BASICS OF GENERAL RELATIVITY

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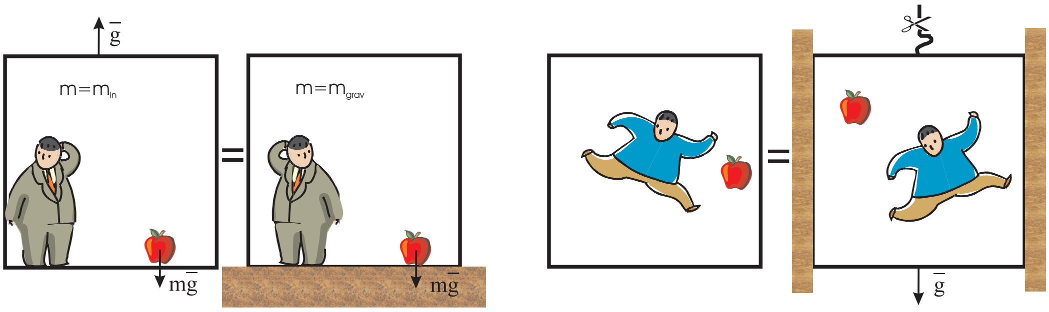

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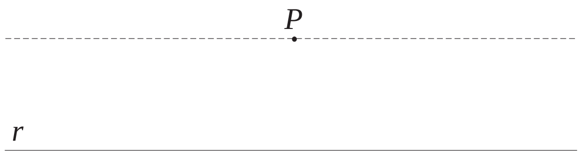
1 Non-euclidean geometry

1.1 Introduction

The attempts to "prove" the so-called Euclid's fifth postulate from the other postulates, considered the most obvious, did not achieve this, but they brought, in 19^{th} century, to the birth of *non-Euclidean geometries* (Gauss, Bolyai, Lobachevski, Klein).

Gauss, Bolyai, Lobachevski e Klein

The fifth postulate can be enunciated as follows: Given a straight line r and a point P outside of it, through the point P passes one and only one line parallel to r (we can give to the term *parallel* the meaning of *that meets* r *only to infinite*, in an *improper* point).

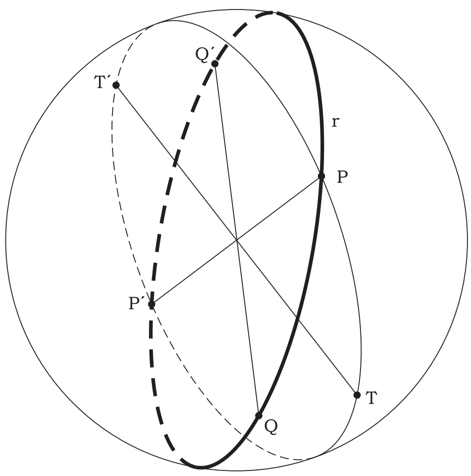


This postulate result *independent* of the others, in the sense that we can construct planar geometries (i.e. in 2 dimensions) in which all the other original Euclid's postulates are still working, but the fifth postulate is different:

1. Through P outside the line r does not pass any parallel line
2. Through P outside the line r pass two (or even infinite) parallel (or *not secant*) lines

In case 1, in addition, the sum of the interior angles of a triangle is$> 180°$, while in case 2 it is$< 180°$.

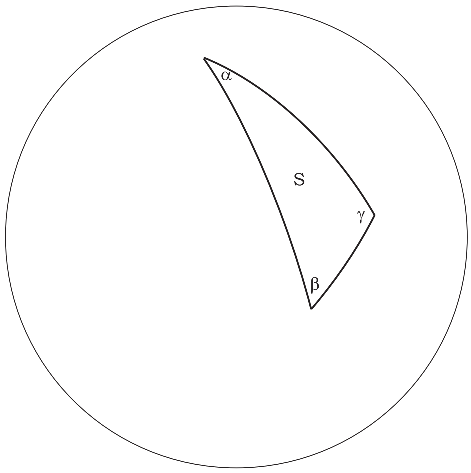
To build a "model" of these geometries we must define, in an appropriate way, points, lines, etc..



For the case 1 (**elliptical plane geometry**) we define a point as the pair of diametrically opposite points (P, P'), and a straight line is a great circle passing through P and P'. We see that through two points (P, P')

and (Q, Q') passes a straight line r and that through a point (T, T'), external to r, does not pass any parallel to r since all the straight lines passing through (T, T') intersect r at a point.

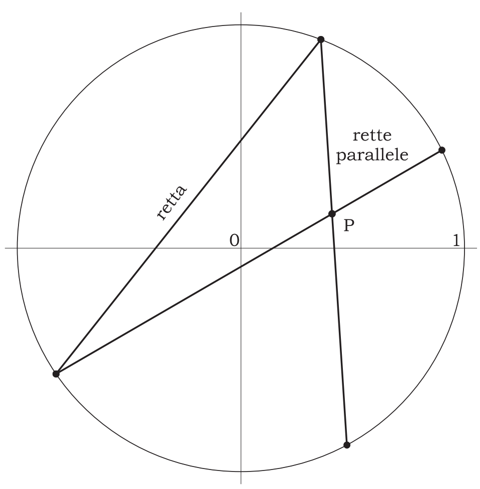
If we define on the sphere a triangle with sides formed by arcs of great circles, the sum of the angles $\alpha + \beta + \gamma$ is always $> \pi$, so that the area S, if the sphere has a radius R, can be expressed as $S = R^2(\alpha + \beta + \gamma - \pi)$. If $S \to 0$ (keeping R fixed) we see that $(\alpha + \beta + \gamma) \to \pi$. If the spherical triangle is much smaller than the radius R, its difference from a plane triangle tends to disappear.

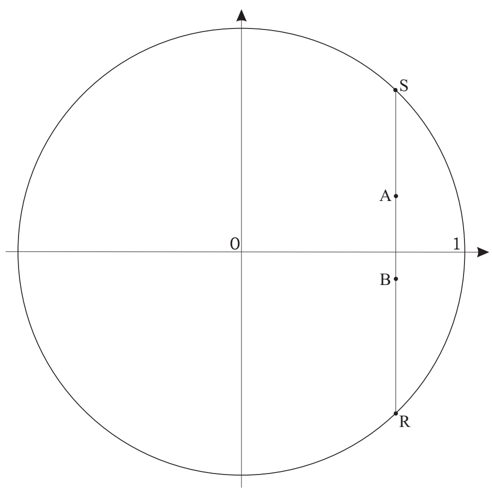


To build a model of elliptical plane geometry we have resorted to the use of a sphere (a two-dimensional surface we will denote by $\mathbf{S}^2$) immersed (*embedded*) in a three-dimensional Euclidean space $\mathbf{E}^3$.

We also note that in order to represent the postulate V 1) we had to use a “curved” surface, i.e. the the sphere. This "curvature" must also be constant throughout the "plan" because the other postulates describe the space as homogeneous, and if the curvature varied this property would be lost.

With the analytic geometry, Descartes has shown that, by identifying the points with pairs of real numbers and defining the distance between two points (x_1, y_1) e (x_2, y_2) as $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ all the postulates of Euclid reduce to theorems about real numbers. The definition of point and distance is therefore essential to describe a geometry.





Similarly to what was done for postulate V 1), we can define a model for the postulate V 2), the so-called **hyperbolic plane geometry** ($\mathbf{H}^2$).

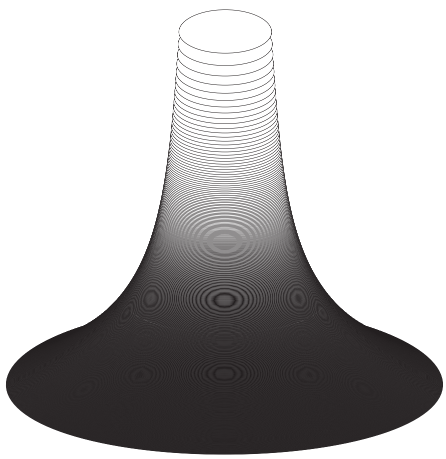
At variance with the “elliptic” plane, the "hyperbolic" plan can not be completely embedded in a 3-D Euclidean space (we will understand later why). But you can build some limited models, such as the circle of Klein (1870): the points are those inside the circle of radius 1. The straight lines are the chords of the circle. The points of the circle are improper points. A straight line is parallel to another if it has in common with the first one an improper point. Yuo can see that trough P pass two straight lines parallel to a given straight line, and there are infinitely many lines through P not intersecting it. The distance between two points A and B (see figure) is given by

$$d(AB) = \frac{1}{2}\log\frac{RA \cdot SB}{RB \cdot SA}.$$

As you can see $d(AB) \to \infty$ if one of the two points tends to the circumference (to the points R or S).

A partial representation of $\mathbf{H}^2$ "embedded" in a Euclidean 3-D space $\mathbf{E}^3$is given by the so-called Beltrami's pseudosphere, which has the shape of a trumpet. This surface has a constant curvature, as the sphere, but negative (later we will understand the meaning of the negative curvature). The fact that the circle of Klein and the pseudosphere are partial representations, i.e. incomplete, of the hyperbolic plane is due to the fact that the points of the edge of Klein's circle, as the edge points of the pseudosphere, are singular points of the surface.

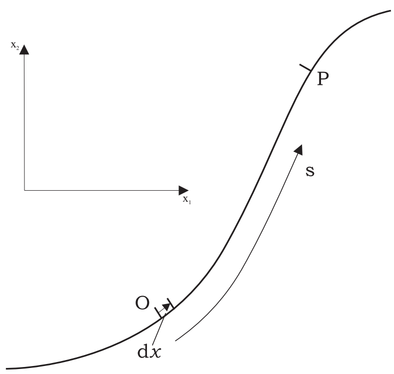
To better understand the meaning of the term "curvature", we must refresh and deepen some concepts of differential geometry directly linked to the distance between points. This is an essential concept to describe the geometry of a surface, even if is not possible to give an intuitive representation of it in an Euclidean 3-D space. This then serves as generalization to switch from 2-D surfaces to spaces with 3 or more dimensions.

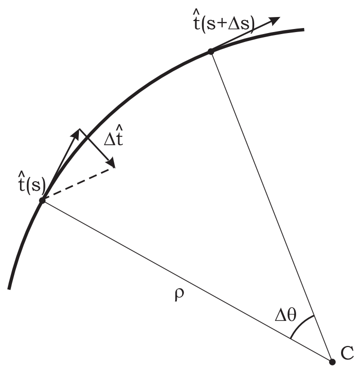


1.2 Curves in the plane

A plane curve can be parametrized in the following way: $\overline{x}(t) = \big(x_1(t), x_2(t)\big)$ where t is a parameter, not necessarily time; the tangent vector (velocity) is $\frac{\mathrm{d}\overline{x}}{\mathrm{d}t}$. The curvilinear abscissa is defined as s:

$$O \equiv \overline{x}(t=0) \qquad P \equiv \overline{x}(t) \qquad \mathrm{d}s = |\mathrm{d}\overline{x}| = \left|\frac{\mathrm{d}\overline{x}}{\mathrm{d}t}\right|\mathrm{d}t \quad \to \quad s = \int_0^t \left|\frac{\mathrm{d}\overline{x}}{\mathrm{d}t}\right|\mathrm{d}t = s(t)$$





If we switch parameter from t to s, we notice that $\frac{\mathrm{d}\overline{x}}{\mathrm{d}s} = \dot{\overline{x}}(s)$ has magnitude 1: it's the tangent versor $\hat{t}(s)$.

Since $|\dot{\overline{x}}(s)| = |\hat{t}(s)| = 1$, we have $\hat{t}\cdot\hat{t} = 1$ and, performing the derivative, $2\hat{t}\cdot\dot{\hat{t}} = 0$, i.e. $\hat{t}\perp\dot{\hat{t}}$. (notice that $\dot{\hat{t}}(s)$ is not a versor!)

By defining $\kappa(s) = |\dot{\hat{t}}(s)|$ and $\hat{n}(s) = \frac{\dot{\hat{t}}(s)}{|\dot{\hat{t}}(s)|}$ I get $\dot{\hat{t}}(s) = \kappa(s)\cdot\hat{n}$. Let's see how we can express $\dot{\hat{t}}(s)$

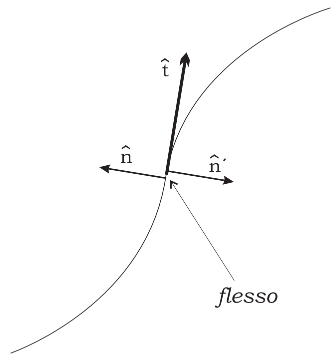
$$\Delta\hat{t} = \hat{t}(s+\Delta s) - \hat{t}(s) \qquad |\Delta\hat{t}| = 2|\hat{t}|\sin\frac{\Delta\theta}{2} \sim \Delta\theta \qquad \Delta s \simeq \rho\Delta\theta \quad \rightarrow \quad \left|\frac{\Delta\hat{t}}{\Delta s}\right| \simeq \frac{\Delta\theta}{\rho\Delta\theta} = \frac{1}{\rho}$$

Then (notice: $\Delta\widehat{t}$ points to the center C of the osculating circle)

$$\frac{\mathrm{d}\hat{t}}{\mathrm{d}s} = \kappa\hat{n} = \frac{1}{\rho}\hat{n} \qquad \begin{cases} \kappa: & \text{curvature} \\ \rho: & \text{curvature radius} \end{cases}$$

If we measure θ with reference to a fixed direction (for instance the x_1 axis)

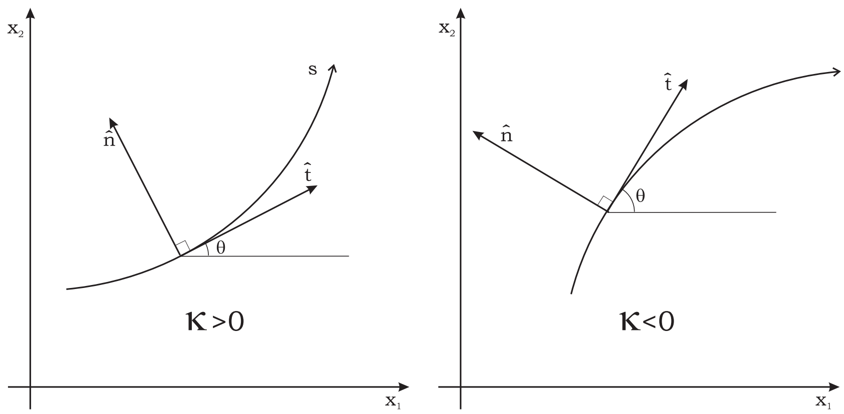
$$\Delta s = \rho\Delta\theta = \frac{\Delta\theta}{\kappa} \qquad \rightarrow \qquad \kappa = \left|\frac{\mathrm{d}\theta}{\mathrm{d}s}\right|$$



We defined κ as being $\geqslant 0$. But, in this way, we get a discontinuity in $\hat{n}$ at an inflection point. In order to avoid this problem, after the definition of a curvilinear abscissa s on the curve, the tangent versor $\hat{t}$ is also defined and we can take the versor $\hat{n}$ obtained from a rotation of $\hat{t}$ by 90° in a positive direction (consistent with O, x_1, x_2). Since $\hat{t}\perp\dot{\hat{t}}$, we still have $\dot{\hat{t}} = \kappa\hat{n}$, but now we can also have $\kappa < 0$. According to the sign of κ, the curve is located to the left or to the right of $\hat{t}$; at the inflection point $\hat{n}$ doesn't change, but κ changes his sign and now it is given by

$$\kappa = \frac{\mathrm{d}\theta(s)}{\mathrm{d}s}$$

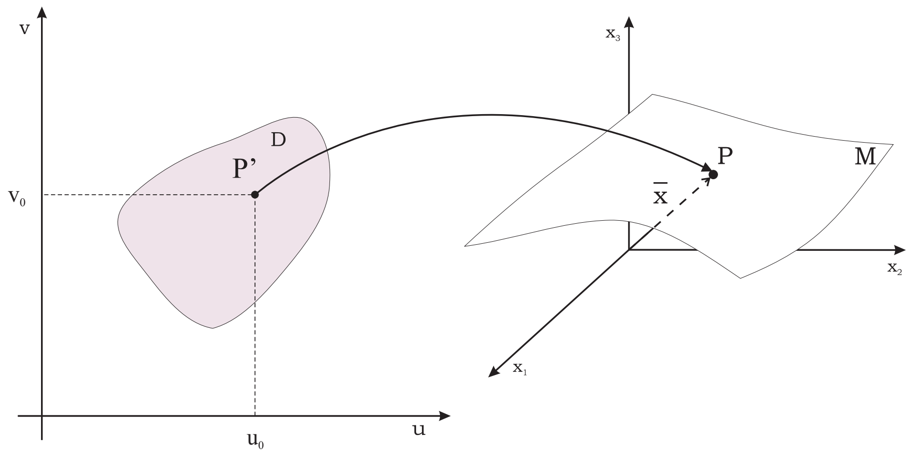
(and not by the absolute value).



1.3 Surface elements

To be more specific, rather than about surfaces, we will talk about surface elements, as we are interested in their local properties.

Also in this case we resort to a parametric representation: we consider a bijective function $\overline{x} : D \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}^3$ (we work in a three-dimensional Euclidean space $\mathbf{E}^3$).



We define $\overline{x}(u,v) \equiv \big(x_1(u,v), x_2(u,v), x_3(u,v)\big)$. If the surface is expressed in the way $z = f(x,y)$ its parameterization becomes $\overline{x}(u,v) = \big(u, v, f(u,v)\big)$.

We speack about a *regular (smooth) surface* if, having defined the vectors

$$\overline{x}_u(u,v) = \frac{\partial \overline{x}}{\partial u} = \left(\frac{\partial x_1}{\partial u}, \frac{\partial x_2}{\partial u}, \frac{\partial x_3}{\partial u}\right)$$
$$\overline{x}_v(u,v) = \frac{\partial \overline{x}}{\partial v} = \left(\frac{\partial x_1}{\partial v}, \frac{\partial x_2}{\partial v}, \frac{\partial x_3}{\partial v}\right)$$

, everywhere (within the domain) $\overline{x}_u \times \overline{x}_v \neq 0$ (cross product).

While keeping fixed $v = v_0$ and by varying u in the neighborhood of a point P' ($\rightarrow P$ on the surface element M) I get a curve on M , whose tangent vector is $\overline{x}_u$. In a similar way, also $\overline{x}_v$ is tangent to a curve on M. These two vectors define the tangent plane to M at the point P.

We can now define a versor $\hat{N}$ perpendicular (normal) to the surface

$$\hat{N} = \frac{\overline{x}_u \times \overline{x}_v}{|\overline{x}_u \times \overline{x}_v|} \qquad \text{and } \hat{N}, \overline{x}_u, \overline{x}_v \text{ form a trihedron.}$$

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Example: *sphere (in geographic coordinates)*

One can describe the surface of the sphere, using the variable u for the longitude ($-\pi \leq u \leq \pi$) and the variable v for the latitude ($-\frac{\pi}{2} \leq v \leq \frac{\pi}{2}$), in the following way (most commonly using the colatitude, $\frac{\pi}{2} - v$): $\overline{x}(u,v) = \big(R\cos u\ \cos v, R\ \sin u\ \cos v, R\sin v\big)$.

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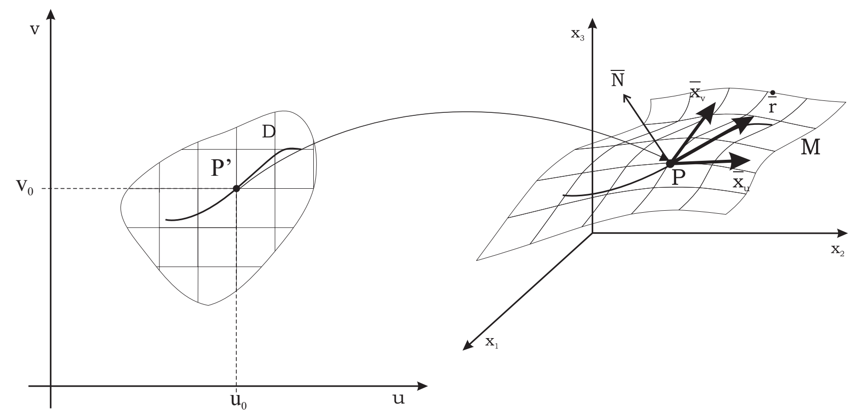
Since in a neighborhood of a point P on M (and of a corresponding point $P' \in D$) the correspondence is bijective, we can think that u and v form, in a neighborhood of P, a system of curvilinear coordinates (like parallels and meridians on a sphere).

