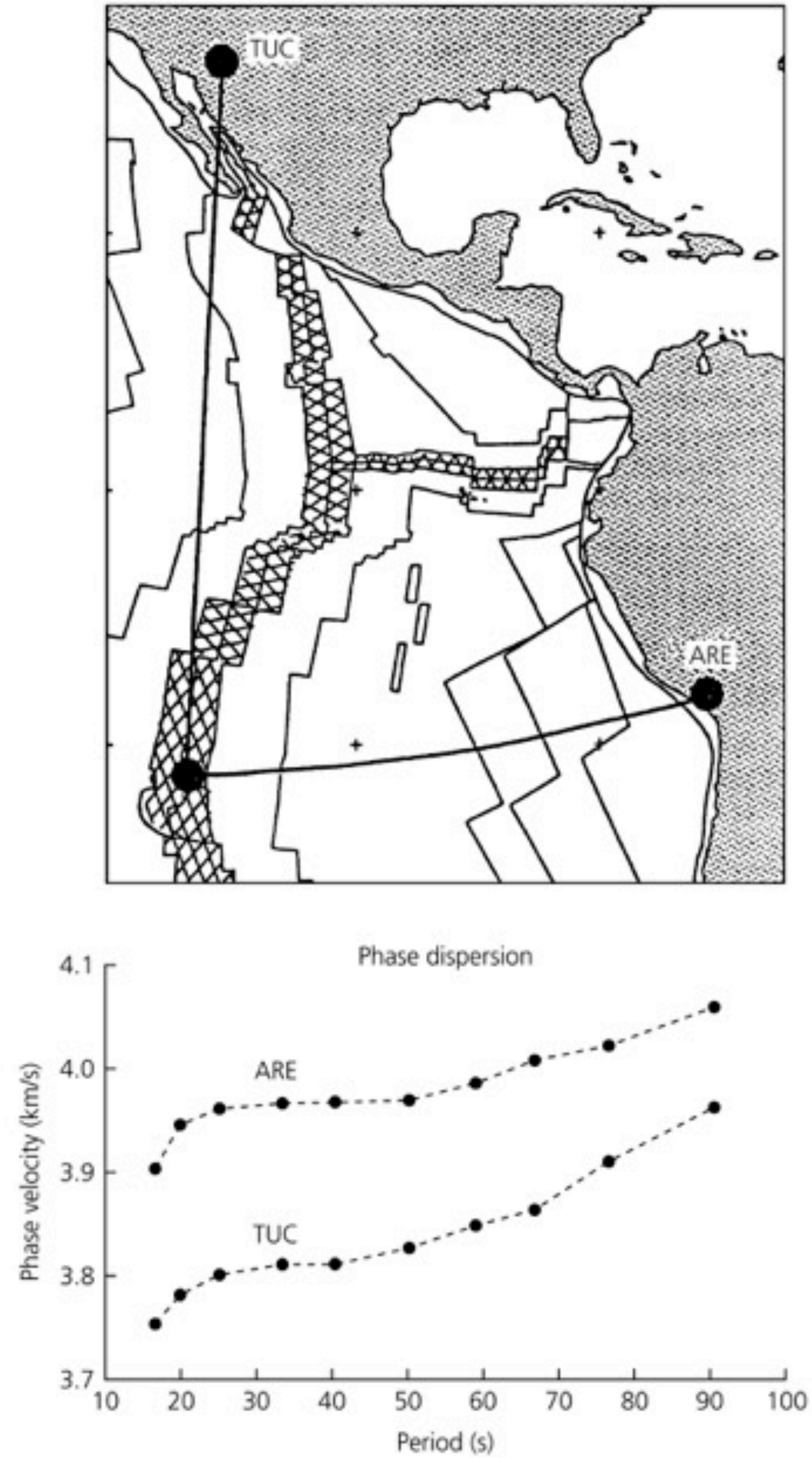
$$= [\omega t - \omega x / c(\omega)] + \phi_i(\omega) + 2n\pi$$

$\omega t - k(\omega)x$ is the phase due to the propagation of the wave in time and space.

$\phi_i(\omega)$ includes the initial phase at the earthquake and any phase shift introduced by the seismometer.

$2n\pi$ reflects the periodicity of the complex exponential, because adding an integral multiple of 2π to the argument yields the same value.