If $u = u(t)$, $v = v(t)$ is a curve in D through $P'(u_0, v_0)$, then $\overline{r}(t) = \overline{x}\big(u(t), v(t)\big)$ is a curve on M through $\overline{x}(u_0, v_0)$. The "velocity" vector $\dot{\overline{r}} = \frac{\mathrm{d}\overline{r}}{\mathrm{d}t}$ will be

$$\frac{\mathrm{d}\overline{r}}{\mathrm{d}t} = \dot{\overline{r}} = \frac{\partial \overline{x}}{\partial u}\frac{\mathrm{d}u}{\mathrm{d}t} + \frac{\partial \overline{x}}{\partial v}\frac{\mathrm{d}v}{\mathrm{d}t} \qquad \rightarrow \qquad \dot{\overline{r}} = \overline{x}_u \frac{\mathrm{d}u}{\mathrm{d}t} + \overline{x}_v \frac{\mathrm{d}v}{\mathrm{d}t}$$



The vector $\dot{\overline{r}}$ is also tangent to M and is therefore contained in the tangent plane. Any vector belonging to the tangent plane at P is a linear combination of $\overline{x}_u$ e $\overline{x}_v$ (in $\overline{x}(u_0, v_0)$); conversely, any linear combination $\overline{v} = a\overline{x}_u(u_0, v_0) + b\overline{x}_v(u_0, v_0)$) is the "velocity" vector of a curve on M. The vectors $\overline{x}_u$ e $\overline{x}_v$ form a basis in the tangent plane at the point P.

1.4 The first fundamental form

If $\overline{r}(t) = \overline{x}\big(u(t), v(t)\big)$, with $a \le t \le b$, is a curve on a surface, and if $s = s(t)$ is the arc length (curvilinear abscissa) along $\overline{r}$, from $\overline{r}(a)$ to $\overline{r}(b)$, then the total length L of this curve is obtained by integrating $\frac{\mathrm{d}s}{\mathrm{d}t} = |\frac{\mathrm{d}\overline{r}}{\mathrm{d}t}|$ on the interval $[a, b]$:

$$L \equiv s(b) = \int_a^b |\frac{\mathrm{d}\overline{r}}{\mathrm{d}t}|\mathrm{d}t$$

but, since $\dot{\overline{r}} = \overline{x}_u \cdot \dot{u} + \overline{x}_v \cdot \dot{v}$ (with $\dot{u} = \frac{\mathrm{d}u}{\mathrm{d}t}$ e $\dot{v} = \frac{\mathrm{d}v}{\mathrm{d}t}$)

$$\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2 = \left|\frac{\mathrm{d}\overline{r}}{\mathrm{d}t}\right|^2 = \dot{\overline{r}} \cdot \dot{\overline{r}} = (\overline{x}_u\dot{u} + \overline{x}_v\dot{v}) \cdot (\overline{x}_u\dot{u} + \overline{x}_v\dot{v}) = \dot{u}^2(\overline{x}_u \cdot \overline{x}_u) + 2\dot{u}\dot{v}(\overline{x}_u \cdot \overline{x}_v) + \dot{v}^2(\overline{x}_v \cdot \overline{x}_v)$$

Now let $E \equiv \overline{x}_u \cdot \overline{x}_u$, $F \equiv \overline{x}_u \cdot \overline{x}_v$, $G \equiv \overline{x}_v \cdot \overline{x}_v$; $(E = E(u, v,)...)$; we obtain:

$$\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2 = E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2$$

$$L = \int_a^b \left[E\left(\frac{\mathrm{d}u}{\mathrm{d}t}\right)^2 + 2F\frac{\mathrm{d}u}{\mathrm{d}t}\frac{\mathrm{d}v}{\mathrm{d}t} + G\left(\frac{\mathrm{d}v}{\mathrm{d}t}\right)^2\right]^{\frac{1}{2}} \mathrm{d}t$$

which is shortened writing (it's understood that what matters is the curve, not the parameters used to describe it)

$$L = \int_{\overline{r}} \mathrm{d}s = \int_{\overline{r}} \left[E\mathrm{d}u^2 + 2F\mathrm{d}u\mathrm{d}v + G\mathrm{d}v^2\right]^{\frac{1}{2}}$$

or, in differential form,

$$\mathrm{d}s^2 = E\mathrm{d}u^2 + 2F\mathrm{d}u\mathrm{d}v + G\mathrm{d}v^2$$

This is the so called **first fundamental form** or **metric form** of a surface.

As we shall see, the metric form determines completely the intrinsic geometry of the surface, including its curvature. Speaking of intrinsic geometry we refer to the geometric properties that can be assessed through measures (e.g. distances, but not only) conducted by remaining within the surface, without "going out" from it (that is, without looking at the two-dimensional surface from an Euclidean three-dimensional space). The possibility to define intrinsic properties is essential because, if going from 2 to 3 dimensions, we want to understand the geometry of space that characterizes our universe, we cannot observe it from "outside"!

Notice: Due to the bijective correspondence between the domain $D \in \mathbb{R}^2$ and the surface element M, the curves $u = \text{const}$ e $v = \text{const}$ form a grid on the surface, and one can think at E, F, e G as functions defined on the surface (and then intrinsic). We may think that the inhabitants of the two-dimensional surface make various measurements of distances between points of the surface to discover the form of the three functions E, F e G, expressed as a function of the curvilinear coordinate grid, perhaps by making assumptions about their possible shape and looking for the best solution.

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***Example**: the plane*

$\overline{x}(u, v) = (u, v, 0)$ is the plane $z = 0$ in $\mathbf{E}^3$ with $x = u$ and $y = v$ as cartesian coordinates.

$\overline{x}_u = (1, 0, 0)$, $\overline{x}_v = (0, 1, 0)$, $E = \overline{x}_u \cdot \overline{x}_u = 1$, $F = \overline{x}_v \cdot \overline{x}_u = 0$, $G = \overline{x}_v \cdot \overline{x}_v = 1$;

$\mathrm{d}s^2 = \mathrm{d}u^2 + \mathrm{d}v^2 = \mathrm{d}x^2 + \mathrm{d}y^2$ (Pythagoras theorem)

$$L = \int_a^b \left[\left(\frac{\mathrm{d}u}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}v}{\mathrm{d}t}\right)^2\right]^{\frac{1}{2}} \mathrm{d}t$$

and, if the curve can be represented in the form $y = f(x)$, defining $x = t$ and $y = f(x)$ we get

$$L = \int_a^b \left[1 + f'(x)^2\right]^{\frac{1}{2}} \mathrm{d}x$$

. . . .

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***Exemple**: the sphere in geographical coordinates:*

$$\begin{aligned}
\overline{x}(u, v) &= (R\ \mathrm{cos}u\ \mathrm{cos}v, R\ \mathrm{sin}u\ \mathrm{cos}v, R\ \mathrm{sin}v) \\
\overline{x}_u &= (-R\ \mathrm{sin}u\ \mathrm{cos}v, R\ \mathrm{cos}u\ \mathrm{cos}v, 0) \\
\overline{x}_v &= (-R\ \mathrm{cos}u\ \mathrm{sin}v, -R\ \mathrm{sin}u\ \mathrm{sin}v, R\ \mathrm{cos}v)
\end{aligned}$$

$$\begin{aligned}
E = \overline{x}_u \cdot \overline{x}_u &= R^2\ \cos^2 v\ \sin^2 u + R^2\ \cos^2 v\ \cos^2 u = R^2\ \cos^2 v \\
G = \overline{x}_v \cdot \overline{x}_v &= R^2\ \sin^2 v\ \cos^2 u + R^2\ \sin^2 v\ \sin^2 u + R^2\ \cos^2 v = R^2 \\
F = \overline{x}_u \cdot \overline{x}_v &= R^2\ \mathrm{cos}v\ \mathrm{cos}u\ \mathrm{sin}v\ \mathrm{sin}u - R^2\ \mathrm{cos}u\ \mathrm{cos}v\ \mathrm{sin}u\ \mathrm{sin}v = 0
\end{aligned}$$

$$\mathrm{d}s^2 = R^2\ \cos^2 v \mathrm{d}u^2 + R^2 \mathrm{d}v^2$$

If we remember that, for $a \le t \le b$, $L = \int \mathrm{d}s = \int_a^b \sqrt{(\frac{\mathrm{d}s}{\mathrm{d}t})^2}\mathrm{d}t$, we can write

$$L = \int_a^b \sqrt{R^2 \cos^2 v(\frac{\mathrm{d}u}{\mathrm{d}t})^2 + R^2(\frac{\mathrm{d}v}{\mathrm{d}t})^2}\mathrm{d}t = R\int_a^b \sqrt{\cos^2 v(\frac{\mathrm{d}u}{\mathrm{d}t})^2 + (\frac{\mathrm{d}v}{\mathrm{d}t})^2}\mathrm{d}t$$

and, given the paths $u = u(t)$ and $v = v(t)$, we can compute their lengths.

. . . .

If $\overline{v} = a\overline{x}_u + b\overline{x}_v$, $\overline{w} = c\overline{x}_u + d\overline{x}_v$, with $a, b, c, d \in \mathbb{R}$, are two vectors tangent to the surface M, then $\overline{v} \cdot \overline{w} = (a\overline{x}_u + b\overline{x}_v) \cdot (c\overline{x}_u + d\overline{x}_v) = acE + adF + bcF + bdG$ which can be written in the matrix form

$$(a, b) \begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} \qquad \text{where} \quad \begin{pmatrix} E & F \\ F & G \end{pmatrix} \quad \text{is the matrix of the first fundamental form}$$

So, if we know the first fundamental form, we are able to compute scalar (dot) products on M, then not only lenghts, but also angles.

We remind that being $\overline{x}_u \times \overline{x}_v$ normal to the plane tangent to the surface, the versor $\hat{N} = \frac{\overline{x}_u \times \overline{x}_v}{|\overline{x}_u \times \overline{x}_v|}$ is normal to the surface.

Lagrange identity (important): $|\overline{x}_u \times \overline{x}_v|^2 = (\overline{x}_u \cdot \overline{x}_u)(\overline{x}_v \cdot \overline{x}_v) - (\overline{x}_u \cdot \overline{x}_v)^2 = EG - F^2 = \det\begin{pmatrix} E & F \\ F & G \end{pmatrix}$

Proof: remember that

$$\begin{aligned}
|\overline{x}_u \times \overline{x}_v| &= |\overline{x}_u||\overline{x}_v|\ \mathrm{sin}\theta \\
\overline{x}_u \cdot \overline{x}_v &= |\overline{x}_u||\overline{x}_v|\ \mathrm{cos}\theta
\end{aligned}$$

so (if we remember that $\sin^2\theta = 1 - \cos^2\theta$) $|\overline{x}_u \times \overline{x}_v|^2 = |\overline{x}_u|^2|\overline{x}_v|^2\ \sin^2\theta = (\overline{x}_u \cdot \overline{x}_u)(\overline{x}_v \cdot \overline{x}_v) - (\overline{x}_u \cdot \overline{x}_v)^2$ Q.E.D.

By the requirement that the surface is smooth it follows that $EG - F^2 \neq 0$

At this point we make a change in the symbology used; as we shall see this will lead to a considerable simplification of formulas.

Let's call $g_{11} \equiv E$ $\quad g_{12} = g_{21} \equiv F$ $\quad g_{22} \equiv G$ $\quad \overline{x}_1 \equiv \overline{x}_u$ $\quad \overline{x}_2 \equiv \overline{x}_v$

and let's write $u^1 \equiv u$ $\quad u^2 \equiv v$ (where the superscripts 1 and 2 are upper indices and not exponents).

Then we will have $g_{ij} = \overline{x}_i \cdot \overline{x}_j \quad (i,j = 1,2)$ and the matrix of the metric form will be:

$$\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} E & F \\ F & G \end{pmatrix}$$

Remember that $g_{ij} = g_{ij}(u,v) = g_{ij}(u^1,u^2)$.

By defining $g \equiv \det(g_{ij}) = EG - F^2$, from Lagrange identuty $|\overline{x}_1 \times \overline{x}_2|^2 = g$.

In the new notation, the first fundamental form can then be written:

$$\mathrm{d}s^2 = g_{11}(\mathrm{d}u^1)^2 + 2g_{12}\mathrm{d}u^1\mathrm{d}u^2 + g_{22}(\mathrm{d}u^2)^2 = \sum_{i,j} g_{ij}\mathrm{d}u^i\mathrm{d}u^j$$

We used $2g_{12} = g_{12} + g_{21}$ since $g_{12} = g_{21}$; moreover, we will soon understand the reason for we write u^i instead of u_i.

A vector, tangent in P to M, $\overline{v} = a\overline{x}_1 + b\overline{x}_2$ can be written as $\overline{v} = v^1\overline{x}_1 + v^2\overline{x}_2 = \sum_i v^i\overline{x}_i$ (notice that i is a "dummy" variable, and any other letter can be used instead of it.)

If $\overline{v} = \sum_i v^i\overline{x}_i$ and $\overline{w} = \sum_j w^j\overline{x}_j$ are two vectors tangent to M at the same point P, then

$$\overline{v} \cdot \overline{w} = \sum_{i,j}(v^i\overline{x}_i) \cdot (w^j\overline{x}_j) = \sum_{i,j} v^i w^j \overline{x}_i \cdot \overline{x}_j = \sum_{i,j} g_{ij}v^i w^j$$

The vectors $\overline{v}$ and $\overline{w}$ are orthogonal if and only if $\sum_{i,j} g_{ij}v^i w^j = 0$.

We define as g^{ij} the elements of the inverse matrix of (g_{ij}), such that

$$\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}\begin{pmatrix} g^{11} & g^{12} \\ g^{21} & g^{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

which, in a more compact way, can be written

$$\sum_j g_{ij}g^{jk} = \delta_i^k$$

where δ_i^k (Kronecker δ) is defined in the following way

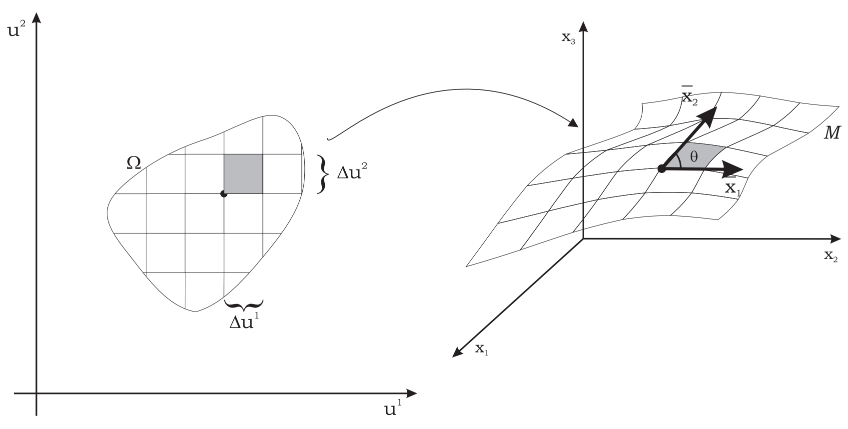
$$\delta_i^k = \begin{cases} 1 & i = k \\ 0 & i \neq k \end{cases}$$

Remembering that the elements of the inverse of a matrix are given by the algebraic complements divided by the determinant of the original matrix, we get:

$$g^{11} = \frac{g_{22}}{g} \qquad g^{12} = g^{21} = -\frac{g_{21}}{g} \qquad g^{22} = \frac{g_{11}}{g}$$

We will now see that the first fundamental form not only allows you to measure distances and angles, but also areas.

Let be $\overline{x} : D \to E^3$ a surface in E^3 and let be $\Omega \in D$ a region of the domain where $\overline{x}$ is bijective. To find the area of $\overline{x}(\Omega)$, we subdivide Ω into rectangular elements by means of lines parallel to the axes u^1 e u^2.



To a small area belonging to Ω, having as sides Δu^1 and Δu^2 corresponds approximately a piece of surface parallelogram-shaped, with sides parallel to the vectors $\overline{x}_1$ e $\overline{x}_2$. These sides have lengths given by $\Delta l_1 \simeq |\overline{x}_1|\Delta u^1$ and $\Delta l_2 \simeq |\overline{x}_2|\Delta u^2$ (Remember that $\overline{x}_1 = \frac{\partial \overline{x}}{\partial u^1}$, and then $\Delta \overline{x}_1 = \frac{\partial \overline{x}}{\partial u^1}\Delta u^1$)

The measure of the small area is given by:

$$\Delta A = |\overline{x}_1|\Delta u^1 \cdot |\overline{x}_2|\Delta u^2 \sin\theta = |\overline{x}_1 \times \overline{x}_2|\Delta u^1 \Delta u^2 = \sqrt{g}\Delta u^1 \Delta u^2$$

where θ is the angle between $\overline{x}_1$ and $\overline{x}_2$, and $g = det(g_{ij})$ as seen above.

Adding all these area elements covering Ω and going to the limit $\Delta u^i \to 0$ we obtain the area of $\overline{x}(\Omega)$:

$$A = \iint_\Omega \sqrt{g}\ \mathrm{d}u^1 \mathrm{d}u^2$$

We observe that, working in two dimensions, the measure of a set is precisely its area; if we work in three dimensions, the measure will be a volume, and an n-dimensional volume in n dimensions. In all cases, even if we don't prove it here, the measure is obtained by integrating $\sqrt{g}$, where g is the determinant of the n-dimensional metric. This applies in the so-called Riemannian spaces (*manifolds*), in which the $ds^2 > 0$. In the pseudo-Riemannian spaces, where ds^2 can be positive, negative or equal to zero (such as Minkowski space-time of Special Relativity), some elements of the metric tensor can be negative; since in this case it can be that (and so is in the space-time) $g < 0$, we will use in general the absolute value of g, and we will write $\sqrt{|g|}$.

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Exemple*: sphere in geographical coordinates:*

$$\mathrm{d}s^2 = R^2 \cos^2 v\ \mathrm{d}u^2 + R^2\ \mathrm{d}v^2 \qquad \begin{array}{c} -\pi \le u \le \pi \\ -\pi/2 \le v \le \pi/2 \end{array}$$

$$g_{ij} = \begin{pmatrix} R^2\cos^2 v & 0 \\ 0 & R^2 \end{pmatrix} \qquad g = R^4 \cos^2 v \quad \to \quad \sqrt{g} = R^2 \cos v$$

$$A = \int_{-\pi/2}^{\pi/2}\left(\int_{-\pi}^{\pi} R^2 \cos v\ \mathrm{d}u\right)\mathrm{d}v = 2\pi R^2 \int_{-\pi/2}^{\pi/2} \cos v\ \mathrm{d}v = 4\pi R^2$$

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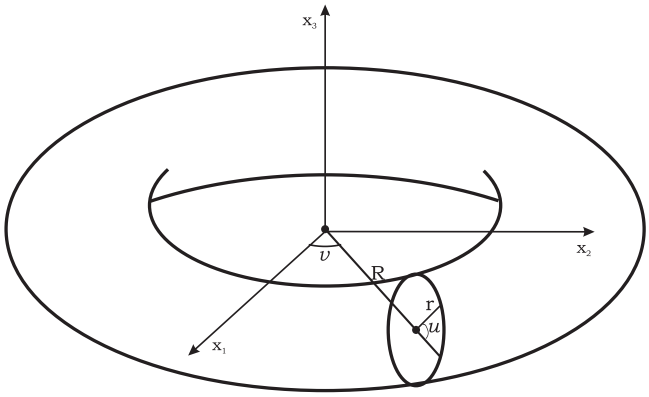
E*xample: area of the torus*:

$$\overline{x}(u,v) = \left[\,(R + r\cos u)\cos v, (R + r\cos u)\sin v,\ r\ \sin u\right] \qquad \sqrt{g} = r\,(R + r\cos u)$$

$$0 \le v \le 2\pi \quad 0 \le u \le 2\pi \quad 0 < r < R$$

$$\begin{aligned} S &= \int_0^{2\pi}\left[\int_0^{2\pi} r\,(R + r\cos u)\,\mathrm{d}u\right]\mathrm{d}v = 2\pi r\left[\int_0^{2\pi} R\,\mathrm{d}u + \int_0^{2\pi} r\cos u\,\mathrm{d}u\right] = \\ &= 2\pi r\left[2\pi R + r\int_0^{2\pi}\cos u\,\mathrm{d}u\right] = 4\pi^2 Rr \end{aligned}$$



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2 Tensors

2.1 Introduction

Why did we write things like g_{ij} and $\mathrm{d}u^i$ and $\mathrm{d}u^j$? Because we are dealing with tensor quantities, quantities whose properties are related to the way they transform when changing the reference system.

If I switch from the coordinate (generally curvilinear) system u^i $(i = 1, 2, ...) \to u'^j (j = 1, 2, ...)$ we will get (by means of ... we begin to see how things can be generalized to more than two dimensions)

$$\mathrm{d}u'^j = \sum_i \frac{\partial u'^j}{\partial u^i} \mathrm{d}u^i \qquad (i, j = 1, 2, ...)$$

$$\mathrm{d}u'^j = A'^j{}_k$$

Every quantity V^j which transforms according to the rule

$$V'^j = \sum_i \frac{\partial u'^j}{\partial u^i} V^i$$

is a *contravariant* tensor (or, to be more precise, its components transforms as a *contravariant* tensor); so, also du^i, or u^i, are *contravariant* tensors. A vector is tensor or rank one. A scalar quantity, the value of which does not change at a given point if we change the coordinate system, is a tensor of rank zero.

We consider now the gradient of a scalar field$\Phi(u^i) = \Phi(u'^j)$. We have:

$$\frac{\partial \Phi}{\partial u'^j} = \sum_i \frac{\partial \Phi}{\partial u^i} \cdot \frac{\partial u^i}{\partial u'^j} = \sum_i \frac{\partial u^i}{\partial u'^j} \cdot \frac{\partial \Phi}{\partial u^i}$$

We see that the gradient of Φ changes differently from du^i! We say that $\frac{\partial \Phi}{\partial u^i}$ is a *covariant* vector, and often we simply write $\partial_i \Phi$ instead of $\frac{\partial \Phi}{\partial u^i}$, with a lower index. When the same index appears both as an upper and a lower index, the sum on that index is implied (Einstein convenction), and we simply write:

$$V'^j = \frac{\partial u'^j}{\partial u^i} V^i \quad \text{and} \quad \frac{\partial \Phi}{\partial u'^j} = \frac{\partial u^i}{\partial u'^j} \frac{\partial \Phi}{\partial u^i}$$

The quantity $\mathrm{d}s^2 = g_{ij} \mathrm{d}u^i \mathrm{d}u^j$ (implying the summation on i and j) is the length, squared, of a segment, and is therefore independent on the reference frame used (it is a scalar). In two different referece frames we will then have

$$\mathrm{d}s^2 = g'_{kl} \mathrm{d}u'^k \mathrm{d}u'^l = g_{ij} \mathrm{d}u^i \mathrm{d}u^j = g_{ij} \frac{\partial u^i}{\partial u'^k} \frac{\partial u^j}{\partial u'^l} \mathrm{d}u'^k \mathrm{d}u'^l$$

$$\text{since} \quad \mathrm{d}u^i = \frac{\partial u^i}{\partial u'^k} \mathrm{d}u'^k \quad \text{and} \quad \mathrm{d}u^j = \frac{\partial u^j}{\partial u'^l} \mathrm{d}u'^l$$

$$\text{we have} \qquad g'_{kl} = \frac{\partial u^i}{\partial u'^k} \frac{\partial u^j}{\partial u'^l} g_{ij}$$

i.e., the metric tensor is a covariant tensor of rank two.

On the contrary, the g^{ij} tensor is contravariant tensor (of rank two).

We have seen that $g_{ij} g^{jk} = \delta_i^k$ (implying also here the summation on the repeated index j). δ_i^k is a *mixed tensor* of rank two, because

$$\delta_i^k \cdot \frac{\partial u'^l}{\partial u^k} \cdot \frac{\partial u^i}{\partial u'^m} = \frac{\partial u'^l}{\partial u^k} \cdot \frac{\partial u^k}{\partial u'^m} = \frac{\partial u'^l}{\partial u'^m} = \delta_m^l$$

(aside from the scalars and zero, δ_i^k is the only tensor that retains the same components in all coordinate systems).

We have also seen that the inner product $\overline{v} \cdot \overline{w}$ can be expressed as $\overline{v} \cdot \overline{w} = g_{ij} v^i w^j$.

If we multiply two tensors we also get a tensor:$A_{ij} \cdot C^k = D^k_{ij}$.

If we *contract* a tensor we still have a tensor, but with its rank reduced by two: $T^j_{kmj} = B_{km}$. In fact, e.g.,

$$A'^k_i = \frac{\partial u'^k}{\partial u^j} \cdot \frac{\partial u^l}{\partial u'^i} \cdot A^j_l \Longrightarrow A'^k_k = \frac{\partial u'^k}{\partial u^j} \cdot \frac{\partial u^l}{\partial u'^k} \cdot A^j_l = \frac{\partial u^l}{\partial u^j} \cdot A^j_l = \delta^l_j \cdot A^j_l = A^j_j = A \quad \text{(a scalar)}.$$

If D_i and D^j are the covariant and contravariant components of the same vector (tensor) and we consider a generic vector C^j such that

$$D_i = g_{ij} C^j \; / \cdot D^i \qquad \to \qquad D_i D^i = g_{ij} C^j D^i,$$

by performing this inner product we obtain, on the left, a scalar that depends on the vector$\overline{D}$, while the right side depends on both $\overline{C}$ and $\overline{D}$; since these two quantities are equal, necessarily $\overline{C} \equiv \overline{D}$, i.e. $D_i = g_{ij} D^j$. We can get this result also in another way. We have seen that a vector $\overline{v}$ can be written as $\overline{v} = v^i \overline{x}_i$, by using its contravariant components; we now define its covariat componentsv_k in the following way:

$$v_k \equiv \overline{v} \cdot \overline{x}_k = v^i \overline{x}_i \cdot \overline{x}_k = v^i g_{ik} = g_{ik} v^i = g_{ki} v^i$$

In a similar way we have $D^j = g^{ij} D_i$. **We see that the metric tensor can be used to transform contravariant components into covariant componenets (and vice versa).**

If g_{ij} lowers an upper index, we can use it also to lower an upper index of g^{jk}, obtaining

$$g_{ij} g^{jk} = g^k_i \equiv \delta^k_i$$

on the basis of what has been said above: the metric tensor in the mixed form (i.e. with an upper index and a lower one) is equal to the Kronecker delta.

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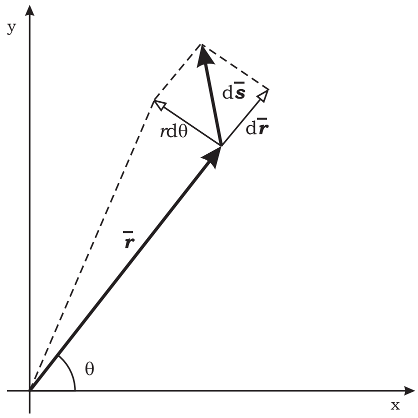
Example: *vectors in the plane in polar coordinates*



$$u^1 \equiv r \qquad \mathrm{d}s^2 = \mathrm{d}r^2 + r^2 \mathrm{d}\theta^2 \qquad g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix} \qquad g = r^2 \quad \to \quad \sqrt{g} = r$$

$$u^2 \equiv \theta \qquad g^{ij} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{r^2} \end{pmatrix}$$

At a point in the plane we have two vectors, whose components are $A_i = (5, 9)$ e $B^i = (3, 7)$. Their inner product is $A_i B^i = A_1 B^1 + A_2 B^2 = 5 \cdot 3 + 9 \cdot 7 = 78$

$$\begin{aligned}
A^i = g^{ij} A_j \qquad &\to \qquad A^1 = g^{11} A_1 + g^{12} A_2 = 1 \cdot 5 + 0 \cdot 9 = 5 \\
&\to \qquad A^2 = g^{21} A_1 + g^{22} A_2 = 0 \cdot 5 + 1/r^2 \cdot 9 = 9/r^2 \\
B_i = g_{ij} B^j \qquad &\to \qquad B_1 = g_{11} B^1 + g_{12} B^2 = 1 \cdot 3 + 0 \cdot 7 = 3 \\
&\to \qquad B_2 = g_{21} B^1 + g_{22} B^2 = 0 \cdot 3 + r^2 \cdot 7 = 7r^2
\end{aligned}$$

then $\qquad A^i B_i = A^1 B_1 + A^2 B_2 = 5 \cdot 3 + 9/r^2 \cdot 7r^2 = 78 = A_i B^i$

We see that $A^i B_i = g_{ij} A^i B^j = g^{ij} A_j B_i = A_i B^i$ is invariant.

Notice: The surface element is given by $\mathrm{d}S = \sqrt{g}\mathrm{d}u^1\mathrm{d}u^2 \rightarrow r\,\mathrm{d}r\mathrm{d}\theta$.

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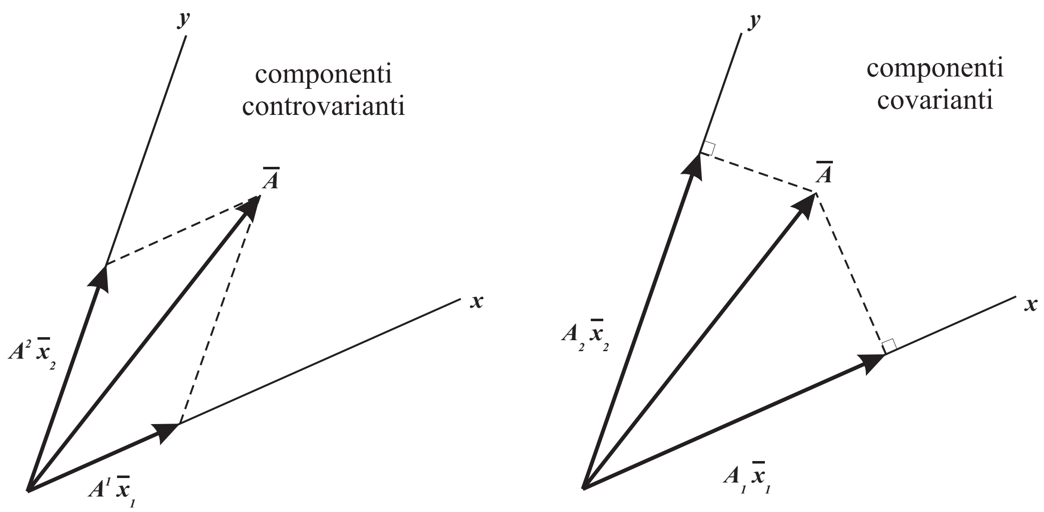
It is not easy to represent the covariant and contravariant components of a vector in general, but one can give a graphic description in some particular case, for example in the case of rectilinear coordinates. Consider, in the plane, a rectilinear non-orthogonal coordinate system Oxy. Let $\bar{x}_i$ be the basis vectors. If we write the vector $\bar{A}$ as $\overline{A} = A^i \bar{x}_i$, I realize that A^i are the ususal components of a vector, such that the component vectors, having magnitude A^i and direction and versus given by $\bar{x}_i$, add according to the parallelogram rule to give the vector $\bar{A}$. The *contravariant components* correspond to the *parallel projections* on the axes.



Conversely, if I write, as done above, the *covariant components* as $A_i = \bar{A} \cdot \bar{x}_i$, I realize that they are the projections of di $\bar{A}$ along the $\bar{x}_i$ direction; they correspond to the *normal projections* on the axes.

It follows that, if the reference frame is *rectilinear and orthogonal* parallel and perpendicular projections are the same thing, and covariant and contravariant components are equal. It's no more necessary to distinguish between upper and lower indices.

Notice that a vector (or more generally a tensor), *per se*, is neither covariant nor contravariant, but *its components* are covariant or contravariant.

But every quantity with indices is not necessarily a tensor. For instance, as we shall see, the affine connections Γ^i_{jk} do not represent a tensor, since they do not transform like a tensor.

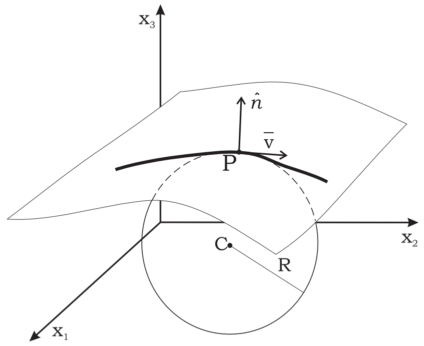
We can draw an *important conclusion*: **each equation is invariant under a general coordinate transformation if it is expressed as the equality between two tensors with the same upper and lower indexes** :

$$A^{\alpha}_{\beta\gamma} = B^{\alpha}_{\beta\gamma} \rightarrow A'^{\alpha}_{\beta\gamma} = B'^{\alpha}_{\beta\gamma} \quad \text{if} \quad A^{\alpha}_{\beta\gamma} \quad \text{and} \quad B^{\alpha}_{\beta\gamma} \quad \text{are tensors.}$$

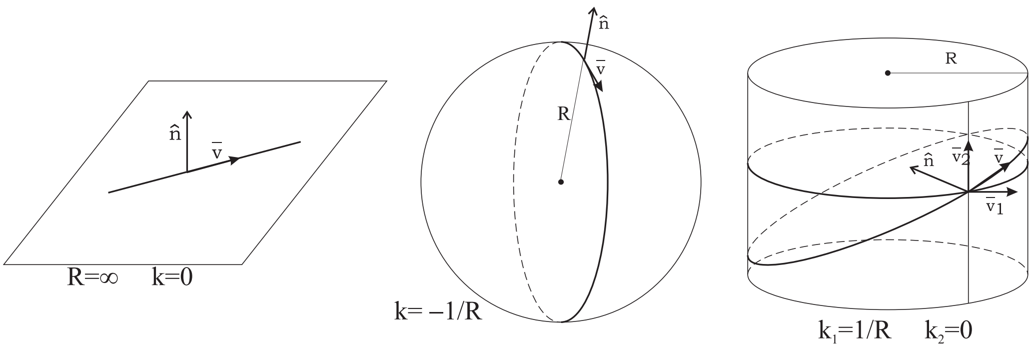
Since also the zero is a tensor of whatever rank (just think that it transforms always into a zero), a relation like $A^{\alpha}_{\beta\gamma} = 0$ will be satisfied in any reference frame.

On the contrary, an equality between quantities that are not tensors with the same upper and lower indices (e.g. $T_{\mu\nu} = 5$; $V^i = B_i$) can be true in some reference frame, but not in all of them.

2.2 Curvature of a surface



Let's see how one can extend the notion of curvature to a surface. Let us then consider a point P on a surface, and let $\hat{n}$ be the unit vector normal to the surface in P. If $\overline{v}$ is a vector tangent to the surface at the point P,$\overline{v}$ and $\hat{n}$ define a plane that cuts the surface along a curve which will have, in P, a certain radius of curvature. The curvature in P is given by $k = \pm\frac{1}{R}$, where the sign is taken positive or negative depending on whether the center of curvature C is, with respect to P, on the same side of $\hat{n}$ or on the opposite side (you can also take the opposite choice but, as we will see, things do not change). Let's see, as examples of surfaces, the plane, the sphere and the right cylinder.

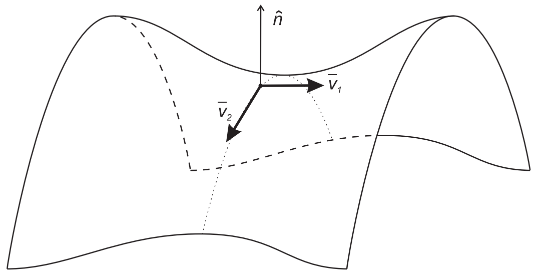


In the case of the cylinder, it can be seen that there are two directions perpendicular to each other and corresponding to the vectors $\overline{v}_1$ and $\overline{v}_2$ which, in turn, correspond to the maximum and the minimum value (k_1 and k_2) of k, the so-called *principal curvatures.* This applies in general, for all smooth surfaces.

The **Gauss curvature** K is defined as the product $k_1 \cdot k_2$. From this we see that K does not depend on the convention adopted for signs of k.

For the plane $K = 0$, for the sphere $K = 1/R^2$, for the cylinder $K = 0$, as for the plane! Although this may appear strange at first sight, actually it reflects the fact that by cutting a right cylinder along a segment parallel to its axis, it can lie on a plane without deforming and without changing lengths and angles of figures drawn on it . The geometry of a cylinder cannot be locally distinguished from that of a plane when we measure angles, lengths, areas, i.e. all those properties that can be measured by moving only along its surface. However, an overall view allows to distinguish a plane from a cylinder: an insect that moves along a circular cross-section (perpendicular to the axis of the cylinder) without turning neither to the right nor to the left, will eventually retrace his steps, but this does not happen on the plane. Also a right circular cone has $K = 0$.

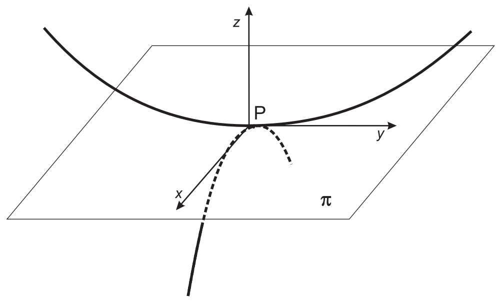
An example of a surface with $K < 0$is given by a hyperbolic paraboloid (a surface shaped like a saddle) $z = x^2 - y^2$: the two centers of curvature are located on opposite sides with respect to P and then we have $K < 0$.



In general a surface will have $K > 0$ if, with respect to the tangent plane in P, it is "all on one side" (at least locally), while it will have $K < 0$ if the surface is on both sides with respect to the tangent plane in P.

For a torus we have the outer zone with $K > 0$, the inner one with $K < 0$, separated by a circumference above and below with zero curvature.

Let us try to understand why, in the neighborhood of a point P, two principal curvatures, in two directions perpendicular to each other, are defined. In the neighborhood of a point P, with respect to the tangent plane π and to its normal, we can write (expanding in Taylor series)



$$z = f(x, y) \qquad P \equiv \text{origin}$$

$$\begin{aligned} z &= \left.\frac{\partial f}{\partial x}\right|_P x + \left.\frac{\partial f}{\partial y}\right|_P y + \frac{1}{2}\left.\frac{\partial^2 f}{\partial x^2}\right|_P x^2 + \frac{1}{2}\left.\frac{\partial^2 f}{\partial y^2}\right|_P y^2 + \left.\frac{\partial^2 f}{\partial x \partial y}\right|_P x \cdot y + \mathcal{O}(3) \\ &= \frac{1}{2}\left[\left.\frac{\partial^2 f}{\partial x^2}\right|_P x^2 + \left.\frac{\partial^2 f}{\partial y^2}\right|_P y^2 + 2\left.\frac{\partial^2 f}{\partial x \partial y}\right|_P x \cdot y\right] + \mathcal{O}(3) \end{aligned}$$

which (neglecting terms of 3^{rd} order or higher) can be rewritten as

$$z = \frac{1}{2}\big[ax^2 + 2bxy + cy^2\big].$$

This, for a given, fixed value of z, represents a conic.

[1]In general, we will have:

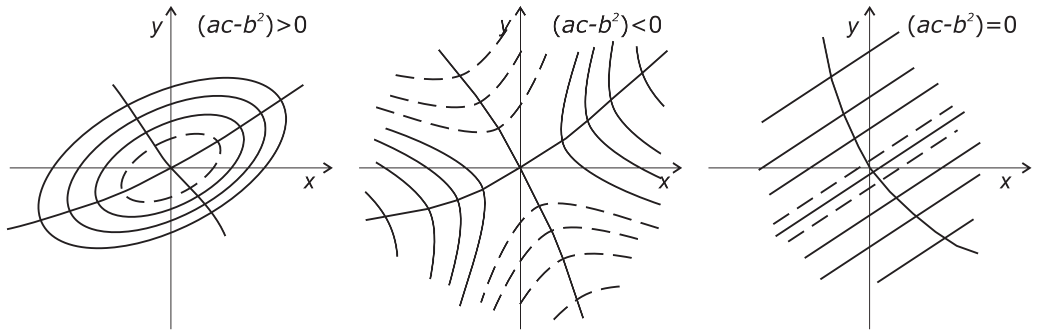
$$\begin{array}{lcl} (ac - b^2) > 0 & \Rightarrow & \text{ellipse} \\ (ac - b^2) = 0 & \Rightarrow & \text{parabola} \\ (ac - b^2) < 0 & \Rightarrow & \text{hyperbola} \end{array}$$

For an ellipse, the contour levels $z = cost$ describe a set of ellipses all centered in the origin. The maximum/minimum radii of curvature of the surface in P are in the direction of the maximum/minimum axes, and are then perpendicular to each other.

In the hyperbolic case the surface is saddle-shaped, and also in this case thera are two orthogonal directions corresponding to maximum and minimum curvatures (with opposite sign).

In the parabolic case the curvature is zero in one direction.