Figure 2.8-6: Example of Rayleigh wave phase velocities for ocean lithosphere.



c measures

Two station method:

$$\Phi_1(\omega) = \omega t_1 - \omega x_1/c(\omega) + \phi_i(\omega) + 2n\pi$$

$$\Phi_2(\omega) = \omega t_2 - \omega x_2/c(\omega) + \phi_i(\omega) + 2m\pi$$

Take the difference $\Phi_{21} = \Phi_2 - \Phi_1$, and solve for c:

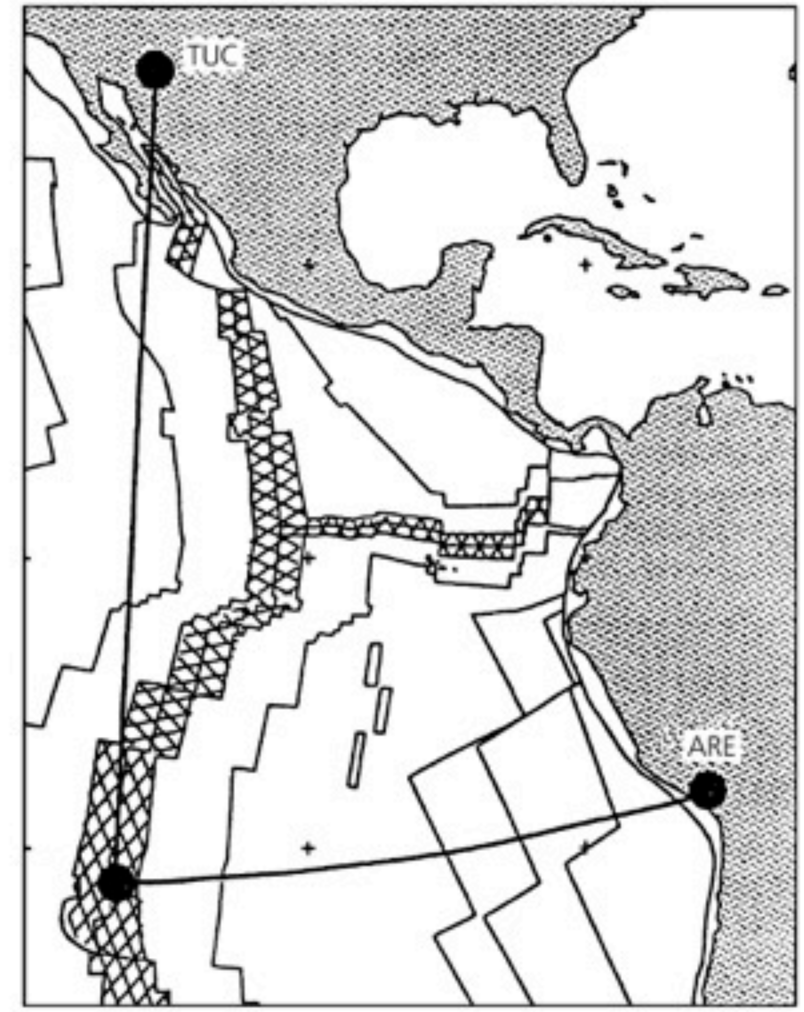
$$c(\omega) = \omega(x_2 - x_1)/[\omega(t_2 - t_1) + 2(m - n)\pi - \Phi_{21}(\omega)].$$

The $2(m - n)\pi$ term is found empirically by ensuring that the phase velocity at long periods is reasonable.

Single station method:
Predict the phase at the earthquake from its focal mechanism.
If $\phi_i(\omega)$ is known, c is:

$$c(\omega) = \omega x/[\omega t + \phi_i(\omega) + 2n\pi - \Phi(\omega)]$$

Figure 2.8-6: Example of Rayleigh wave phase velocities for ocean lithosphere.