[1]If we write this relation, changing symbols, as $z \equiv L\frac{x^2}{2} + M\,xy + N\frac{y^2}{2}$, the quadratic form $L\,dx^2 + 2M\,dx\,dy + N\,dy^2$ is the so-called *second fundamental form* of a surface.

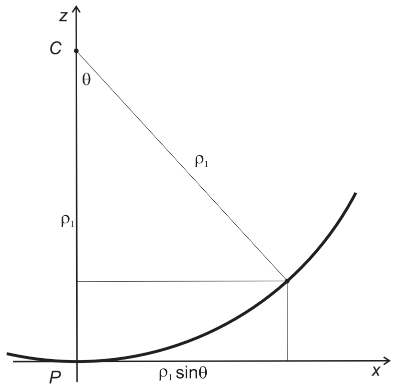


If we make a rotation of axes in the tangent plane and let x and y axes coincide with the principal axes of conics, then we can write the surface $z = f(x, y)$ in the simpler form

$$z = a'x^2 + b'y^2 + \mathcal{O}(3)$$

where a' and b' are functions of the extreme curvature radii. Let's see how.

If we move in the $y = 0$ plane, ed we approximate the section of the surface by an arc of circle in a neighborhood of P, we get



$$\begin{cases} z = \rho_1 - \rho_1 \cos\theta = \rho_1(1 - \cos\theta) \simeq \rho_1[1 - (1 - \theta^2/2 + ...)] \\ x = \rho_1 \sin\theta \simeq \rho_1\theta \end{cases}$$

and then

$$z \sim \rho_1 \frac{\theta^2}{2} \sim \frac{\rho_1}{2}\left(\frac{x}{\rho_1}\right)^2 \sim \frac{x^2}{2\rho_1}$$

We do the same in the zy plane. Therefore, in a neighborhood of P, chosen appropriately the reference system, we can write

$$z = f(x, y) = \frac{x^2}{2\rho_1} + \frac{y^2}{2\rho_2} = \frac{k_1 x^2}{2} + \frac{k_2 y^2}{2}$$

One could define the curvature of a surface in other ways, for example $K' = k_1 + k_2$. In this case, plane and cylinder would be different locally. But the main advantage of the Gauss curvature lies in the fact that, as we shall see, it may be determined by resorting only to measurements carried out on the surface, without the need to "see" the surface in 3 dimensions (as would happen instead for $K' = k_1 + k_2$).

The Gauss curvature is an *intrinsic property* of the surface, and can be determined by knowing the metric tensor $g_{ij} \quad (i, j = 1, 2)$. This is the result of the so-called **Theorema Egregium**, so named by the same Gauss.

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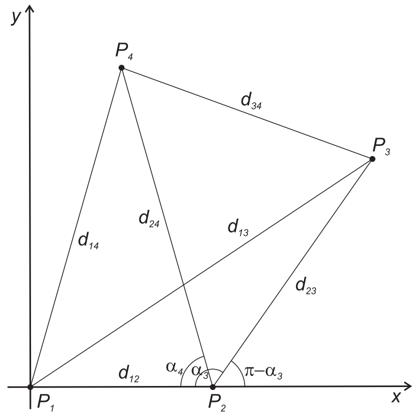
Exemple: *The mysterious planet* (from Weinberg, 1972)

To have an intuitive idea of how this is possible, consider this example: suppose we have measured on the surface of a celestial body on which we were transported blindfolded (so without seeing it from space!) the distances between four locations P_1, P_2, P_3, P_4 as shown in the figure. Given the values of the six segments, can I tell if the planet's surface is flat or not?



$d_{12} = 780km \qquad d_{13} = 1498km \qquad d_{14} = 1112km \qquad d_{23} = 735km \qquad d_{24} = 960km \qquad d_{34} = 813km$

By Carnot theorem: $d_{13}^2 = d_{12}^2 + d_{23}^2 - 2d_{12}d_{23}\cos\alpha_3$, so that

$$\cos\alpha_3 = \frac{d_{12}^2 + d_{23}^2 - d_{13}^2}{2d_{12}d_{23}}$$

In a similar way

$$\cos\alpha_4 = \frac{d_{12}^2 + d_{24}^2 - d_{14}^2}{2d_{12}d_{24}}$$

Coordinates of the points: $P_1 = (0,0)$; $P_2 = (d_{12}, 0)$;

$P_3 = (d_{12} + d_{23}\cos(\pi - \alpha_3), d_{23}\sin(\pi - \alpha_3)) = (d_{12} - d_{23}\cos\alpha_3, d_{23}\sin\alpha_3)$; $P_4 = (d_{12} - d_{24}\cos\alpha_4, d_{24}\sin\alpha_4)$.

$d_{34}^2 = [d_{12} - d_{23}\cos\alpha_3 - d_{12} + d_{24}\cos\alpha_4]^2 + [d_{23}\sin\alpha_3 - d_{24}\sin\alpha_4]^2 = d_{23}^2 + d_{24}^2 - 2d_{23}d_{24}\cos(\alpha_3 - \alpha_4)$

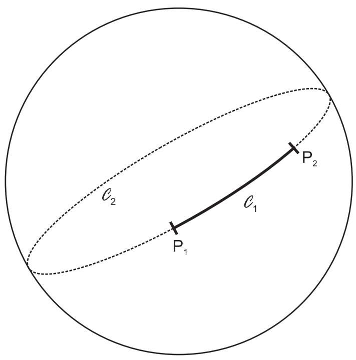
So, if the surface was flat, we would get $d_{34} = 1147.6$, but this is different from the measured value (813! So I can say thet I'm not on a flat planet (if I assume that the surface is a sphere, I could even derive the radius of the planet).

. . . .

2.3 Geodesics

Let $\overline{r}(s) = \left(u^i(s)\right)$, con $a \leq s \leq b$, be a curve on a surface (s being the curvilinear abscissa s) between two points P_1 and P_2 ($P_1 = \overline{r}(a)$; $\quad P_2 = \overline{r}(b)$). We say that this curve is a *geodesic* between P_1 and P_2 if its length is stationary for small variations of the curve which cancel the extremes. The curve that connects, on the surface, P_1 e P_2 along the shortest path is a geodesic, but the opposite is not always true.

For example, on a sphere both $\mathcal{C}_1$ and $\mathcal{C}_2$ (both arcs of a great circle) are geodesics between P_1 e P_2, but the shortest path corresponds to $\mathcal{C}_1$.



From the relation $ds^2 = g_{jk}du^j du^k$, if we express the coordinates u^i in parametric form by means of the parameter t (not necessarily the time) we get:

$$ds^2 = \left(g_{jk}\frac{du^j}{dt}\frac{du^k}{dt}\right)dt^2$$

By defining $L(u^i, \dot{u}^i, t) = (g_{jk}\dot{u}^j\dot{u}^k)^{1/2}$ (with $g_{jk} = g_{jk}(u^i)$ and $\dot{u}^i \equiv \frac{du^i}{dt}$) the length of a curve between P_1 and P_2 is:

$$S = \int_{P_1}^{P_2} L dt = \int_{P_1}^{P_2} ds$$

To find the condition for S to be stationary we use *Euler-Lagrange equations* (see variational calculus):

$$\frac{\partial L}{\partial u^i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{u}^i}\right) = 0$$

$$L = \sqrt{g_{jk}\dot{u}^j\dot{u}^k} \equiv \sqrt{F}$$

$$\frac{\partial L}{\partial u^i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{u}^i}\right) = \frac{1}{2\sqrt{F}}\frac{\partial g_{jk}}{\partial u^i}\dot{u}^j\dot{u}^k - \frac{d}{dt}\left[\frac{1}{2\sqrt{F}}\left(g_{ik}\dot{u}^k + g_{ji}\dot{u}^j\right)\right] = 0$$

but $g_{ik}\dot{u}^k + g_{ji}\dot{u}^j = 2g_{ji}\dot{u}^j$for the symmetry of $g_{ij}$and for the fact that k e j are dummy (summed) indices and can be exchanged; then we have:

$$\frac{1}{2\sqrt{F}}\frac{\partial g_{jk}}{\partial u^i}\dot{u}^j\dot{u}^k - \left\{-\frac{1}{2F^{3/2}}\frac{dF}{dt}g_{ji}\dot{u}^j + \frac{1}{\sqrt{F}}\left(\frac{\partial g_{ji}}{\partial u^l}\dot{u}^l\dot{u}^j + g_{ji}\ddot{u}^j\right)\right\} = 0$$

If we assume that$t$ is proportional (or even equal) to the curvilinear abscissa s, then F is stationary and $\frac{dF}{dt} = 0$. This happens because:

$$ds = L dt \quad \rightarrow \quad ds^2 = L^2(dt)^2 = F dt^2 \quad \rightarrow \quad F = \left(\frac{ds}{dt}\right)^2 \quad \rightarrow \quad \frac{dF}{dt} = 2\left(\frac{ds}{dt}\right)\frac{d^2s}{dt^2} = 0$$

if $s = \alpha t + \beta$, with α and β real numbers (we will simply asuume $s = t$). Going on we have

$$g_{ji}\ddot{u}^j + \frac{\partial g_{ji}}{\partial u^l}\dot{u}^l\dot{u}^j - \frac{1}{2}\frac{\partial g_{jk}}{\partial u^i}\dot{u}^j\dot{u}^k = 0$$

and if we set, for the symmetry of the summed indices l and j, we get

$$\frac{\partial g_{ji}}{\partial u^l}\dot{u}^l\dot{u}^j = \frac{1}{2}\left[\frac{\partial g_{ji}}{\partial u^l} + \frac{\partial g_{li}}{\partial u^j}\right]\dot{u}^l\dot{u}^j$$

If we replace, in this relation, the index l with the index k we obtain

$$g_{ji}\ddot{u}^j + \frac{1}{2}\left[\frac{\partial g_{ji}}{\partial u^k} + \frac{\partial g_{ki}}{\partial u^j} - \frac{\partial g_{jk}}{\partial u^i}\right]\dot{u}^j\dot{u}^k = 0 \qquad \text{and multiplying by } g^{il}$$

$$\delta^l_j\ddot{u}^j + \frac{1}{2}g^{il}\left[\frac{\partial g_{ji}}{\partial u^k} + \frac{\partial g_{ki}}{\partial u^j} - \frac{\partial g_{jk}}{\partial u^i}\right]\dot{u}^j\dot{u}^k = 0 \qquad \text{which can be written}$$

(recalling the properties of δ^l_j)

$$\frac{\mathrm{d}^2u^l}{\mathrm{d}s^2}+\Gamma^l_{jk}\frac{\mathrm{d}u^j}{\mathrm{d}s}\frac{\mathrm{d}u^k}{\mathrm{d}s}=0$$

This expresses the condition of stationarity, i.e. it is the differential equation that defines a geodesic. The symbol with three indices Γ^l_{jk} is the so-called *affine connection* or *Christoffel symbol of* 2^{nd} *type*, defined as:

$$\Gamma^i_{jk}=\frac{1}{2}g^{il}\left(\frac{\partial g_{lj}}{\partial u^k}+\frac{\partial g_{lk}}{\partial u^j}-\frac{\partial g_{jk}}{\partial u^l}\right)$$

This quantity depends on g_{ij} and on its first derivatives. Moreover, notice that $\Gamma^l_{jk}=\Gamma^l_{kj}$. Often, in order to simplify even more the notation, we use to write:

$$\frac{\partial g_{ij}}{\partial u^k}\equiv\partial_k g_{ij}\equiv g_{ij},_k$$

You can verify that Γ^i_{jk} is not a tensor, as

$$\Gamma'^l_{mn}\neq\frac{\partial u'^l}{\partial u^i}\frac{\partial u^j}{\partial u'^m}\frac{\partial u^k}{\partial u'^n}\Gamma^i_{jk}$$

In the geodesic equation the term on the left hand side is a tensor of rank 1 (a contravariant vector), although Γ^l_{jk} is not a tensor. So, if it is null in a reference system, it will also be null in a generic reference system.

. . . .

Example*: the plane in cartesian coordinates*

$\mathrm{d}s^2=\mathrm{d}u^2+\mathrm{d}v^2$; since g_{ij} is constant, the Γ are all zero, and geodesics are solutions of

$$\frac{\mathrm{d}^2u}{\mathrm{d}s^2}=0 \quad\text{and}\quad \frac{\mathrm{d}^2v}{\mathrm{d}s^2}=0 \quad\rightarrow\quad \begin{array}{l}u=as+b\\ v=cs+d\end{array}$$

(with a, b, c, d real numbers): those are the parametric equations of a straight line.

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In a similar, but more complicate, way one can show that arcs of great circle are geodesic lines on the sphere (we shall prove it later on).

. . . .

Example*: geodesics in the plane in polar coordinates*

$$ds^2=\mathrm{d}r^2+r^2\mathrm{d}\theta^2$$

$$g_{ij}=\begin{pmatrix}1&0\\0&r^2\end{pmatrix}\qquad g=r^2\qquad g^{ij}=\begin{pmatrix}1&0\\0&1/r^2\end{pmatrix}\qquad\begin{array}{l}u^1=r\\u^2=\theta\end{array}$$

$$\frac{\mathrm{d}^2u^i}{\mathrm{d}s^2}+\Gamma^i_{jk}\frac{\mathrm{d}u^j}{\mathrm{d}s}\frac{\mathrm{d}u^k}{\mathrm{d}s}=0$$

$$\begin{aligned}
\Gamma^i_{jk}&=\frac{1}{2}g^{ir}\left(\frac{\partial g_{jr}}{\partial u^k}+\frac{\partial g_{kr}}{\partial u^j}-\frac{\partial g_{jk}}{\partial u^r}\right) &&\text{remember the symmetry on } j \text{ and } k\\
\Gamma^1_{jk}&=\frac{1}{2}g^{11}\left(\frac{\partial g_{j1}}{\partial u^k}+\frac{\partial g_{k1}}{\partial u^j}-\frac{\partial g_{jk}}{\partial u^1}\right) &&\text{since } g^{12}=0\\
\Gamma^1_{22}&=\frac{1}{2}g^{11}\left(-\frac{\partial g_{22}}{\partial u^1}\right)=-\frac{1}{2}g^{11}\frac{\partial g_{22}}{\partial r}=-\frac{1}{2}\cdot1\cdot2r=-r\\
\Gamma^2_{jk}&=\frac{1}{2}g^{22}\left(\frac{\partial g_{j2}}{\partial u^k}+\frac{\partial g_{k2}}{\partial u^j}-\frac{\partial g_{jk}}{\partial u^2}\right)\\
\Gamma^2_{12}&=\frac{1}{2}g^{22}\left(\frac{\partial g_{22}}{\partial u^1}\right)=\frac{1}{2}\cdot\frac{1}{r^2}\cdot2r=\frac{1}{r}=\Gamma^2_{21}\\
\Gamma^1_{11}&=\Gamma^1_{12}=\Gamma^2_{11}=\Gamma^2_{22}=0
\end{aligned}$$

$$\frac{\mathrm{d}^2 r}{\mathrm{d}s^2} + (-r)\cdot\left(\frac{\mathrm{d}\theta}{\mathrm{d}s}\right)^2 = 0 \qquad \text{(I)}$$

$$\frac{\mathrm{d}^2 \theta}{\mathrm{d}s^2} + \frac{2}{r}\left(\frac{\mathrm{d}r}{\mathrm{d}s}\right)\left(\frac{\mathrm{d}\theta}{\mathrm{d}s}\right) = 0 \qquad \text{(II)}$$

(if $\mathrm{d}\theta/\mathrm{d}s = 0$ we get the staight line passing through the origin); if we put $\mathrm{d}\theta/\mathrm{d}s \equiv \theta'$ and divide (II) by θ' we get:

$$\frac{1}{\theta'}\frac{\mathrm{d}\theta'}{\mathrm{d}s} + \frac{2}{r}\frac{\mathrm{d}r}{\mathrm{d}s} = 0 \qquad \to \qquad \ln\theta' + \ln r^2 = \ln(\theta' r^2) = \text{const}$$

and then

$$r^2\frac{\mathrm{d}\theta}{\mathrm{d}s} = h = cost$$

Instead of integrating (I), we use another method. From $\mathrm{d}s^2 = \mathrm{d}r^2 + r^2\mathrm{d}\theta^2$, dividing by $\mathrm{d}s^2$, we get

$$1 = \left(\frac{\mathrm{d}r}{\mathrm{d}s}\right)^2 + r^2\left(\frac{\mathrm{d}\theta}{\mathrm{d}s}\right)^2 = \left(\frac{\mathrm{d}r}{\mathrm{d}s}\right)^2 + \frac{h^2}{r^2}$$

You can verify that this relation is an integral of (I). From this we get

$$\frac{\mathrm{d}r}{\mathrm{d}s} = \pm\sqrt{1-\frac{h^2}{r^2}} = \pm\frac{\sqrt{r^2-h^2}}{r} \qquad \text{together with} \qquad \frac{\mathrm{d}\theta}{\mathrm{d}s} = \frac{h}{r^2}$$

Dividing the second equation by the first one, to eliminate s, we obtain

$$\frac{\mathrm{d}\theta}{\mathrm{d}r} = \pm\frac{h}{r\sqrt{r^2-h^2}} = \pm\frac{\mathrm{d}}{\mathrm{d}r}\left[\arccos\left(\frac{h}{r}\right)\right]$$

that is

$$\theta = \pm\arccos\left(\frac{h}{r}\right) + \theta_0 \qquad \to \qquad \frac{h}{r} = \cos(\theta-\theta_0) \qquad \to \qquad r\cos(\theta-\theta_0) = h$$

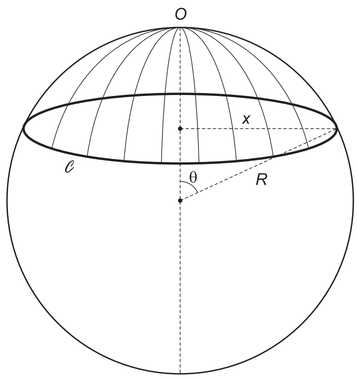
which is precisely the equation of a line in polar coordinates (h is the minimum distance of the line from the origin, obtained for $\theta = \theta_0$).

....

We have defined the geodesics on a surface (which correspond to the lines in the Cartesian plane). We know that, on the plane, the circumference $\mathcal{C}$ of a circle of radius a is $\mathcal{C} = 2\pi a$.

In a similar way, on any surface, to define a circle of radius a and center$O$, let's draw from this point all the geodesics and let's mark on each of them the point at a distance from O equal to a curvilinear abscissa a; the geometric locus of all these points is the requested circumference. We can now move along this circle (always staying on the surface) and, with the same ruler with which we measured $s = a$, we can measure the length $\mathcal{C}$.

Let's see this for a sphere of radius R.



We will obviously have (we know this because we "see” the sphere in $\mathbf{E^3}$)

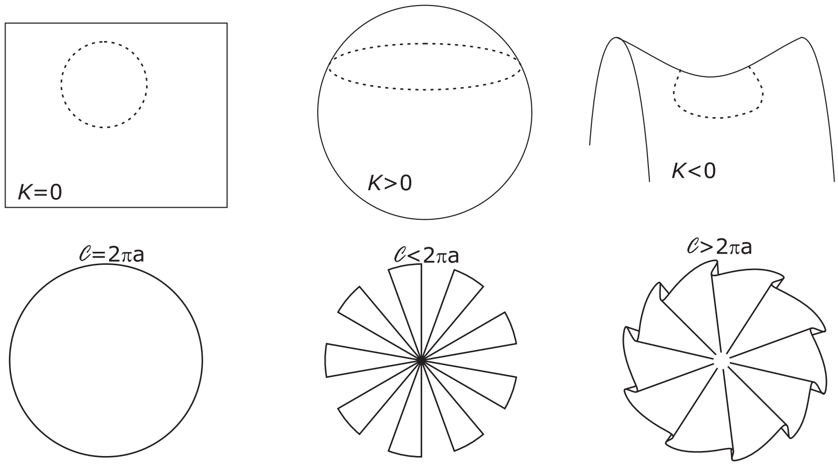
$$C = 2\pi x = 2\pi R\,\sin\big(\frac{a}{R}\big) \simeq 2\pi R\big[\frac{a}{R} - \frac{1}{6}\frac{a^3}{R^3} + ...\big] = 2\pi a - \frac{\pi}{3}\frac{a^3}{R^2} + \mathcal{O}(a^5)$$

But we also know that, for the sphere,$1/R^2 = K$ and, if $a \to 0$, neglecting higher order terms, we can write:

$$K = \frac{3}{\pi}\lim_{a\to 0}\left(\frac{2\pi a - C}{a^3}\right)$$

This result, which is true in general, shows us how to actually derive the Gauss curvature K, with measurements carried out *on the surface.*

For the plane $2\pi a = C$ and $K = 0$; for the sphere $2\pi a > C$ and $K > 0$; around a saddle point $2\pi a < C$ and $K < 0$.