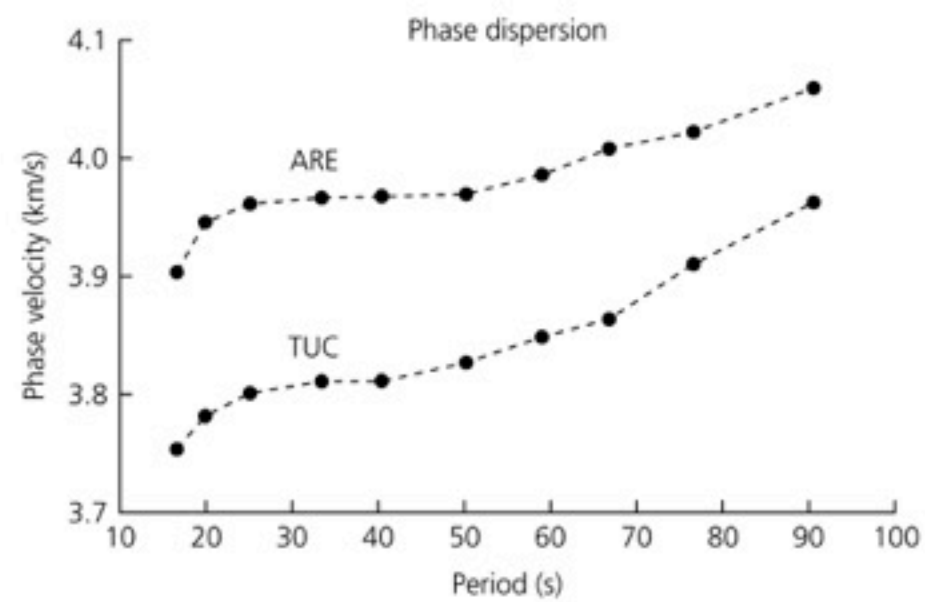
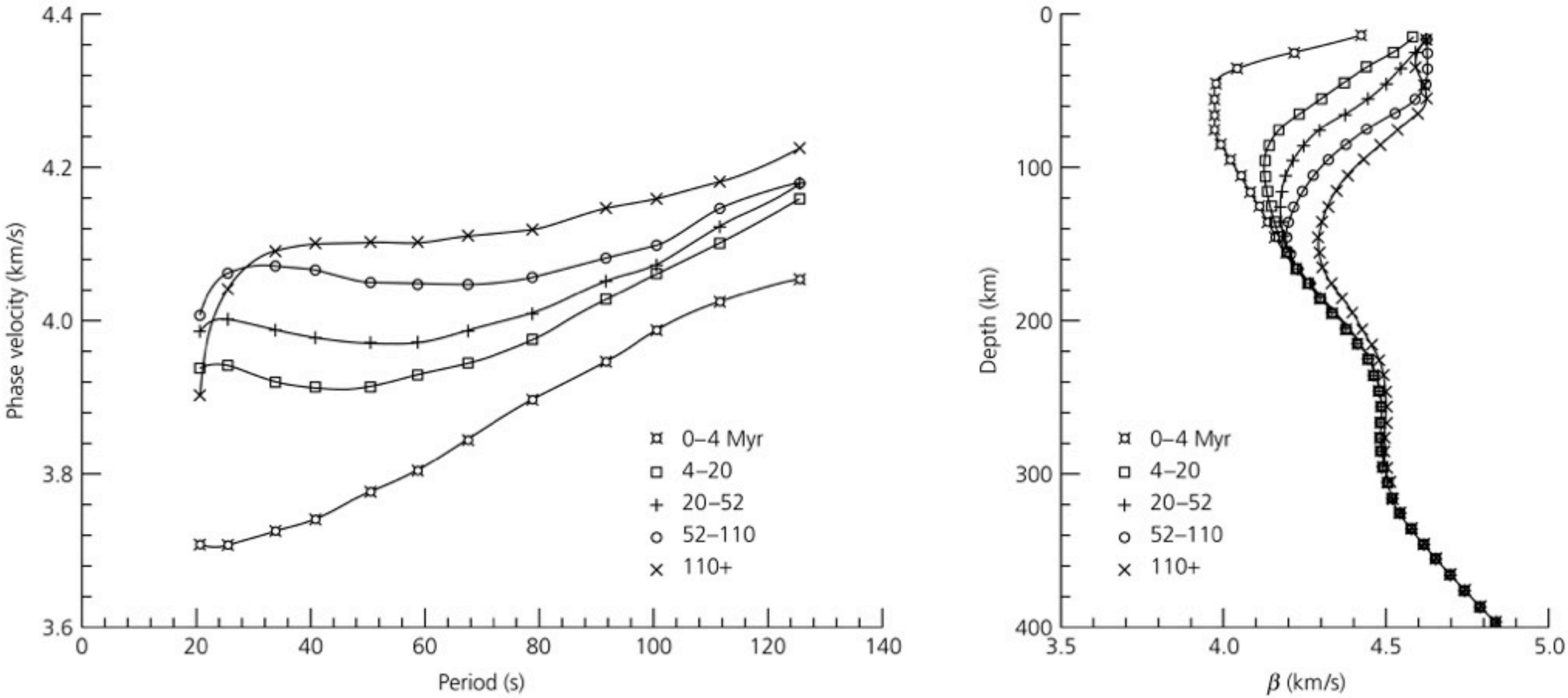


Figure 2.8-7: Rayleigh wave phase velocity dispersion as a function of oceanic plate age.



Elastodynamic equations

Considering an elastic body of volume V and surface S, the application of body forces, as well as the application of tractions, will generate a displacement field that is constrained to satisfy the equations of motion:

$$\rho \ddot{u}_i = f_i + \frac{\partial \sigma_{ij}}{\partial x_j} = f_i + \sigma_{ij,j}$$

The equation for elastic displacement can be written also using the vector differential operator, $(L(\mathbf{u}))_i = \rho \ddot{u}_i - \left(c_{ijkl} u_{k,l}\right)_{,j} = \rho \ddot{u}_i - \sigma_{ij,j}$ as:

$$L(\mathbf{u}) = 0 \quad \text{homogeneous}$$

$$L(\mathbf{u}) = \mathbf{f} \quad \text{inhomogeneous}$$

Isotropic medium

And for an isotropic medium, in absence of body forces, the equations of motion become:

$$(\mathbf{L}(\mathbf{u}))_i = \rho \ddot{\mathbf{u}}_i - \frac{\partial}{\partial_j}\left(\lambda \partial_k \mathbf{u}_k \delta_{ij} + \mu(\partial_i \mathbf{u}_j + \partial_j \mathbf{u}_i)\right) = 0$$

i.e. a linear system of three differential equations with three unknowns: the components of the displacement vector, whose coefficients depend upon the elastic parameters of the material. It is not possible to find the analytic solution for this system of equations, therefore it is necessary to add further approximations, chosen according to the adopted resolving method. Two ways can be followed:

a) an **exact definition of the medium is given**, and a **direct numerical integration technique** is used to solve the set of differential equations;

b) **exact analytical techniques** are applied to an **approximated model** of the medium that may have the elastic parameters varying along one or more directions of heterogeneity.