The Gauss curvature is therefore an *intrinsic, local property* of a surface. As the result does not depend on the particular coordinate system used on the surface, K is an *invariant* quantity (such as $\mathrm{d}s^2$, for example), although it may change from point to point on the surface (invariant doesn't mean constant).

How does one determine K from g_{ij}? Since the metric tensor contains the information about distances, and measuring these we get K, there must be a link between these two quantities. We will see that K should depend on the second derivatives (at least) of g_{ij} at a selected point. This comes from the fact that K is invariant, i.e. does not depend on the coordinate system used, and is a local quantity, that is it depends on the behavior of g_{ij} in an infinitesimal region around the selected point.

But in an infinitesimal neighborhood of an point we can always choose a coordinate system in which g_{ij} is like $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, and in which the derivatives $g_{ij},_k$ are zero. We name it *locally Euclidean system* .

Let's see how this is, in principle, possible. Remember that the transformation from g_{ij} to g'_{kl} is:

$$g'_{kl} = \frac{\partial u^i}{\partial u'^k}\cdot\frac{\partial u^j}{\partial u'^l}g_{ij}$$

and let's expand g'_{kl} arounf the point x_0:

$$g'_{kl}(x) = g'_{kl}(x_0) + g'_{kl},_m(x_0)(x^m - x_0^m) + \frac{1}{2}g'_{kl},_{mn}(x_0)(x^m - x_0^m)(x^n - x_0^n) + \dots$$

where

$$g'_{kl}(x_0) = \left[\frac{\partial u^i}{\partial u'^k}\cdot\frac{\partial u^j}{\partial u'^l}\cdot g_{ij}\right]_{x_0}$$

$$g'_{kl},_m(x_0) = \left[\frac{\partial u^i}{\partial u'^k}\frac{\partial u^j}{\partial u'^l}g_{ij},_m\right]_{x_0} + \left[\frac{\partial^2 u^i}{\partial u'^m\partial u'^k}\frac{\partial u^j}{\partial u'^l}g_{ij}\right]_{x_0} + \left[\frac{\partial u^i}{\partial u'^k}\frac{\partial^2 u^j}{\partial u'^m\partial u'^l}g_{ij}\right]_{x_0}$$

$$=\left[\frac{\partial u^i}{\partial u'^k}\frac{\partial u^j}{\partial u'^l}g_{ij},_m\right]_{x_0}+\left[2\frac{\partial^2 u^i}{\partial u'^m\partial u'^k}\frac{\partial u^j}{\partial u'^l}g_{ij}\right]_{x_0}\quad\text{due to the symmetry between } i \text{ and } j \text{ and } k \text{ and } l$$

$$g'_{kl},_{mn}(x_0)=\left[\frac{\partial u^i}{\partial u'^k}\frac{\partial u^j}{\partial u'^l}g_{ij},_{mn}\right]_{x_0}+\left[2\frac{\partial^3 u^i}{\partial u'^m\partial u'^n\partial u'^k}\frac{\partial u^j}{\partial u'^l}g_{ij}\right]_{x_0}+\text{ first, second and third drivatives}$$

If, with an appropriate coordinate transformation, we want to put g'_{kl} into a desired form in a neighborhood of x_0, we have to specify the following quantities in the transformation:

	$2-D$	$3-D$	$4-D$	$N-D$
$\left(\frac{\partial u^i}{\partial u'^k}\right)_{x_0}$	$2\times 2=4$	9	16	N^2
$\left(\frac{\partial^2 u^i}{\partial u'^m\partial u'^k}\right)_{x_0}$	$2\times 3=6$	18	40	$\frac{N^2(N+1)}{2}$
$\left(\frac{\partial^3 u^i}{\partial u'^m\partial u'^n\partial u'^k}\right)_{x_0}$	$2\times 4=8$	30	80	$\frac{N^2(N+1)(N+2)}{6}$

On the other side, the number of independent derivatives of the metric tensor (i.e. the number of conditions to be satisfied) is the following:

	$2-D$	$3-D$	$4-D$	$N-D$
$g'_{kl}(x_0)$	3	6	10	$\frac{N(N+1)}{2}$
$g'_{kl},_m(x_0)$	6	18	40	$\frac{N^2(N+1)}{2}$
$g'_{kl},_{mn}(x_0)$	9	36	100	$\left[\frac{N(N+1)}{2}\right]^2$

Let's draw the appropriate conclusions from this for two, three and four dimensions:

- **2-D**: If we want to set the values of $g'_{kl}(x_0)$ we have 3 equations for 4 coefficients: we are left with a degree of freedom that correctly corresponds to a rotation of the axes around x_0 in the plane. If we want $g'_{kl},_m(x_0)\equiv 0$, we have 6 equations and 6 parameters, then we can manage it. If we wanted also $g_{kl},_{mn}(x_0)=0$, we notice that there are 9 equations but only 8 parameters, and in general the system is too conditioned to admit a solution: we cannot then cancel locally also the second derivatives of the metric.
- **3-D**: We have 6 equations for 9 parameters to fix $g'_{kl}(x_0)$: we are left with 3 degrees of freedom corresponding to the rotation of the coordinate system in space (for instance: the three Euler angles). We can put $g'_{kl},_m(x_0)=0$ (18 equations for 18 unknowns), but not $g'_{kl},_{mn}(x_0)=0$ (36 equations and 30 unknowns).
- **4-D** (Minkowski space): 10 equations for 16 parameters to fix $g'_{kl}(x_0)$: we are left with 6 degrees of freedom corresponding to 3 rotations in space and 3 Lorentz transformations of velocity. We can have $g'_{kl},_m(x_0)=0$ with 40 equations and 40 unknowns, but we cannot have $g'_{kl},_{mn}(x_0)=0$ (100 equations and 80 unknowns).

Since we can always put g_{ij} in the form δ_{ij}, and have $g_{ij},_k=0$ at a point, the curvature must necessarily depend on the second derivatives of g_{ij}. And the simplest form of dependence would be linear: let's see if we can find some suitable expression. Before doing so, however, we must address another issue.

2.4 Covariant derivative

We have seen that the derivative (the gradient) of a scalar field ϕ, $\partial\phi/\partial u^i$, is a covariant vector. We could then think to perform the derivative of vectoer field $A_i(u^k)$, obtaining in this way a rank two tensor. *But this is not correct!* The differential $\mathrm{d}A_i$ of a vector A_i, basic ingredient of the difference quotient, doesn't in general behave like a tensor. In fact, from the transformation rule

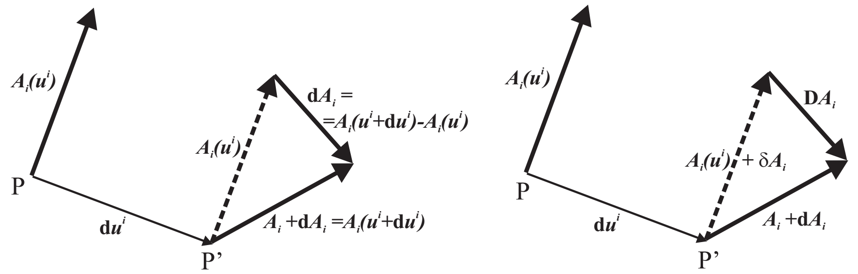
$$A_i=\frac{\partial u'^k}{\partial u^i}A'_k$$

it comes that

$$\mathrm{d}A_i = \frac{\partial u'^k}{\partial u^i}\mathrm{d}A'_k + A'_k \mathrm{d}\frac{\partial u'^k}{\partial u^i} = \frac{\partial u'^k}{\partial u^i}\mathrm{d}A'_k + \frac{\partial^2 u'^k}{\partial u^i \partial u^l}A'_k \mathrm{d}u^l$$

We see that $\mathrm{d}A_i$ is a vector only if $\frac{\partial^2 u'^k}{\partial u^i \partial u^l} = 0$, that is if the u'^i are linear functions of u^i (as it is when we go from a rectilinear coordinate system to another).

But why isn't $\mathrm{d}A_i$ a vector? The reason is that the difference $\mathrm{d}A_i = A_i(u^i + \mathrm{d}u^i) - A_i(u^i)$ is the difference of two vectors that are located in two different points (although infinitely close). The two vectors $A_i(u^i + \mathrm{d}u^i)$ and $A_i(u^i)$ transform then in a different way as the coefficients of the transformations depend on the position. If we want that the difference between two vectors is a tensor, it is necessary that the two vectors are compared at the same point (in this case both, and therefore also their difference, transform in the same way). In order to have a derivative that behaves as a tensor it is necessary to define a new type of derivative, the so-called *covariant derivative.*



Caso Euclideo **Caso generale**

In a Euclidean space, the derivative of the vector $A_i(u^i)$ is performed by moving $A_i(u^i)$ parallel to itself and leaving unchanged magnitude and direction, and by making its application point to coincide with that of $A_i(u^i + \mathrm{d}u^i)$. Then, at the point P', you run the difference and calculate the limit of the difference quotient

$$\lim_{\mathrm{d}u^i \to 0} \frac{A_i(u^i + \mathrm{d}u^i) - A_i(u^i)}{\mathrm{d}u^i}$$

How can we do something similar in a non-Euclidean space? In this case we define *parallel transport* from u^i to $u^i + \mathrm{d}u^i$ the displacement that produces a change in the vector A_i by the amount δA_i, such that moving to a locally Euclidean system (which, as we have seen, is always possible - *locally*), it vanishes: $\delta A_i = 0$. So in P', we have both $A_i + \mathrm{d}A_i \equiv A_i(u^i + \mathrm{d}u^i)$ and $A_i + \delta A_i$, corresponding to the parallel transport of $A_i(u^i)$ from P to P'. The difference

$$\mathrm{D}A_i = (A_i + \mathrm{d}A_i) - (A_i + \delta A_i) = \mathrm{d}A_i - \delta A_i$$

is a vector since it is the difference between two vectors that are at the same point. We can then use $\mathrm{D}A_i$ to define a new kind of derivative.

Now δA_i must be found. If we impose that $\mathrm{D}A_i$ (*absolute differential*) is linear as the usual differentials, δA_i must linearly depend on both the transported vector A_i and the displacement $\mathrm{d}u^i$, we can write

$$\delta A_i = \Delta^m_{il} A_m \mathrm{d}u^l$$

where the quantities Δ^m_{il} are functions of coordinates and depend on the reference frame. In the locally Euclidean frame the Δ^m_{il} vanish, but generally they will do not, and this tells us theat the Δ^m_{il} do not represent a tensor (remember that a tensor vanishing in a reference frame will vanish in all reference frames). This makes us think of another object with three indices which is not a tensor, i.e. the affine connection. As we will check in a while, it is in fact $\Delta^m_{il} \equiv \Gamma^m_{il}$, so that$\delta A_i = \Gamma^m_{il} A_m \mathrm{d}u^l$. It follows that

$$\mathrm{D}A_i = \mathrm{d}A_i - \delta A_i = \frac{\partial A_i}{\partial u^l}\mathrm{d}u^l - \Gamma^m_{il} A_m \mathrm{d}u^l$$

and the covariant derivative $\mathrm{D}A_i/\partial u^l$, also written as $A_{i;l}$, is

$$\frac{\mathrm{D}A_i}{\partial u^l} = A_{i;l} = \frac{\partial A_i}{\partial u^l} - \Gamma^m_{il} A_m$$

The covariant derivative of a tensor can be derived by considering this as the product of two vectors and requesting that it meets Leibniz rule for the derivation of a product. Then, if $T_{ik} \equiv A_i B_k$

$$\begin{aligned} T_{ik;l} &= B_k A_{i;l} + A_i B_{k;l} \\ &= B_k \left(\frac{\partial A_i}{\partial u^l} - \Gamma^m_{il} A_m \right) + A_i \left(\frac{\partial B_k}{\partial u^l} - \Gamma^m_{kl} B_m \right) = \\ &= B_k \frac{\partial A_i}{\partial u^l} + A_i \frac{\partial B_k}{\partial u^l} - \Gamma^m_{il} A_m B_k - \Gamma^m_{kl} A_i B_m = \\ &= \frac{\partial T_{ik}}{\partial u^l} - \Gamma^m_{il} T_{mk} - \Gamma^m_{kl} T_{im} \end{aligned}$$

This relation holds in general. Let us look at the expression

$$A_{i;l} = \left(g_{ik} A^k \right)_{;l} = g_{ik;l} A^k + g_{ik} A^k_{;l}$$

Since $A_{i;l}$ is a tensor, we can use the metric tensor to write it as $A_{i;l} = g_{ik} A^k_{;l}$; if we compare this expression with that one written above we realize that $g_{ik;l} = 0$. Let's now use the relation for the covariant derivative of a tensor to write explicitly this result:

$$g_{ik;l} = 0 \;\rightarrow\; \frac{\partial g_{ik}}{\partial u^l} - \Gamma^m_{il} g_{mk} - \Gamma^m_{kl} g_{im} = 0 \qquad (1)$$

We do now, in this relation, a clockwise rotation of the indices i, k, l ($i \to k,\ k \to l,\ l \to i$) and we get

$$\frac{\partial g_{kl}}{\partial u^i} - \Gamma^m_{ki} g_{ml} - \Gamma^m_{li} g_{km} = 0 \qquad (2)$$

And again another rotation of indices:

$$\frac{\partial g_{li}}{\partial u^k} - \Gamma^m_{lk} g_{mi} - \Gamma^m_{ik} g_{lm} = 0 \qquad (3)$$

If we now perform $(1) + (3) - (2)$we get, by using the symmetry in the lower indices of both Γ^m_{il} and g_{ik},

$$\frac{\partial g_{ik}}{\partial u^l} + \frac{\partial g_{li}}{\partial u^k} - \frac{\partial g_{kl}}{\partial u^i} - \Gamma^m_{il} g_{mk} - \Gamma^m_{kl} g_{im} - \Gamma^m_{lk} g_{mi} - \Gamma^m_{ik} g_{lm} + \Gamma^m_{ki} g_{ml} + \Gamma^m_{li} g_{km} \quad = \quad 0$$

which can be simplified:

$$\frac{\partial g_{ik}}{\partial u^l} + \frac{\partial g_{li}}{\partial u^k} - \frac{\partial g_{kl}}{\partial u^i} - 2\Gamma^m_{kl} g_{im} = 0$$

Multiplying this relation by $\frac{1}{2} g^{ij}$ we obtain

$$\frac{1}{2} g^{ij} \left(\frac{\partial g_{ik}}{\partial u^l} + \frac{\partial g_{li}}{\partial u^k} - \frac{\partial g_{kl}}{\partial u^i} \right) = \Gamma^m_{kl} g_{im} g^{ij} = \Gamma^m_{kl} \delta^j_m = \Gamma^j_{kl}$$

We find again the relationship that defines the affine connection, and thus we have verified the assumption $\Delta^m_{il} \equiv \Gamma^m_{il}$.

Let's now consider the scalar product$A_i B^i$; being a scalar quantity it does not change by parallel transport: $\delta \left(A_i B^i \right) = 0$ and then

$$B^i \delta A_i + A_i \delta B^i = 0 \;\rightarrow\; A_i \delta B^i = -B^i \delta A_i$$

$$A_i \delta B^i = -B^i \Gamma^m_{il} A_m \mathrm{d}u^l$$

Since the indices i and m are dummy indices, we exchange them with each other

$$A_i \delta B^i = -B^m \Gamma^i_{ml} A_i \mathrm{d}u^l$$

and, being A_i a generic vector, this means that

$$\delta B^i = -\Gamma^i_{ml} B^m \mathrm{d}u^l$$

From this result we can expresses the covariant derivative for a contravariant vector:

$$\frac{\mathrm{D}B^i}{\partial u^l} = B^i_{;l} = \frac{\partial B^i}{\partial u^l} + \Gamma^i_{ml} B^m$$

The general rule for the covariant derivative of a tensor of arbitrary rank is to make the partial derivative and then add a term of the type $+\Gamma$ for each contravariant index and a term of type $-\Gamma$ for each covariant index.

2.5 Parallel transport and curvature tensor

Let $u^i = u^i(s)$ be the parametric equation of a curve, with s curvilinear abscissa measured by starting at a given point on the curve. We know that $\mathrm{d}u^i$ is a vector (from the definition of contravariant vector), $\mathrm{d}s$ is a scalar, and $\mathrm{d}u^i/\mathrm{d}s \equiv v^i$ is then a vector. In particular, v^i is a unit vector, the *versor*[2] *tangent to the* curve.

If we were in an Euclidean space, to define a geodesic as a segment of a straight line, we would say that the tangent versor does not change with s:

$$\frac{\mathrm{d}v^i}{\mathrm{d}s} = 0$$

If we want now to generalize this relation to any space, 2 or more dimensional, we must not use the normal derivative, but the covariant one, since it is a tensor quantity:

$$\frac{\mathrm{D}v^i}{\mathrm{d}s} = 0$$

Expanding

$$\frac{\mathrm{D}v^i}{\mathrm{d}s} = \frac{\mathrm{D}v^i}{\mathrm{d}u^l}\frac{\mathrm{d}u^l}{\mathrm{d}s} = \frac{\mathrm{d}u^l}{\mathrm{d}s}\left(\frac{\partial v^i}{\partial u^l} + \Gamma^i_{ml}v^m\right) = 0$$

that is

$$\begin{aligned} \frac{\partial v^i}{\partial u^l}\frac{\mathrm{d}u^l}{\mathrm{d}s} + \Gamma^i_{ml}v^m\frac{\mathrm{d}u^l}{\mathrm{d}s} &= 0 \\ \frac{\mathrm{d}v^i}{\mathrm{d}s} + \Gamma^i_{ml}v^m v^l &= 0 \end{aligned}$$

from this, by remembering that $\mathrm{d}u^i/\mathrm{d}s \equiv v^i$, we have

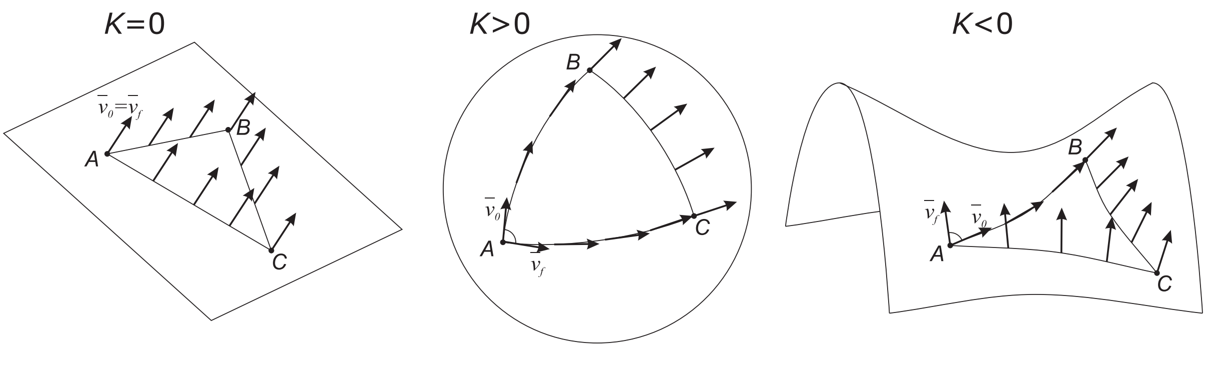
$$\frac{\mathrm{d}^2u^i}{\mathrm{d}s^2} + \Gamma^i_{ml}\frac{\mathrm{d}u^m}{\mathrm{d}s}\frac{\mathrm{d}u^l}{\mathrm{d}s} = 0$$

We find again the geodesic equation (and this is another proof of the fact that, when we leave the Euclidean space, we must switch from usual derivatives to covariant derivatives).

We see that, along a geodesics $\mathrm{D}v^i = 0$, i.e. $\mathrm{d}v^i = \delta v^i$: the unit vector v^i, parallel transported from a point u^i on the geodesic to a point $u^i + \mathrm{d}u^i$ on the same geodesic coincides with the vector $v^i + \mathrm{d}v^i$, tangent to the geodesic at the point $u^i + \mathrm{d}u^i$.

Now consider a vector A_i that is parallel transported along the same geodesic. The angle it forms with v^i, tangent versor, will be given by the scalar product $A_i v^i$. But a scalar does not change for parallel transport and so, along the geodesic, the angle between A_i and v^i remains constant: *a vector parallel transported along a geodesic always form the same angle with the tangent to the curve.*

Now imagine we parallel transport a vector $\overline{v}_0$ along a triangle formed by pieces of geodesic. If we are in a Euclidean space (e.g. on a plane) the vector $\overline{v}_f$ we get after closing the path coincides with $\overline{v}_0$.