1D heterogeneity

- Let us consider a halfspace in a system of Cartesian coordinates with the vertical z axis positive downward and the **free surface**, where vertical stresses (σ_{xz}, σ_{yz}, σ_{zz}) are null, is defined by the plane z=0.

- Let us assume that ρ, λ and μ are piecewise continuous functions of z, that displacement and stress components are continuous along z, and that body wave velocities, α and β, assume their largest value, α_H and β_H, when z=H, remaining constant for greater depths.

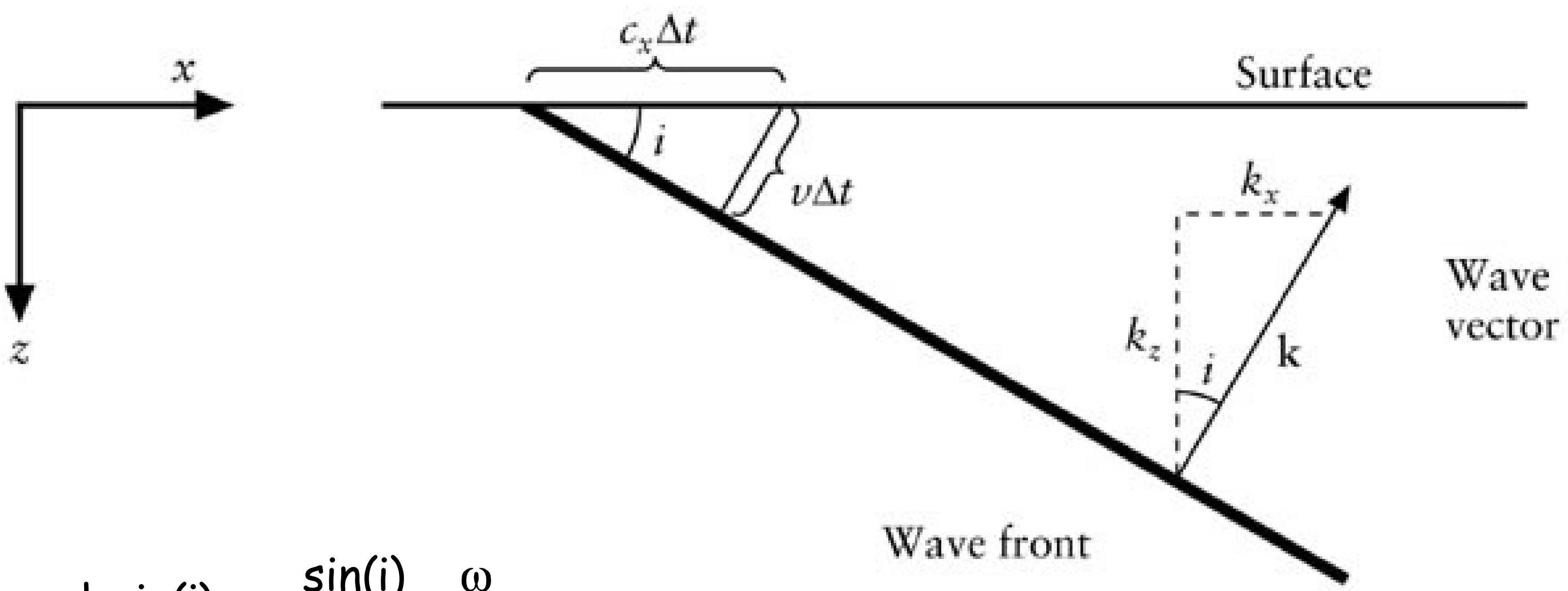
If the parameters depend only upon the vertical coordinate, the equations become:

$$\rho\ddot{\mathbf{u}} = (\lambda+\mu)\nabla(\nabla\cdot\mathbf{u}) + \mu\nabla^2\mathbf{u} + \frac{\partial\lambda}{\partial z}(\hat{\mathbf{z}}\nabla\cdot\mathbf{u}) + \frac{\partial\mu}{\partial z}\left[(\nabla\cdot\hat{\mathbf{z}})\mathbf{u} + \nabla(\hat{\mathbf{z}}\cdot\mathbf{u})\right]$$

we can consider solutions of having the form of plane harmonic waves propagating along the positive x axis:

$$\mathbf{u}(\mathbf{x},t) = \mathbf{F}(z)e^{i(\omega t - kx)}$$

Apparent horizontal (phase) velocity



$$k_x = k\sin(i) = \omega\frac{\sin(i)}{\alpha} = \frac{\omega}{c}$$

$$k_z = k\cos(i) = \sqrt{k^2 - k_x^2} = \omega\sqrt{\left(\frac{1}{\alpha}\right)^2 - \left(\frac{1}{c}\right)^2} = \frac{\omega}{c}\sqrt{\left(\frac{c}{\alpha}\right)^2 - 1} = k_x r_\alpha$$

$$k_x = k\sin(i) = \omega\frac{\sin(i)}{\beta} = \frac{\omega}{c}$$

$$k_z = k\cos(i) = \sqrt{k^2 - k_x^2} = \omega\sqrt{\left(\frac{1}{\beta}\right)^2 - \left(\frac{1}{c}\right)^2} = \frac{\omega}{c}\sqrt{\left(\frac{c}{\beta}\right)^2 - 1} = k_x r_\beta$$

Remember: when c is less then the body wave velocity k_z is imaginary and represent **inhomogeneous** waves, i.e. waves exponentially **decaying** or increasing with depth;
examples are Rayleigh waves in a homogenous halfspace, or Love waves in low velocity layer over a homogeneous halfspace

In current terminology, k_x is k

P-SV problem

We have to solve two independent eigenvalue problems for the three components of the vector **F**=(F_x,F_y,F_z). The first one describes the motion in the plane **(x,z)**, i.e., **P-SV** waves and it has the form:

$$\frac{\partial}{\partial z}\left[\mu\frac{\partial F_x}{\partial z}-ik\mu F_z\right]-ik\lambda\frac{\partial F_z}{\partial z}+\left[\omega^2\rho-k^2(\lambda+2\mu)\right]F_x=0$$

$$\frac{\partial}{\partial z}\left[(\lambda+2\mu)\frac{\partial F_z}{\partial z}-ik\lambda F_x\right]-ik\mu\frac{\partial F_x}{\partial z}+\left[\omega^2\rho-k^2\mu\right]F_z=0$$

and must be solved with the free surface boundary condition at z = 0

$$\sigma_{zz}=\left[(\lambda+2\mu)\frac{\partial F_z}{\partial z}-ik\lambda F_x\right]_{z=0}=0$$

$$\sigma_{xz}=\left[\mu\frac{\partial F_z}{\partial z}-ik\mu F_z\right]_{z=0}=0$$

SH problem

The second eigenvalue problem describes the case when the particle motion is limited to the **y-axis**, and determines phase velocity and amplitude of **SH waves**. It has the (Sturm-Liouville) form:

$$\frac{\partial}{\partial z}\left(\mu \frac{\partial F_y}{\partial z}\right) + \left(\omega^2 \rho - k^2 \mu\right) F_y = 0$$

and must be solved with the free surface boundary condition at z = 0

$$\left[\mu \frac{\partial F_y}{\partial z}\right]_{z=0} = 0$$

Layered halfspace

Let us now assume that the vertical heterogeneity in the halfspace is modelled with **a series of N-1 homogeneous flat layers**, parallel to the free surface, overlying a homogeneous halfspace.
Let ρ_m, α_m, β_m, and d_m, respectively be the density, P-wave and S-wave velocities, and the thickness of the m-th layer.
Furthermore, let us define:

$$r_{\alpha m} = \begin{cases} \sqrt{\left(\dfrac{c}{\alpha_m}\right)^2 - 1} & \text{if } c > \alpha_m \\ -i\sqrt{1 - \left(\dfrac{c}{\alpha_m}\right)^2} & \text{if } c < \alpha_m \end{cases} \qquad r_{\beta m} = \begin{cases} \sqrt{\left(\dfrac{c}{\beta_m}\right)^2 - 1} & \text{if } c > \beta_m \\ -i\sqrt{1 - \left(\dfrac{c}{\beta_m}\right)^2} & \text{if } c < \beta_m \end{cases}$$

Love (SH) problem

The SH solutions (displacement and stress) for the m-th layer are:

$u_x = u_z = 0$

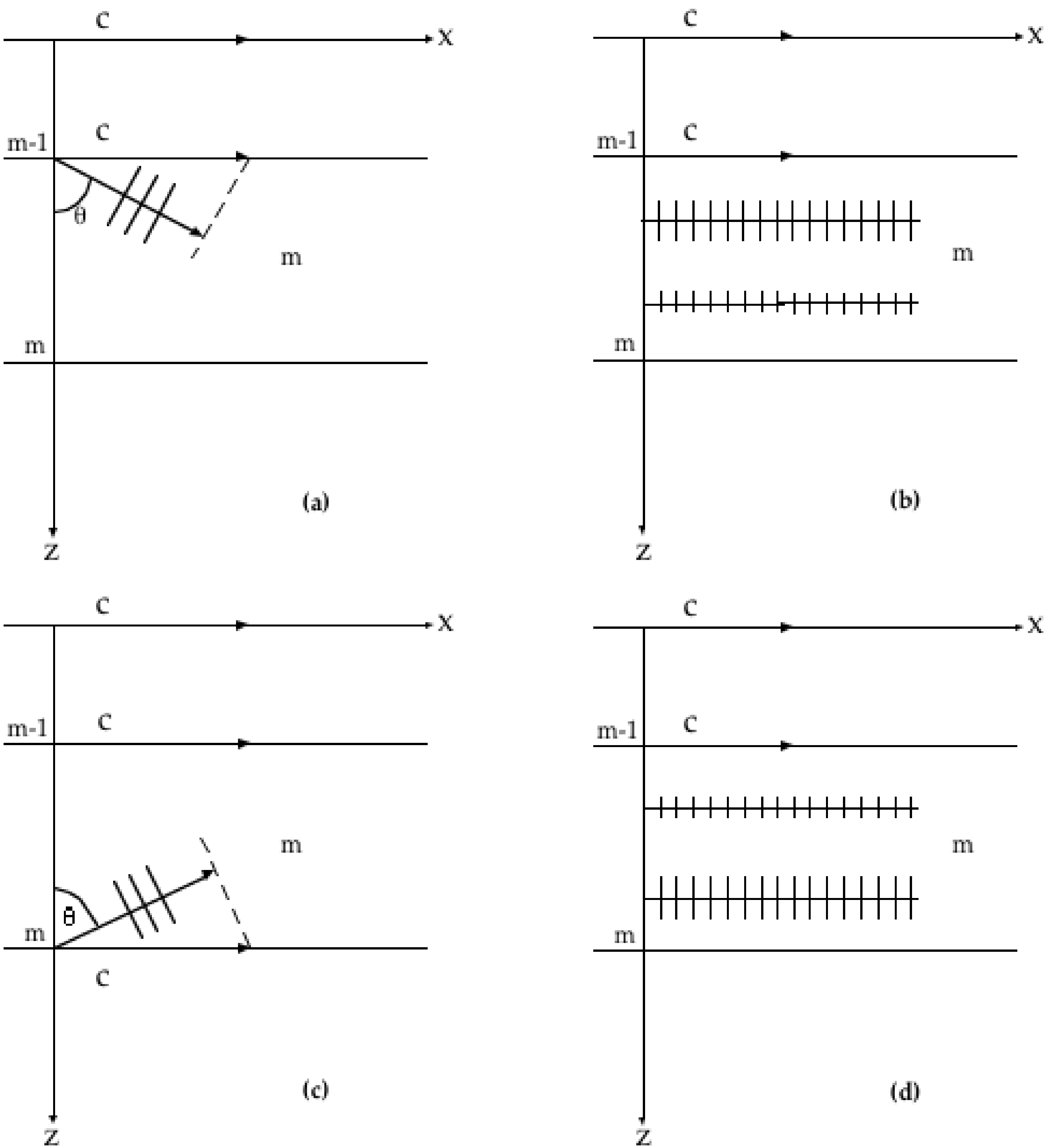
$$u_y = \left(v'_m e^{-ikr_{\beta m}z} + v''_m e^{+ikr_{\beta m}z}\right) e^{i(\omega t - kx)}$$

$$\sigma_{zy} = \mu \frac{\partial u_y}{\partial z} = ik\mu r_{\beta m}\left(-v'_m e^{-ikr_{\beta m}z} + v''_m e^{+ikr_{\beta m}z}\right) e^{i(wt - kx)}$$

where v_m' and v_m'' are constants.

Given the sign conventions adopted, the term in v' represents a plane wave whose direction of propagation makes an angle $\cot^{-1} r_{\beta m}$ with the +z direction when $r_{\beta m}$ is real, and a wave propagating in the +x direction with amplitude diminishing exponentially in the +z direction when $r_{\beta m}$ is imaginary. Similarly the term in v'' represents a plane wave making the same angle with the direction -z when $r_{\beta m}$ is real and a wave propagating in the +x direction with amplitude increasing in the +z direction when $r_{\beta m}$ is imaginary.