[2] *To check that v^i is a versor, let's see what is his magnitude by means of the sclar product $v_i v^i$:*

$$v_i v^i = g_{ij}v^i v^j = g_{ij}\frac{du^i}{ds}\frac{du^j}{ds} \equiv 1 \quad \Longleftarrow \quad ds^2 = g_{ij}\,du^i du^j$$

The same thing does not happen along a spherical triangle: the vector is rotated by an angle which has the same direction of rotation of the direction in which we moved along the spherical triangle. The opposite happens if $K < 0$. We can look at it in another way: imagine we have to go from point A to point B, either directly or through a point C, always along geodesic arcs. In Euclidean space the result of the parallel transport along the two paths is the same, but the same thing does not happen on curved surfaces (what said here for a triangle formed by arcs of geodesic applies to a generic path, which can be thought as consisting of a large number of arches of geodesic). The result is that, unless we are in a Euclidean space, *there is no natural and not ambiguous way to move a vector from one point to another*; we can move it in parallel, but the result depends on the path, and there is no a natural choice for this. So *we can compare two vectors only if they are applied at the same point*. For example, two particles that pass alongside one another have a well-defined relative velocity (and less than c, with c the speed of light), but two particles in different points of a generic space do not have a well-defined, relative velocity.

Let's now quantify what we said above in a qualitative way. Moving along a closed path formed by arcs of geodesic, a vector A_k parallel transported will undergo, returning to the starting point, a variation

$$\Delta A_k = \oint \delta A_k = \oint \Gamma^i_{km} A_i \mathrm{d}u^m$$

To solve the integral we apply Stokes Theorem[3]

$$\oint A_m \mathrm{d}u^m = \frac{1}{2} \int_{Surface} \left(\frac{\partial A_m}{\partial u^l} - \frac{\partial A_l}{\partial u^m} \right) \mathrm{d}f^{lm}$$

where $\mathrm{d}f^{lm}$ is a tensor which corresponds to the projection of the element of surface area on the coordinate planes. In our case $A_m \mathrm{d}u^m \to \Gamma^i_{km} A_i \mathrm{d}u^m$ so that

$$\Delta A_k = \frac{1}{2} \int_{Surface} \left[\frac{\partial \left(\Gamma^i_{km} A_i \right)}{\partial u^l} - \frac{\partial \left(\Gamma^i_{kl} A_i \right)}{\partial u^m} \right] \mathrm{d}f^{lm}$$

If we assume that the surface element bounded by the closed curve is infinitesimal (any finite surface element can be divided into infinitesimal elements), the integrand is constant and, by neglecting infinitesimals of higher order, we can write

$$\Delta A_k = \frac{1}{2} \left[\frac{\partial \Gamma^i_{km}}{\partial u^l} A_i - \frac{\partial \Gamma^i_{kl}}{\partial u^m} A_i + \Gamma^i_{km} \frac{\partial A_i}{\partial u^l} - \Gamma^i_{kl} \frac{\partial A_i}{\partial u^m} \right] \Delta f^{lm}$$

Since A_i is parallel transported on the curve

$$\frac{\partial A_i}{\partial u^l} = \frac{\delta A_i}{\partial u^l} = \Gamma^n_{il} A_n$$

Then

$$\begin{aligned} \Delta A_k &= \frac{1}{2} \Delta f^{lm} \left[\frac{\partial \Gamma^i_{km}}{\partial u^l} A_i - \frac{\partial \Gamma^i_{kl}}{\partial u^m} A_i + \Gamma^i_{km} \Gamma^n_{il} A_n - \Gamma^i_{kl} \Gamma^n_{im} A_n \right] = \\ &= \frac{1}{2} A_i \Delta f^{lm} \left[\frac{\partial \Gamma^i_{km}}{\partial u^l} - \frac{\partial \Gamma^i_{kl}}{\partial u^m} + \Gamma^n_{km} \Gamma^i_{nl} - \Gamma^n_{kl} \Gamma^i_{nm} \right] \end{aligned}$$

where the second step, in which A_i is made explicit, was obtained by interchanging the dummy indices i and n in the terms containing the products of affine connections. The quantity in braces is a tensor, as A_i, Δf^{lm} and ΔA_k (difference of two vectors applied at the same point) are tensors. It is named **Riemann-Christoffel tensor** or **curvature tensor**:

$$R^i_{\ klm} = \frac{\partial \Gamma^i_{km}}{\partial u^l} - \frac{\partial \Gamma^i_{kl}}{\partial u^m} + \Gamma^n_{km} \Gamma^i_{nl} - \Gamma^n_{kl} \Gamma^i_{nm}$$

(*Warning*: you can find it defined with the signs interchanged!) If, in a volume of space, $R^i_{\ klm} = 0$, then $\Delta A_k = 0$: The parallel transport along a closed curve keeps the vector unchanged, and that volume of space is said to be **flat**. This happens in a Euclidean space, as well as in any (volume of) space in which g_{ij} is constant, because the affine connections are null and so also the curvature tensor; and since a tensor equal to zero in a coordinate system is zero in any coordinate system, then $R^i_{\ klm} = 0$ in any refernce frame. On the contrary, if $R^i_{\ klm} \neq 0$ the parallel transport depends on the path, and the space (or the volume of space) is said, by contrast, **curved** (this is the reason for the Riemann-Christoffel tensor is also named curvature tensor).

[3]se, e.g., Landau Lifsic, *The Classical Theory of Fields*, eq. (6.19)

2.6 Properties of the curvature tensor

It can be proved[4] that $R^i_{\ klm}$ is the only tensor that can be constructed from the metric tensor and its first and second derivatives, and which is linear in the second derivatives (and also quadratic in the first derivatives). The metric tensor allows to write it in the totally covariant form $R_{jklm} = g_{ji}R^i_{\ klm}$.

Let's consider the tensor $R^i_{\ klm}$ as written few lines above, and let's move to the locally Euclidean system; in this reference frame, at the point u^i, it is $\frac{\partial g_{ij}}{\partial u^k} = 0$. Then the affine connections are zero, and the covariant derivatives are reduced to simple partial derivatives. In this system

$$R^i_{\ klm;j} = \frac{\partial}{\partial u^j}\left(R^i_{\ klm}\right) = \frac{\partial^2 \Gamma^i_{km}}{\partial u^j \partial u^l} - \frac{\partial^2 \Gamma^i_{kl}}{\partial u^j \partial u^m}$$

(at u^i the affine connection vanish, but not, in general, their derivatives). Cyclically permuting the indices l, m and j , we get:

$$R^i_{\ kmj;l} = \frac{\partial^2 \Gamma^i_{kj}}{\partial u^l \partial u^m} - \frac{\partial^2 \Gamma^i_{km}}{\partial u^l \partial u^j}$$

and also

$$R^i_{\ kjl;m} = \frac{\partial^2 \Gamma^i_{kl}}{\partial u^m \partial u^j} - \frac{\partial^2 \Gamma^i_{kj}}{\partial u^m \partial u^l}$$

Adding the three relations, we easily get:

$$R^i_{\ klm;j} + R^i_{\ kmj;l} + R^i_{\ kjl;m} = 0.$$

These are the so-called ***Bianchi Identities***. The tensorial nature of these reltions tell us that, although we have obtained them in the locally Euclidean system , they hold in all reference systems.

Lowering the contravariant iindex we get

$$R_{iklm;j} + R_{ikmj;l} + R_{ikjl;m} = 0.$$

The Riemann tensor has its own properties, let's see them in the fully covariant form$R_{jklm} = g_{ji}R^i_{\ klm}$:

- *Symmetry* properties
$$R_{jklm} = R_{lmjk}$$
- *Antisymmetry* properties
$$R_{jklm} = -R_{kjlm} = -R_{jkml} = R_{kjml}$$
- *Cyclic* properties
$$R_{jklm} + R_{jmkl} + R_{jlmk} = 0.$$

From the Riemann tensor, by contraction, we can get a rank 2 tensor, the *Ricci tensor*, defined as:

$$R_{km} \equiv R^i_{\ kim}$$

(indices i and l of $R^i_{\ klm}$ are contracted). Considering the antisymmetry properties, if in $R^i_{\ klm}$ we instead contractc i and m we get again Ricci tensor, but with its sign changed:

$$R^i_{\ kli} = -R^i_{\ kil} = -R_{kl}$$

Ricci tensor is symmetric:

$$R_{mk} = R^i_{\ mik} = g^{ir}R_{rmik} = g^{ir}R_{ikrm} = R^r_{\ krm} = R_{km}$$

It is the only symmetric tensor of rank 2 that can be obtained from $R^i_{\ klm}$. From the Ricci tensor one can obtain the *Ricci scalar* or *curvature scalar*:

$$R = g^{km}R_{km}$$

It is the only scalar that can be obtained from $R^i_{\ klm}$.

All these properties of the Riemann tensor reduce the number of its independent components and, in N dimensions, this number is $\mathcal{N} = \frac{N^2(N^2-1)}{12}$. In particular:

[4]See, e.g., *Weinberg*, 1972

- For $N = 1$, $\mathcal{N} = 0$ and $R_{1111} \equiv 0$ always: a curve has always (intrinsic) curvature zero, we do not have information on how the curve is "embedded" in a space with 2 or more dimensions.
- For $N = 2$, $\mathcal{N} = 1$. There is only one independent componenet, for instance R_{1212}.
- For $N = 3$, $\mathcal{N} = 6$, as many as the components of the (symmetrical) Ricci tensor. So for $N = 3$ it is sufficient to know R_{km} to describe the space cuvatura.
- For $N = 4$, $\mathcal{N} = 20$, while R_{km} has only 10 components. One must use the complete $R^i_{\ klm}$ tensor (apart from situations of particular symmetry, and we'll see that it is so in the case of the homogeneous and isotropic universe).

From Bianchi identities, in the covariant form, by exploiting the properties of antisymmetry of the Riemann tensor, we have

$$R_{iklm;j} - R_{kimj;l} - R_{iklj;m} = 0 \qquad / \cdot g^{il} g^{km}$$

$$g^{km} R^l_{\ klm;j} - g^{il} R^m_{\ imj;l} - g^{km} R^l_{\ klj;m} = 0$$

that is

$$g^{km} R_{km;j} - g^{il} R_{ij;l} - g^{km} R_{kj;m} = 0$$

and them

$$R_{;j} - R^l_{\ j;l} - R^m_{\ j;m} = R_{;j} - 2R^l_{\ j;l} = 0$$

anf finally

$$R^l_{\ j;l} = \frac{1}{2} R_{;j} = \frac{1}{2} \frac{\partial R}{\partial u^j}$$

where the last step is due to the fact that R is a scalar, then does not depend on the reference system used, and its covariant derivative coincides with the simple partial derivative. The quantity $R^l_{\ j;l}$ is the (covariant) *divergence* of the Ricci tensor. Now consider the mixed tensor

$$R^l_{\ j} - \frac{1}{2} \delta^l_{\ j} R$$

Its divergence is (for the rule of the derivation of a product and being $\delta^l_{\ j;l} = 0$ [5]

$$R^l_{\ j;l} - \frac{1}{2} \delta^l_{\ j} \frac{\partial R}{\partial u^l} = R^l_{\ j;l} - \frac{1}{2} \frac{\partial R}{\partial u^j} = 0$$

as seen just above. So the (covariant) divergence of this tensor is equal to zero. If we switch to its covariant components we get

$$g_{il} R^l_{\ j} - \frac{1}{2} g_{il} \delta^l_{\ j} R = R_{ij} - \frac{1}{2} g_{ij} R \equiv G_{ij}$$

where G_{ij} is the so-called *Einstein tensor.* This tensor has very relevant properties: it is *symmetric*, has *vanishing divergence* and, since it comes from Riemann tensor, it contains *terms linear in the second derivatives* of the metric and *quadratic in its first derivatives.*

2.7 The Theorema Egregium

In 2 dimensions the *Theorema Egregium* of Gauss states that the Gauss cirvature K can be derived from the metric tensor; in particular $K = R_{1212}/g$.

Here we give a justification of Theorema egregium. We have seen that, locally, in a neighborhood of the point P, a surface element can be written in the form

$$z = f(x, y) = \frac{x^2}{2\rho_1} + \frac{y^2}{2\rho_2}$$

[5] $\delta^l_{\ j;l} = \frac{\partial \delta^l_j}{\partial u^l} + \Gamma^l_{lk} \delta^k_{\ j} - \Gamma^m_{jl} \delta^l_{\ m} = \Gamma^l_{lj} - \Gamma^l_{jl} = 0$

which, put in the form $\overline{x}(u,v)$, can be written as ($x \equiv u$, $y \equiv v$):

$$\overline{x}(u,v) = \left(u, v, \frac{u^2}{2\rho_1} + \frac{v^2}{2\rho_2}\right); \qquad \overline{x}_u = \left(1, 0, \frac{u}{\rho_1}\right) \qquad \overline{x}_v = \left(0, 1, \frac{v}{\rho_2}\right)$$

$$\begin{aligned} E &= \overline{x}_u \cdot \overline{x}_u = 1 + \frac{u^2}{\rho_1^2} \\ F &= \overline{x}_u \cdot \overline{x}_v = \frac{uv}{\rho_1\rho_2} \\ G &= \overline{x}_v \cdot \overline{x}_v = 1 + \frac{v^2}{\rho_2^2} \end{aligned}$$

$$\mathrm{d}s^2 = \left(1 + \frac{u^2}{\rho_1^2}\right)\mathrm{d}u^2 + 2\frac{uv}{\rho_1\rho_2}\mathrm{d}u\ \mathrm{d}v + \left(1 + \frac{v^2}{\rho_2^2}\right)\mathrm{d}v^2$$

$$g_{ij} = \begin{pmatrix} 1 + \frac{u^2}{\rho_1^2} & \frac{uv}{\rho_1\rho_2} \\ \frac{uv}{\rho_1\rho_2} & 1 + \frac{v^2}{\rho_2^2} \end{pmatrix} \qquad \rightarrow \qquad g = det(g_{ij}) = 1 + \frac{u^2}{\rho_1^2} + \frac{v^2}{\rho_2^2}$$

Notice that, in P, $g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $g_{ij},_k = 0$ and second derivatives do not vanish

$$g^{ij} = \frac{1}{g}\begin{pmatrix} 1 + \frac{v^2}{\rho_2^2} & -\frac{uv}{\rho_1\rho_2} \\ -\frac{uv}{\rho_1\rho_2} & 1 + \frac{u^2}{\rho_1^2} \end{pmatrix} \qquad\qquad K\big|_P = \frac{R_{1212}|_P}{g|_P}$$

$R_{1212} = g_{1k}R^k_{212} = g_{11}R^1_{212} + g_{12}R^2_{212}$ but in P $g_{12} = 0$ e $g_{11} = 1$, that is $R_{1212}|_P \equiv R^1_{212}|_P$

$$R^1_{212} = \frac{\partial\Gamma^1_{22}}{\partial x^1} - \frac{\partial\Gamma^1_{21}}{\partial x^2} + \Gamma^r_{22}\Gamma^1_{r1} - \Gamma^r_{21}\Gamma^1_{r2}$$

but the Γ contain the $g_{ij,k}$ which in P vanish and we are left with

$$R^1_{212}\big|_P = \frac{\partial\Gamma^1_{22}}{\partial u}\bigg|_P - \frac{\partial\Gamma^1_{21}}{\partial v}\bigg|_P$$

$$\begin{aligned} \Gamma^1_{22} &= \frac{1}{2}g^{1\sigma}\left(\frac{\partial g_{\sigma 2}}{\partial x^2} + \frac{\partial g_{\sigma 2}}{\partial x^2} - \frac{\partial g_{22}}{\partial x^\sigma}\right) = \\ &= \frac{1}{2}\left[g^{11}\left(\frac{\partial g_{12}}{\partial v} + \frac{\partial g_{12}}{\partial v} - \frac{\partial g_{22}}{\partial u}\right) + g^{12}\left(\frac{\partial g_{22}}{\partial v} + \frac{\partial g_{22}}{\partial v} - \frac{\partial g_{22}}{\partial v}\right)\right] = \\ &= \frac{1}{2g}\left[\left(1 + \frac{v^2}{\rho_2^2}\right)\left(2 \cdot \frac{u}{\rho_1\rho_2}\right) + \left(-\frac{uv}{\rho_1\rho_2}\right)\left(\frac{2v}{\rho_2^2}\right)\right] = \frac{u}{g\rho_1\rho_2} \end{aligned}$$

$$\begin{aligned} \Gamma^1_{21} &= \frac{1}{2}g^{1\sigma}\left(\frac{\partial g_{\sigma 2}}{\partial x^1} + \frac{\partial g_{\sigma 1}}{\partial x^2} - \frac{\partial g_{21}}{\partial x^\sigma}\right) = \\ &= \frac{1}{2}\left[g^{11}\left(\frac{\partial g_{12}}{\partial u} + \frac{\partial g_{11}}{\partial v} - \frac{\partial g_{21}}{\partial u}\right) + g^{12}\left(\frac{\partial g_{22}}{\partial u} + \frac{\partial g_{21}}{\partial v} - \frac{\partial g_{21}}{\partial v}\right)\right] = 0 \end{aligned}$$

$$\frac{\partial\Gamma^1_{22}}{\partial u} = \frac{\partial}{\partial u}\left[\frac{u}{g\rho_1\rho_2}\right] = \frac{1}{\rho_1\rho_2} \cdot \frac{g - u \cdot \frac{2u}{\rho_1^2}}{g^2} = \frac{1}{\rho_1\rho_2}\left[\frac{1}{g} - \frac{2u^2}{g^2\rho_1^2}\right]$$

At the end

$$K|_P = \frac{R_{1212}|_P}{g|_P} = \frac{R^1_{212}|_P}{g|_P} = \frac{\frac{\partial\Gamma^1_{22}}{\partial u}\big|_P}{g|_P} = \frac{1}{\rho_1\rho_2} \qquad\qquad \text{Q.E.D.}$$

To divide by$g$ seems apparently not essential. But remember that K is a scalar, that is a tensor of rank zero, while R_{1212} is a component of a tensor, which is not invariant under coordinate transformations, and the same holds for g, which is not a scalar. However, their ratio behaves like a scalar. To divide by$g$ is also useful for normalization [if, e.g., we transform $x \to \alpha u \quad y \to \beta v$ we get $\overline{x}(u,v) = (\alpha u, \beta v, \frac{\alpha^2 u^2}{2\rho_1} + \frac{\beta^2 v^2}{2\rho_2})$ and if we redo the calculations we find that the factor $1/g$ in the formula for K is now essential: $g|_P = \alpha^2\beta^2$].

We observe that the relation which expresses the Theorema Egregium, $K = R_{1212}/g$, is a relationship between tensors of rank zero, ie scalars. If, as we showed, it is true in a particular frame of reference, it applies in any frame of reference, and the particular result obtained can be extended in general.

This method of proving that a relationship between tensors holds in a particular frame of reference and therefore, having to do with tensor objects, it is valid in any frame of reference, is a method routinely used in tensor calculus.
The curvature tensor is related to the Gauss curvature even in spaces with any number of dimensions. Given a point P in one of these spaces, and two vectors a^μ and b^μ applied at the point P, we can draw a family of geodesic curves $x^\mu(s, \alpha, \beta)$ through P, with α and β real numbers. All these geodesics, which have as their initial tangent vector $\mathrm{d}x^\mu/\mathrm{d}s = \alpha a^\mu + \beta b^\mu$, form a two-dimensional surface for P, with Gauss curvature given by[6]

$$K(a,b) = \frac{R_{\lambda\mu\nu\kappa} a^\lambda b^\mu a^\nu b^\kappa}{(g_{\lambda\nu} g_{\mu\kappa} - g_{\lambda\kappa} g_{\mu\nu})\, a^\lambda b^\mu a^\nu b^\kappa}$$

....

***Example**: estimation of Gauss curvature*

Given, for a surface element, the following metric

$$\mathrm{d}s^2 = \mathrm{d}u^2 + e^{\frac{2u}{k}}\,\mathrm{d}v^2$$

estimate K (intrinsic Gauss curvature).