Love (SH) problem



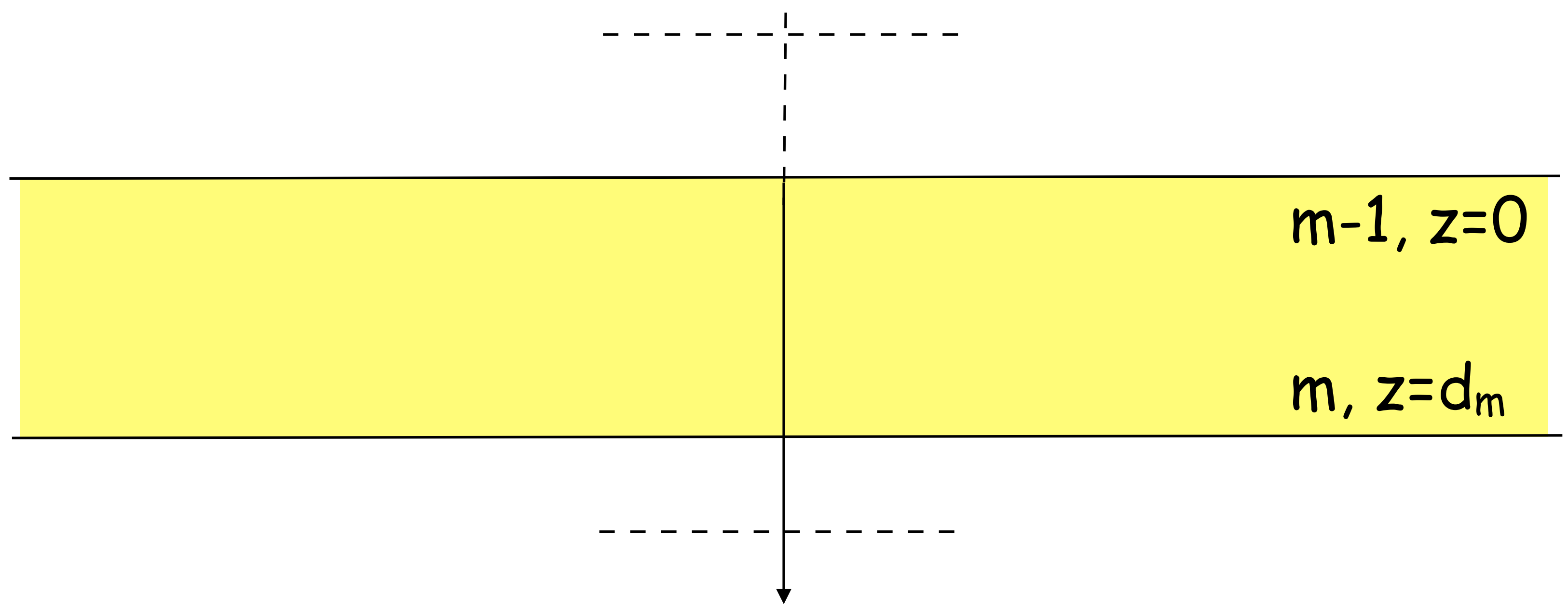
the term in v' represents a plane wave whose direction of propagation makes an angle $\cot^{-1} r_{\beta m}$ with the +z direction when $r_{\beta m}$ is real (a),
and a wave propagating in the +x direction with amplitude diminishing exponentially in the +z direction when $r_{\beta m}$ is imaginary (b).

Similarly the term in v'' represents a plane wave making the same angle with the direction -z when $r_{\beta m}$ is real (c)
and a wave propagating in the +x direction with amplitude increasing in the +z direction
when $r_{\beta m}$ is imaginary d).

Consider the m-th layer and the (m-1) interface, set temporarily as the origin of the coordinate system. It is convenient to use $[(du_y/dt)/c]=iku_y$ instead of displacement, to deal with adimensional quantities.

$$\left(\frac{\dot{u}_y}{c}\right)_{m-1} = ik(v'_m + v''_m)$$

$$\left(\sigma_{zy}\right)_{m-1} = ik\mu_m r_{\beta_m}(v''_m - v'_m)$$



$$\left(\frac{\dot{u}_y}{c}\right)_m = ik(v'_m + v''_m)\cos Q_m - k(v''_m - v'_m)\sin Q_m$$

$$Q_m = kr_{\beta m}d_m$$

$$\left(\sigma_{zy}\right)_m = -k\mu_m r_{\beta_m}(v''_m + v'_m)\sin Q_m + ik\mu_m r_{\beta_m}(v''_m - v'_m)\cos Q_m$$

Love layer matrix

$$\left(\frac{\dot{u}_y}{c}\right)_m = \left(\frac{\dot{u}_y}{c}\right)_{m-1} \cos Q_m + i\left(\sigma_{zy}\right)_{m-1}\left(\mu_m r_{\beta_m}\right)^{-1} \sin Q_m$$

$$\left(\sigma_{zy}\right)_m = \left(\frac{\dot{u}_y}{c}\right)_{m-1} i\,\mu_m r_{\beta_m} \sin Q_m + \left(\sigma_{zy}\right)_{m-1} \cos Q_m$$

$$a_m = \begin{bmatrix} \cos Q_m & \dfrac{i \sin Q_m}{\mu_m r_{\beta_m}} \\ i\,\mu_m r_{\beta_m} \sin Q_m & \cos Q_m \end{bmatrix}$$