We know that $K = R_{1212}/g$

$$g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & e^{2u/k} \end{pmatrix} \quad \to \quad g = e^{2u/k} \quad \to \quad g^{ij} = \begin{pmatrix} 1 & 0 \\ 0 & e^{-2u/k} \end{pmatrix}$$

Considering the particular values of g_{ij} and g^{ij} we get $R_{1212} = g_{1k} R^k_{\ 212} = R^1_{\ 212}$

$$R^1_{\ 212} = \frac{\partial \Gamma^1_{22}}{\partial u^1} - \frac{\partial \Gamma^1_{21}}{\partial u^2} + \Gamma^r_{22}\Gamma^1_{r1} - \Gamma^r_{21}\Gamma^1_{r2} = \frac{\partial \Gamma^1_{22}}{\partial u} - \frac{\partial \Gamma^1_{21}}{\partial v} + \Gamma^1_{22}\Gamma^1_{11} + \Gamma^2_{22}\Gamma^1_{21} - \Gamma^1_{21}\Gamma^1_{12} - \Gamma^2_{21}\Gamma^1_{22}$$

Then

$$\Gamma^1_{22} = \frac{1}{2} g^{11} \left(\frac{\partial g_{21}}{\partial u^2} + \frac{\partial g_{21}}{\partial u^2} - \frac{\partial g_{22}}{\partial u^1} \right) = -\frac{1}{k} e^{2u/k}$$

$$\Gamma^1_{21} = 0 \qquad \Gamma^1_{11} = 0 \qquad \Gamma^2_{21} = \frac{1}{k} \qquad\qquad \frac{\partial \Gamma^1_{22}}{\partial u} = -\frac{2}{k^2} e^{2u/k}$$

$$R^1_{\ 212} = -\frac{2}{k^2} e^{2u/k} - \left(\frac{1}{k} \cdot -\frac{1}{k} e^{2u/k}\right) = -\frac{1}{k^2} e^{2u/k} \equiv R_{1212}$$

$$K = \frac{R_{1212}}{g} = -\frac{1}{k^2}\, e^{2u/k} \Big/ e^{2u/k} = -\frac{1}{k^2}$$

....

[6]See *Weinberg* 1972, Section 6.9

. . . .

***Example**: curvature of the pseudosphere*

The pseudosphere can be represented by a surface of revolution: the result of *revolving a tractrix about its asymptote*[7]

$$\overline{x}(u,v) = \Big(a\ \sin u\ \cos v, a\ \sin u\ \sin v, a\big[\cos u + \ln\big(\mathrm{tg}\frac{u}{2}\big)\big] \Big) \qquad \text{with} \quad 0 < u < \pi/2$$

Find the metric of the surface and K

$$\overline{x}_u = \Big(a\ \cos u\ \cos v, a\ \cos u\ \sin v, a\big[-\sin u + \frac{1}{\mathrm{tg} u/2} \cdot \frac{1}{\cos^2 u/2} \cdot \frac{1}{2}\big] \Big) = \Big(a\ \cos u\ \cos v, a\ \cos u\ \sin v, a\frac{\cos^2 u}{\sin u} \Big)$$

$$\overline{x}_v = \Big(-a\ \sin u\ \sin v, a\ \sin u\ \cos v, 0 \Big)$$

$$\begin{aligned} E &= \overline{x}_u \cdot \overline{x}_u = a^2 \frac{\cos^2 u}{\sin^2 u} = \frac{a^2}{\mathrm{tg}^2 u} \\ G &= \overline{x}_v \cdot \overline{x}_v = a^2 \sin^2 u \\ F &= \overline{x}_u \cdot \overline{x}_v = 0 \end{aligned}$$

$$\mathrm{d}s^2 = \frac{a^2}{\mathrm{tg}^2 u} \mathrm{d}u^2 + a^2 \sin^2 u\ \mathrm{d}v^2$$

We now perform a coordinate transformation

$$(u,v) \to (x,y) \qquad \begin{cases} x = a\ \ln(\sin u) \\ y = av \end{cases} \qquad \mathrm{d}x = \tfrac{a}{\mathrm{tg} u}\mathrm{d}u \qquad \mathrm{d}y = a\ \mathrm{d}v \qquad e^{x/a} = \sin u$$

and we realize that $\mathrm{d}s^2 = \mathrm{d}x^2 + e^{2x/a}\mathrm{d}y^2$ is a metric equivalent to the original one.

Moreover, from the previous exercise, we know that $K = -1/a^2$

. . . .

[7] Tractrix (from the Latin verb trahere "pull, drag"; plural: tractrices) is the curve along which an object moves, under the influence of friction, when pulled on a horizontal plane by a line segment attached to a tractor (pulling) point that moves at a right angle to the initial line between the object and the puller at an infinitesimal speed. It is therefore a curve of pursuit. It was first introduced by Claude Perrault in 1670, and later studied by Sir Isaac Newton (1676) and Christiaan Huygens (1692). The revolution of a tractrix about its asymptote produces the surface called pseudosphere. The name derives from the fact that the curvature is constant, as for the sphere, but has the opposite sign.

3 General Relativity

3.1 Minkoswki space

In Special Relativity, passing from one frame of reference to another, the infinitesimal distance between two events:

$$\mathrm{d}s^2 = c^2\mathrm{d}t^2 - (\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2)$$

is preserved (= is invariant). If we define $x^0 = ct$; $x^1 = x$; $x^2 = y$; $x^3 = z$ we can write

$$\mathrm{d}s^2 = \eta_{\alpha\beta}\mathrm{d}x^\alpha\mathrm{d}x^\beta \qquad \text{with} \qquad \eta_{\alpha\beta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

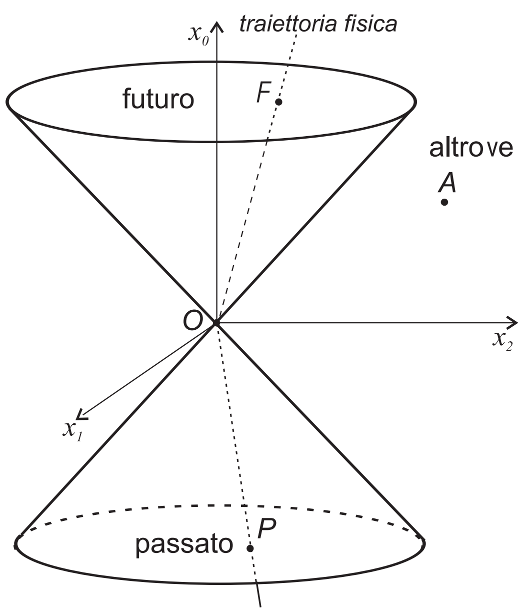
We have then the metric of Minkoswki space, which is "pseudo-Euclidean", but it is flat: in fact the $\eta_{\alpha\beta}$ are constant, therefore Γ^i_{jk} and R^h_{ijk} are zero. In the following we will use, by convention, the Greek indices α, β, γ, ... if these vary from 0 to 3, while we will use italic indices i, j, k, ... if they vary from 1 to 3. Warning: in literature also the opposite convention is used. Even $\eta_{\alpha\beta}$ is often defined with opposite signs, i.e. with the *signature* $(-1, 1, 1, 1)$ instead of $(1, -1, -1, -1)$.

Moreover, we say that the interval$\mathrm{d}s^2$ is:

- ***time like*** if $\mathrm{d}s^2 > 0$ (corresponding to a physical trajectory with $v < c$)
- ***space like*** if $\mathrm{d}s^2 < 0$
- ***light like*, *null*** if $\mathrm{d}s^2 = 0$ (corresponding to the motion of particles, like photons, which move with speed $v = c$)

If we represent the space-time (eliminating one of the spatial coordinates) about an event taken as the origin, we can divide it into three zones defined by the cone in Figure:

- **future:** is the volume of space-time formed by events such that the event at O can interact with them by means of particles that follow a physical trajectory.
- **past:** is the volume of space-time formed by past events which can influence today the event at O.
- **elsewhere:** is the volume of space-time formed by events which cannot affect or be affected by the event at O, since information cannot propagate with $v > c$. An observer in motion with respect to O can see both O and A happen at the same time.



Each observer has with him a ruler and a clock: the time marked by this clock is the proper time τ. An observer, who sees two events (physically connected) occur at different times but at the same place ($\mathrm{d}x = \mathrm{d}y = \mathrm{d}z = 0$) obtains $\mathrm{d}s^2 = c^2\mathrm{d}\tau^2$: $\mathrm{d}s$ and $\mathrm{d}\tau$ are proportional.

The distance $\mathrm{d}s$ between the same two events, both for an observer who sees them occurring at the same point, and for another observer who sees them occurring at a distance $\mathrm{d}l$, is the same:

$$\mathrm{d}s^2 = c^2\mathrm{d}\tau^2 = c^2\mathrm{d}t^2 - |\mathrm{d}\bar{l}^2| \qquad \to \qquad \mathrm{d}\tau^2 = \mathrm{d}t^2\left(1 - \frac{1}{c^2}\frac{\mathrm{d}\bar{l}}{\mathrm{d}t}\cdot\frac{\mathrm{d}\bar{l}}{\mathrm{d}t}\right) = \mathrm{d}t^2\big(1 - \frac{v^2}{c^2}\big)$$

where v is the particle speed for the observer who sees it moving, and also the relative velocity between thw two observers. Defining $\beta \equiv v/c$ and $\gamma \equiv 1/\sqrt{1-\beta^2}$ we get $\mathrm{d}t = \gamma\mathrm{d}\tau$. Since $\gamma \geq 1$, then $\mathrm{d}t \geq \mathrm{d}\tau$: the interval between two "ticks" of a clock is shorter for the “proper” clock; moving clocks appear slower (think about the twin paradox).

The velocity four-vector (**four-velocity**) is defined as $u^\alpha \equiv \frac{\mathrm{d}x^\alpha}{\mathrm{d}s}$; it is a vector since $\mathrm{d}x^\alpha$ is a vector and $\mathrm{d}s$ is a scalar.

In a generic reference frame, not at rest with a particle which has a velocity $\overline{v} \equiv \frac{\mathrm{d}\overline{x}}{\mathrm{d}t}$, we have

$$\begin{aligned} u^0 &= \frac{\mathrm{d}x^0}{\mathrm{d}s} = \frac{\mathrm{d}(ct)}{c\mathrm{d}\tau} = \frac{\mathrm{d}t}{\mathrm{d}\tau} = \gamma \\ u^i &= \frac{\mathrm{d}x^i}{c\mathrm{d}\tau} = \frac{1}{c}\frac{\mathrm{d}x^i}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}\tau} = \gamma\frac{v^i}{c} = \gamma\beta^i \end{aligned}$$

and we can write $u^\alpha = \gamma(1, \overline{\beta})$. If the particle is at rest we have $u^\alpha = (1, 0, 0, 0)$.

The quantity $u^\alpha u_\alpha$ is invariant: $u^\alpha u_\alpha = \eta_{\alpha\beta}u^\alpha u^\beta = u^0u^0 - (u^1u^1 + u^2u^2 + u^3u^3) = \gamma^2 - (\gamma^2v^2/c^2) = 1$; u^α is the unit vector (versor) tangent to the trajectory of the particle (in the 4-D space-time).

The **four-momentum** is defined as $P^\alpha = m_0u^\alpha$where m_0 is the rest mass of the particle. If we remember that $\overline{P} = m\overline{v} = \gamma m_0\overline{v}$; $E = mc^2 = m_0c^2\gamma$ we get:

$$P^0 = \gamma m_0 = E/c^2 \qquad\qquad P^i = \gamma m_0\frac{v^i}{c} = m\frac{v^i}{c}$$

$$P^\alpha P_\alpha = \gamma^2m_0^2 - \gamma^2m_0^2\frac{v^2}{c^2} = \gamma^2m_0^2\left(1 - \frac{v^2}{c^2}\right) = m_0^2$$

$$P^\alpha P_\alpha = m_0^2 = \frac{E^2}{c^4} - \frac{1}{c^2}\overline{P}\cdot\overline{P} \qquad \to \qquad m_0^2c^2 = \frac{E^2}{c^2} - |\overline{P}|^2$$

If $\frac{dP^\alpha}{ds} = 0$ then $P^\alpha = const \quad \Rightarrow \quad E = const$ and $\overline{P} = const$: this is the energy and momentum conservation.

The **four-acceleration** is $\frac{\mathrm{d}^2x^\alpha}{\mathrm{d}s^2} = \frac{\mathrm{d}u^\alpha}{\mathrm{d}s}$. The **geodesic** equation has always the same form:

$$\frac{\mathrm{d}^2x^\alpha}{\mathrm{d}s^2} + \Gamma^\alpha_{\beta\gamma}\frac{\mathrm{d}x^\beta}{\mathrm{d}s}\frac{\mathrm{d}x^\gamma}{\mathrm{d}s} = 0$$

If the metric tensor is simply $\eta_{\alpha\beta}$, then the $\Gamma^\alpha_{\beta\gamma}$ vanish, so that $\mathrm{d}^2x^\alpha/\mathrm{d}s^2 = 0$, i.e. $x^\alpha = a^\alpha\cdot s + b^\alpha$, or

$$\begin{cases} ct = a^0\cdot s + b^0 \\ \overline{x} = \overline{a}\cdot s + \overline{b} \end{cases}$$

and the trajectory is a straight line covered with uniform rectilinear motion. If we write the metric tensor in another way, for instance in planar polar coordinates, $\mathrm{d}s^2 = c^2\mathrm{d}t^2 - (\mathrm{d}r^2 + r^2\mathrm{d}\theta^2)$, the $\Gamma^\alpha_{\beta\gamma}$ are not all zero, but the resulting geodetic curve is always a straight line, covered with uniform rectilinear motion, but written in polar coordinates.

Notice: While in 3-D Euclidean space the geodesic between two points is a straight line, so it is the shortest distance between two points, in special relativity the quantity $\int_A^B \mathrm{d}s$ is maximized with respect to variations in path with the ends fixed. It is $\Delta\tau = \Delta s/c$, and you can think at the twin paradox, where the elapsed time is maximized for the twin who remained on the Earth.

3.2 The energy-momentum tensor

To deal with General Relativity and Cosmology we need an "object" that has the properties of a continuous medium, such as density and velocity, and links them to conservation of energy and momentum.

Let's consider first the case of incoherent matter, whose particles (for the moment) do not interact (*"dust"*). The matter field will be described at any point by the four-velocity $u^\alpha = \gamma(1, \overline{v}/c)$ and by its *proper density* $\rho_0(x)$, i.e. that measured by an observer who follows the fluid. With these quantities one can form a symmetric tensor of rank 2 in the simplest way as:

$$T^{\alpha\beta} = \rho_0 c^2 u^\alpha u^\beta$$

Let's see how this tensor is mede in detail:

$$T^{00} = \rho_0 c^2 \gamma^2 = \gamma^2 \rho_0 c^2 = \rho c^2 \qquad \text{by writing} \quad \rho = \gamma^2 \rho_0$$

To interpret this result remember that the mass is $m = \gamma m_0$(m_0= rest mass) and that a volume element in motion appears contracted by a factor $1/\gamma$, and its density grows by another factor γ. So if the proper density is ρ_0, an observer with respect to which the fluid has velocity $\overline{v}$ measures a density $\gamma^2\rho_0$.

T^{00} represents the mass-energy density (in this case the only contribution to the energy comes from matter motion).

The components of $T^{\alpha\beta}$ can be written:

$$T^{\alpha\beta} = \rho c^2 \cdot \begin{pmatrix} 1 & v_x/c & v_y/c & v_z/c \\ v_x/c & v_x^2/c^2 & v_x v_y/c^2 & v_x v_z/c^2 \\ v_y/c & v_y v_x/c^2 & v_y^2/c^2 & v_y v_z/c^2 \\ v_z/c & v_z v_x/c^2 & v_z v_y/c^2 & v_z^2/c^2 \end{pmatrix} \qquad (**)$$

We now derive the motion equations from the expression $\partial_\beta T^{\alpha\beta} = 0$, the four-divergence of $T^{\alpha\beta}$(remember we are in Minkowski space-time, and covariant derivatives are simply partial derivatives).

- For $\alpha = 0$ we have $\partial_\beta T^{0\beta} = 0 \quad \Leftrightarrow \quad \frac{\partial T^{0\beta}}{\partial x^\beta} = 0$ which can be expanded:

$$\frac{1}{c}\frac{\partial(\rho c^2)}{\partial t} + \frac{\partial(\rho c v_x)}{\partial x} + \frac{\partial(\rho c v_y)}{\partial y} + \frac{\partial(\rho c v_z)}{\partial z} = 0$$

  and then simplified to

$$\frac{\partial \rho}{\partial t} + \overline{\nabla} \cdot (\rho \overline{v}) = 0$$

  which is the *continuity equation* for a fluid, expressing mass-energy conservation.

- For $\alpha = 1, 2, 3$ we have

$$\frac{1}{c}\frac{\partial(\rho c v_x)}{\partial t} + \frac{\partial(\rho v_x v_x)}{\partial x} + \frac{\partial(\rho v_x v_y)}{\partial y} + \frac{\partial(\rho v_x v_z)}{\partial z} = 0 \qquad (\alpha = 1)$$

$$\frac{1}{c}\frac{\partial(\rho c v_y)}{\partial t} + \frac{\partial(\rho v_y v_x)}{\partial x} + \frac{\partial(\rho v_y v_y)}{\partial y} + \frac{\partial(\rho v_y v_z)}{\partial z} = 0 \qquad (\alpha = 2)$$

$$\frac{1}{c}\frac{\partial(\rho c v_z)}{\partial t} + \frac{\partial(\rho v_z v_x)}{\partial x} + \frac{\partial(\rho v_z v_y)}{\partial y} + \frac{\partial(\rho v_z v_z)}{\partial z} = 0 \qquad (\alpha = 3)$$

If we multiply the first by $\hat{\imath}$ (unit vector of the x-axis), the second by $\hat{\jmath}$ and the third by $\hat{k}$ and then add them toghether they can be summarized in the expression

$$\frac{\partial}{\partial t}(\rho \overline{v}) + \frac{\partial}{\partial x}(\rho v_x \overline{v}) + \frac{\partial}{\partial y}(\rho v_y \overline{v}) + \frac{\partial}{\partial z}(\rho v_z \overline{v}) = 0$$

which, by expanding and by using continuity equation, becomes

$$\rho\frac{\partial\overline{v}}{\partial t}+\overline{v}\left[\frac{\partial\rho}{\partial t}+\overline{\nabla}\cdot(\rho\overline{v})\right]+\rho v_x\frac{\partial\overline{v}}{\partial x}+\rho v_y\frac{\partial\overline{v}}{\partial y}+\rho v_z\frac{\partial\overline{v}}{\partial z}=0$$

$$\text{that is}\qquad \rho\left[\frac{\partial\overline{v}}{\partial t}+\left(\overline{v}\cdot\overline{\nabla}\right)\overline{v}\right]=0\qquad (I)\Leftrightarrow\qquad \rho\frac{\mathrm{d}\overline{v}}{\mathrm{d}t}=0\qquad (II)$$

This equation, typical of fluid dynamics, is the motion equation for a fluid without pressure, viscosity and external forces. Therefore it expresses the conservation of momentum. In particular, in the form (I) one imagines to observe the fluid at a fixed point and to see how its motion evolves (the so-called *Eulerian* point of view), while in the form (II) one imagines to follow in their motion the particles of fluid (the so-called *Lagrangian* point of view).

Thus we see that the tensor $T^{\alpha\beta}$ expresses the energy and dynamic properties of the fluid (dust) in this case. $T^{\alpha\beta}$ is the **stress-energy tensor**.

In a *locally inertial frame at rest (LIRF)* with respect to the fluid, in which $u^\alpha=(1,0,0,0)$, $T^{\alpha\beta}$ has the particularly simple form

$$T^{\alpha\beta}_{LIRF}=\begin{pmatrix}\rho_0c^2 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}$$

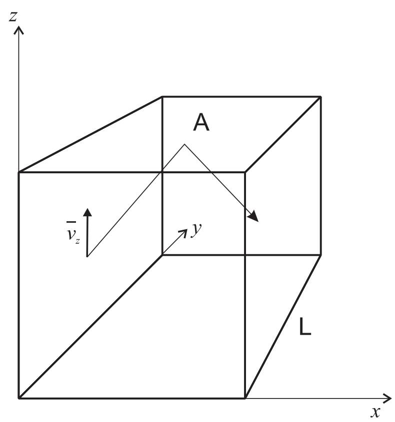
We now come to consider the case in which the particles interact in the simplest way, that is through collisionss due to their thermal motion: in this case the fluid has a *pressure*. We assume that there is no transport of energy by conduction or radiation and there is no viscosity. The **fluid** so defined is said to be **perfect**.