$$\begin{bmatrix} \left(\frac{\dot{u}_y}{c}\right)_m \\ \left(\sigma_{zy}\right)_m \end{bmatrix} = a_m \begin{bmatrix} \left(\frac{\dot{u}_y}{c}\right)_{m-1} \\ \left(\sigma_{zy}\right)_{m-1} \end{bmatrix}$$

$$\begin{bmatrix} \left(\frac{\dot{u}_y}{c}\right)_{N-1} \\ \left(\sigma_{zy}\right)_{N-1} \end{bmatrix} = A \begin{bmatrix} \left(\frac{\dot{u}_y}{c}\right)_0 \\ \left(\sigma_{zy}\right)_0 \end{bmatrix} \qquad A = a_{N-1} a_{N-2} \ldots a_2 a_1$$

Love dispersion equation

remembering that the boundary conditions of a) surface waves and b) the free surface implies that $v_N''=0$ and $\sigma_{zy}(z=0)=0$, we have that:

$$A_{21} + \mu_N r_{\beta_N} A_{11} = 0$$

The left-hand side is the **dispersion function** for Love modes (SH waves), where A_{21} and A_{11} are elements of the matrix A.
The couples (ω,c) for which the dispersion function is equal to zero are its roots and represent the **eigenvalues** of the problem.

Eigenvalues, according to the number of zeroes of the corresponding **eigenfunctions**, uy(z,ω,c) and $\sigma_{zy}(z,\omega,c)$,
can be subdivided in the **dispersion curve** of the fundamental mode (which has no nodal planes), of the first higher mode (having one nodal plane), of the second higher mode and so on.

Once the phase velocity c is determined, we can compute analytically the **group velocity** using the implicit functions theory, and the eigenfunctions.

Rayleigh (P-SV) problem

The P-SV solutions (displacement and stress) for the m-th layer can be found combining dilatational and rotational potentials:

$$\Delta_m = \frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x} = \left(\Delta_m^{'} e^{-ikr_{\alpha m}z} + \Delta_m^{''} e^{+ikr_{\alpha m}z}\right) e^{i(\omega t - kx)}$$

$$\delta_m = \frac{1}{2}\left[\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x}\right] = \left(\delta_m^{'} e^{-ikr_{\beta m}z} + \delta_m^{''} e^{+ikr_{\beta m}z}\right) e^{i(\omega t - kx)}$$

where Δ_m', Δ_m'', δ_m' and δ_m'' are constants.
Given the sign conventions adopted, the term in Δ_m' represents a plane wave whose direction of propagation makes an angle $\cot^{-1} r_{am}$ with the +z direction when r_{am} is real, and a wave propagating in the +x direction with amplitude diminishing exponentially in the +z direction when r_{am} is imaginary. Similarly the term in Δ_m'' represents a plane wave making the same angle with the direction -z when r_{am} is real and a wave propagating in the +x direction with amplitude increasing in the +z direction when r_{am} is imaginary.
The same considerations can be applied to the terms in δ_m' and δ_m'', substituting r_{am} with $r_{\beta m}$.

Rayleigh (P-SV) problem

The P-SV solutions (displacement and stress) components can be written as:

$$u_x = -\frac{\alpha_m^2}{\omega^2}\left(\frac{\partial \Delta_m}{\partial x}\right) - 2\frac{\beta_m^2}{\omega^2}\left(\frac{\partial \delta_m}{\partial z}\right)$$

$$u_z = -\frac{\alpha_m^2}{\omega^2}\left(\frac{\partial \Delta_m}{\partial z}\right) + 2\frac{\beta_m^2}{\omega^2}\left(\frac{\partial \delta_m}{\partial x}\right)$$

$$\sigma_{zz} = \rho_m\left\{\alpha_m^2 \Delta_m + 2\beta_m^2\left[\frac{\alpha_m^2}{\omega^2}\left(\frac{\partial^2 \Delta_m}{\partial x^2}\right) + 2\frac{\beta_m^2}{\omega^2}\left(\frac{\partial^2 \delta_m}{\partial z^2}\right)\right]\right\}$$

$$\sigma_{zx} = 2\beta_m^2 \rho_m\left\{-\frac{\alpha_m^2}{\omega^2}\left(\frac{\partial^2 \Delta_m}{\partial x \partial z}\right) + \frac{\beta_m^2}{\omega^2}\left[\left(\frac{\partial^2 \delta_m}{\partial x^2}\right) - \left(\frac{\partial^2 \delta_m}{\partial z^2}\right)\right]\right\}$$

Starting with the free surface condition ($\sigma_{zz}(z=0)=\sigma_{zx}(z=0)=0$), iterating the continuity boundary conditions at every interface, and applying the condition of no radiation in the final halfspace, one can build up the **dispersion function whose roots are the eigenvalues associated with the Rayleigh modes**.