If we are now in the *LIRF*, $T^{\alpha\beta}$ will be no more that one written just above, with only $T^{00}\neq 0$. The particles now have random motions around the zero of their positions and velocities. We must then refer back to the previous form (**) of $T^{\alpha\beta}$, in which however the terms that appear will be mediated on time and on the distribution of particle velocity.

But this gives us immediately an important information: all the off-diagonal terms contain elements as v_x, v_y or v_z or their products; when we average $\langle v_x\rangle=0$ and also $\langle v_xv_y\rangle=\langle v_x\rangle\langle v_y\rangle=0$ (assuming that v_x and v_y are not correlated). Then $T^{\alpha\beta}$ is diagonal in the *LIRF*.

T^{00}_{LIRF} (expressing the mass-energy density) will be no longer ρ_0c^2, but rather ρc^2, with $\rho>\rho_0$ to take account of the fact that the particles have velocities different from zero even in *LIRF* and their mass-energy density is greater than in the case of pure dust. For the other diagonal terms we have $\langle\rho v_x^2\rangle,\langle\rho v_y^2\rangle,\langle\rho v_z^2\rangle$.

To interpret these terms we make a small digression on the kinetic theory of gases.



Let $\overline{v}$ and $\overline{P}$ be the velocity and the momentum of a particle, and f_z the average force exerted by this particle perpendicularly to the surface A (see figure)

$$\overline{v}=(v_x,v_y,v_z)\qquad\qquad \overline{P}=(P_x,P_y,P_z)$$

$$f_z=\frac{\Delta P}{\Delta t}=\frac{2P_z}{2L/v_z}=\frac{1}{L}P_zv_z$$

for one particle.

For N particles the force is (p is the pressure)

$$F_z = \frac{N}{L} P_z v_z = \frac{N}{L^3} P_z v_z L^2 \equiv p \cdot A$$

so that, by making actually the average on the velocity distribution, we get ($A = L^2$)

$$p = \frac{N}{L^3} \langle P_z v_z \rangle = n \langle P_z v_z \rangle$$

$$\overline{P} \cdot \overline{v} = P_x v_x + P_y v_y + P_z v_z = 3 P_z v_z \qquad (\text{for symmetry, on the average})$$

$$\text{and then} \qquad p = \frac{n}{3} \langle \overline{P} \cdot \overline{v} \rangle$$

which holds also for a degenerate and a relativistic gas. We can rewrite this relation in the form $p = \frac{n}{3} \langle \overline{P} \cdot \overline{v} \rangle = \langle n P_x v_x \rangle = \langle n \cdot m v_x^2 \rangle = \langle \rho \cdot v_x^2 \rangle$. So, for a perfect fluid:

$$T^{\alpha\beta}_{LIRF} = \begin{pmatrix} \rho c^2 & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}$$

where ρ takes into account also the mass-energy due to thermal motions.

It's easy to check that, in the *LIRF*, all this can be summarized in the relation

$$T^{\alpha\beta}_{LIRF} = (p + \rho c^2) u^\alpha u^\beta - p \eta^{\alpha\beta}$$

For instance, for T^{00}, by considering that $u^0 = 1$ and $\eta^{00} = 1$, we get $T^{00} = p + \rho c^2 - p = \rho c^2$

But this expression is a tensor, and then will hold in any frame of reference, with $u^\alpha \neq (1, 0, 0, 0)$ and the appropriate metric tensor instead of $\eta^{\alpha\beta}$. Written with covariant indices the energy-momentum tensor will be:

$$T_{\alpha\beta} = (p + \rho c^2) u_\alpha u_\beta - p g_{\alpha\beta}$$

. . . .

Example: *the relativistic hydrodynamics equations*

Let's see what corresponds in this case to the relation

$$\partial_\beta T^{\alpha\beta} = \left[(p + \rho c^2) u^\alpha u^\beta - p \eta^{\alpha\beta} \right]_{,\beta} = 0$$

$$\frac{\partial}{\partial x^\beta} \left[(p + \rho c^2) u^\alpha u^\beta \right] - \frac{\partial p}{\partial x^\beta} \eta^{\alpha\beta} = 0$$

For $\alpha = 0$, if we remember that $u^\alpha = \gamma\,(1, \overline{v}/c)$, we get

$$\frac{1}{c}\frac{\partial}{\partial t} \left[(p + \rho c^2) \gamma^2 \right] + \frac{\partial}{\partial x} \left[(p + \rho c^2) \gamma^2 \frac{v_x}{c} \right] + \frac{\partial}{\partial y} \left[(p + \rho c^2) \gamma^2 \frac{v_y}{c} \right] + \frac{\partial}{\partial z} \left[(p + \rho c^2) \gamma^2 \frac{v_z}{c} \right] - \frac{1}{c}\frac{\partial p}{\partial t} = 0$$

From this we get

$$\frac{\partial}{\partial t} \left[(p + \rho c^2) \gamma^2 \right] + \nabla \cdot \left[(p + \rho c^2) \gamma^2 \overline{v} \right] = \frac{\partial p}{\partial t}$$

If the overall motion of particles is not relativistic, we have $\gamma \approx 1$ and this relation reduces to the simpler form

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \left[(\rho + \frac{p}{c^2}) \overline{v} \right] = 0$$

Expandig the partial derivative we can also write the initial relation in the alternative form

$$\frac{\partial}{\partial t} \left[(p + \rho c^2) \gamma^2 \right] + (p + \rho c^2) \gamma^2 \overline{\nabla} \cdot \overline{v} + \overline{v} \cdot \overline{\nabla} \left[(p + \rho c^2) \gamma^2 \right] = \frac{\partial p}{\partial t}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[(p+\rho c^2)\gamma^2\right]+(p+\rho c^2)\gamma^2\overline{\nabla}\cdot\overline{v}=\frac{\partial p}{\partial t}$$

This relation is not particularly illuminating. We shall soon see, however, a relation more useful and understandable, obtained by transforming the $\partial_\beta T^{\alpha\beta}$ in a scalar relation.

For $\alpha = 1, 2, 3$, in a way similar to that followed in the *"dust"* case, we get

$$\left(\frac{p}{c^2}+\rho\right)\gamma^2\frac{\mathrm{d}\overline{v}}{\mathrm{d}t}=-\left[\overline{\nabla}p+\overline{v}\frac{\partial(p/c^2)}{\partial t}\right]$$

which is a generalization of the fluid-dynamics relation $\rho\frac{\mathrm{d}\overline{v}}{\mathrm{d}t}=-\overline{\nabla}p$ (*the so-called Euler equation*). As one can see, $(\rho + p/c^2)$ plays the role of "inertial mass density".

We can add the *conservation of the number of particles*, or continuity equation, which can be introduced starting from the quantity

$$J^\alpha \equiv n\,u^\alpha$$

which is a *current*, where n is the number density of particles in a frame at rest with the fluid. Imposing that the divergence of J^α is equal to zero we write the conservation of the number of particles:

$$J^\alpha_{\ ,\alpha}=\frac{\partial\,(n\,u^\alpha)}{\partial x^\alpha}=\frac{1}{c}\frac{\partial}{\partial t}\,(n\gamma)+\frac{\partial}{\partial x}\left(n\gamma\frac{v_x}{c}\right)+\frac{\partial}{\partial y}\left(n\gamma\frac{v_y}{c}\right)+\frac{\partial}{\partial z}\left(n\gamma\frac{v_z}{c}\right)=0$$

$$\frac{\partial}{\partial t}\,(n\gamma)+\overline{\nabla}\cdot(n\gamma\overline{v})=0$$

$$\frac{\partial}{\partial t}\,(n\gamma)+n\gamma\,\overline{\nabla}\cdot\overline{v}+\left(\overline{v}\cdot\overline{\nabla}\right)n\gamma=0\;\leftrightarrow\;\frac{\mathrm{d}}{\mathrm{d}t}\,(n\gamma)+(n\gamma)\,\overline{\nabla}\cdot\overline{v}=0$$

....

....

Example: *conservation of the entropy per particle*

Let's now derive, as anticipated above, a scalar relation from $\partial_\beta T^{\alpha\beta}$; to do that we multiply it by u_α.

We start fron the fact that, as we have seen, $u^\alpha u_\alpha = 1$. So

$$\begin{aligned}\frac{\partial}{\partial x^\beta}\,(u^\alpha u_\alpha) &= u^\alpha\frac{\partial u_\alpha}{\partial x^\beta}+u_\alpha\frac{\partial u^\alpha}{\partial x^\beta}\\ &= \eta^{\alpha\gamma}u_\gamma\frac{\partial u_\alpha}{\partial x^\beta}+u_\alpha\frac{\partial u^\alpha}{\partial x^\beta}\\ &= u_\gamma\frac{\partial u^\gamma}{\partial x^\beta}+u_\alpha\frac{\partial u^\alpha}{\partial x^\beta}=2u_\alpha\frac{\partial u^\alpha}{\partial x^\beta}=0\end{aligned}$$

from this $u_\alpha\frac{\partial u^\alpha}{\partial x^\beta}=0$ (we used the fact that α and γ are dummy indices). If we take the equation expressing the divergence of $T^{\alpha\beta}$ and multiply it by u_α we obtain

$$u_\alpha\,\frac{\partial}{\partial x^\beta}\left[(p+\rho c^2)u^\alpha u^\beta\right]-\frac{\partial p}{\partial x^\beta}\eta^{\alpha\beta}\,u_\alpha=0$$

and, by performing the derivative of the first term, we get

$$u_\alpha\left\{u^\alpha\frac{\partial}{\partial x^\beta}\left[(p+\rho c^2)u^\beta\right]+(p+\rho c^2)u^\beta\frac{\partial u^\alpha}{\partial x^\beta}\right\}-\frac{\partial p}{\partial x^\beta}u^\beta=0$$

If we remember that $u^\alpha u_\alpha = 1$ and $u_\alpha\frac{\partial u^\alpha}{\partial x^\beta}=0$ we can write

$$\frac{\partial}{\partial x^\beta}\left[(p+\rho c^2)u^\beta\right]-u^\beta\frac{\partial p}{\partial x^\beta}=0$$

$$(p+\rho c^2)\frac{\partial u^\beta}{\partial x^\beta}+u^\beta\frac{\partial}{\partial x^\beta}(p+\rho c^2)-u^\beta\frac{\partial p}{\partial x^\beta}=0$$

From conservation of the number of particles we have

$$\frac{\partial\left(n\,u^\beta\right)}{\partial x^\beta}=0\;\Rightarrow\;n\frac{\partial u^\beta}{\partial x^\beta}+u^\beta\frac{\partial n}{\partial x^\beta}=0\;\Rightarrow\;\frac{\partial u^\beta}{\partial x^\beta}=-\frac{u^\beta}{n}\frac{\partial n}{\partial x^\beta}$$

Substituting this result in the previous relation and collecting u^β we have

$$u^\beta\left\{\frac{\partial(p+\rho c^2)}{\partial x^\beta}-\frac{p+\rho c^2}{n}\frac{\partial n}{\partial x^\beta}-\frac{\partial p}{\partial x^\beta}\right\}=0$$

We notice now that

$$\frac{\partial}{\partial x^\beta}\left(\frac{p+\rho c^2}{n}\right)=\frac{1}{n^2}\Big[\frac{\partial(p+\rho c^2)}{\partial x^\beta}n-(p+\rho c^2)\frac{\partial n}{\partial x^\beta}\Big]=\frac{1}{n}\Big[\frac{\partial(p+\rho c^2)}{\partial x^\beta}-\frac{p+\rho c^2}{n}\frac{\partial n}{\partial x^\beta}\Big]$$

Substitute into the previous relation

$$u^\beta\left\{n\frac{\partial}{\partial x^\beta}\left(\frac{p+\rho c^2}{n}\right)-\frac{\partial p}{\partial x^\beta}\right\}=0\quad\Rightarrow\quad u^\beta\left\{n\left[\frac{\partial}{\partial x^\beta}\left(\frac{p}{n}\right)+\frac{\partial}{\partial x^\beta}\left(\frac{\rho c^2}{n}\right)\right]-\frac{\partial p}{\partial x^\beta}\right\}=0$$

$$u^\beta\left\{n\,p\frac{\partial}{\partial x^\beta}\left(\frac{1}{n}\right)+\frac{n}{n}\frac{\partial p}{\partial x^\beta}+n\frac{\partial}{\partial x^\beta}\left(\frac{\rho c^2}{n}\right)-\frac{\partial p}{\partial x^\beta}\right\}=0$$

$$n\,u^\beta\left\{p\frac{\partial}{\partial x^\beta}\left(\frac{1}{n}\right)+\frac{\partial}{\partial x^\beta}\left(\frac{\rho c^2}{n}\right)\right\}=0$$

Recall now the first law of thermodynamics: $dU=dQ+dL$; if we introduce the entropy S we can write: $TdS=dU+pdV$, where the internal energy is $U=\rho c^2$. If we rewrite it referring to a particle we have $Td\sigma=d\left(\frac{\rho c^2}{n}\right)+pd\left(\frac{1}{n}\right)$, with σ entropy per particle. Expanding the differentials

$$T\frac{\partial\sigma}{\partial x^\beta}dx^\beta=\frac{\partial}{\partial x^\beta}\left(\frac{\rho c^2}{n}\right)dx^\beta+p\frac{\partial}{\partial x^\beta}\left(\frac{1}{n}\right)dx^\beta\qquad /\cdot\frac{1}{ds}$$

If we remember that $\frac{dx^\beta}{ds}\equiv u^\beta$ and compare this reltation with the previous one we get

$$u^\beta\frac{\partial\sigma}{\partial x^\beta}=0$$

which, when expanded, becomes

$$\gamma\frac{1}{c}\frac{\partial\sigma}{\partial t}+\gamma\frac{v_x}{c}\frac{\partial\sigma}{\partial x}+\gamma\frac{v_y}{c}\frac{\partial\sigma}{\partial y}+\gamma\frac{v_z}{c}\frac{\partial\sigma}{\partial z}=0$$

$$\frac{\partial\sigma}{\partial t}+\left(\bar{v}\cdot\bar{\nabla}\right)\sigma=0\qquad\Longleftrightarrow\qquad\frac{d\sigma}{dt}=0$$

....

The result is that, in the system in which the fluid is at rest, the entropy per particle (or, if preferred, the entropy for a certain number$N$ of particles contained in a cubic volume V of edge$L$, which can vary but maintain always inside the same number of particles) is constant. This is related to the fact that, in the ideal fluid, there is no exchange of energy by conduction (or radiation), nor is there dissipation. From the first law of thermodynamics, in the frame that follows the fluid, $\mathrm{d}Q=\mathrm{d}U+p\mathrm{d}V$ and $U=\rho c^2\cdot V$. Then

$$\mathrm{d}Q=\rho c^2\mathrm{d}V+V\mathrm{d}(\rho c^2)+p\mathrm{d}V=(p+\rho c^2)\mathrm{d}V+V\mathrm{d}(\rho c^2)=T\mathrm{d}S$$

Since $\mathrm{d}Q=0\quad\rightarrow\quad\mathrm{d}S=0$.

If we write $p=w\rho c^2$ (with w constant, although, in general, may be $w=w(T)$),

$$(1+w)\rho c^2\mathrm{d}V=-V\;\mathrm{d}(\rho c^2)$$

and if $w=const$, we have $\mathrm{d}\rho/\rho=-(1+w)\;\mathrm{d}V/V$, that is $\rho V^{1+w}=$ constant.

We will meet three interesting cases in cosmology:

1. For a non-relativistic gas $p \ll \rho_0 c^2$ $(\rho \approx \rho_0)$ so that $w \simeq 0$ and $\rho_0 V \simeq const$. If L is the edge of a cubic volume $V = L^3$, we obtain $\rho \propto 1/L^3$
2. For a gas of photons (and in general for a relativistic gas) $\rho_{rad} \propto aT^4$ and $p = \frac{1}{3}\rho c^2$; $w = \frac{1}{3}$:

$$T^4 V^{4/3} = const \qquad TV^{1/3} = const \qquad V \propto L^3 \quad \rightarrow \quad T \propto \frac{1}{L}$$

$$\rho_{rad} V^{4/3} = const \qquad V \propto L^3 \quad \rightarrow \quad V^{4/3} \simeq L^4 \quad \rightarrow \quad \rho_{rad} \simeq \frac{1}{L^4}$$

3. If $p = -\rho c^2 \quad (w = -1) \quad \rightarrow \quad \rho V^0 = const$ that is ρ does not depend on V and L and remains constant if V changes.

We can express the first principle in another useful way by writing $V \propto L^3$

$$\left(\rho + \frac{p}{c^2}\right) \mathrm{d}V + V \mathrm{d}\rho = 0 \qquad \rightarrow \qquad \left(\rho + \frac{p}{c^2}\right) \cdot 3L^2 \mathrm{d}L + L^3 \mathrm{d}\rho = 0$$

which gives

$$3\left(\rho + \frac{p}{c^2}\right) \frac{\mathrm{d}L}{L} + \mathrm{d}\rho = 0$$

and, taking into account a possible dependence of L on time,

$$3\left(\rho + \frac{p}{c^2}\right) \frac{\dot{L}}{L} + \dot{\rho} = 0$$

We wrote $\partial_\alpha T^{\beta\alpha} = 0$ in Minkowski space; but, if the $\Gamma^\alpha_{\beta\gamma}$ do not all vanish, and this is the general case, instead of the simple partial derivative we must use the covariant derivative:

$$T^{\alpha\beta}{}_{;\beta} = 0$$

that expresses the conservation laws in a generic frame of reference.

3.3 Mach principle

According to Newton's dynamics the inertial properties of a body depend on its motion with respect to absolute space. *Ernst Mach*[8] suggested instead that the inertia is related to the motion with respect to the total distribution of matter in the universe. The motion is only relative to other bodies: operationally we can only measure the motion of matter in relation to other matter, not with respect to the absolute space of Newton. If there was only one body in the universe, its motion would not be defined: without other matter we can not say if this body is at rest or is accelerating. And since the reaction of matter to the acceleration is the only way to determine the inertia, this body does not possesses inertia. The idea that masses and positions of celestial bodies define the inertia and inertial systems is called **Mach principle**. Several objections can be moved to this idea: for instance, no observer can be in an empty universe and verify the ideas of Mach, and inertia may exist even in an empty universe.

Anyway, the ideas of Mach influenced, by his own admission, Einstein himself. According to Newtonian physics, in an volume without interactions, the bodies should remain at rest or move with uniform motion. But since the universe is permeated by gravitational fields that can not be shielded, all bodies move along curved paths due to these fields. But then the question arises: if we say that a path is curved, we assume that we know how to define a straight line. But how can we do this if no body, not even photons, as we shall see, follows a straight line? So we try to do without the concept of straight line, and assume that there are no physical entities such as "gravitational forces" curving the trajectories of the heavenly bodies, but that the geometry of the space is modified by the gravitation in such a way that the trajectories observed correspond to free, inertial motion of bodies. But how to express this link between inertial motion and gravitation?

Special Relativity can be described by a geometry of Minkowski $\mathrm{d}s^2 = \eta_{\alpha\beta} \mathrm{d}x^\alpha \mathrm{d}x^\beta$ and from the properties of invariance of $\mathrm{d}s^2$ between inertial systems derive the results of this theory (time dilation, length contraction, ...). How do we move to a metric $\mathrm{d}s^2 = g_{\alpha\beta} \mathrm{d}x^\alpha \mathrm{d}x^\beta$ in the presence of a gravitational field? What links are there between $g_{\alpha\beta}$ and the gravitational field, and between $g_{\alpha\beta}$ and the gravitation according to Newton? Set in this way, General Relativity turns out to be a geometrical theory of gravitation.

[8] *Ernst Mach (1838-1916) was professor of physics and then philosophy at the University of Vienna. His ideas have had a precursor in the English bishop and philosopher George Berkeley, in 1710, when Newton was still alive.*

3.4 Locally inertial frames

If our aim is to give a geometric description of space-time, we can use what we have already learned about surface elements and generalize it in 4 dimensions. In particular, we have seen that, in a neighborhood of a generic point, one can transform $g_{\alpha\beta}$ in such a way that it has a given form and that its first derivatives are zero. So, in the neighborhood of an event, we can always put $g_{\alpha\beta} \equiv \eta_{\alpha\beta} + \mathcal{O}(|x|^2)$: at the first order geometry is the same as that of Special Relativity. In the (infinitesimal) neighborhood of each event the laws of physics are the ones that hold in a inertial frame of reference. In a neighborhood of each event we can define a **locally inertial reference frame**.

In the presence of gravitational fields, as mentioned above, local deviations from Special Relativity occur only at the level of the second derivatives of $g_{\alpha\beta}$ which, remember, are related to the curvature tensor $R^{\alpha}_{\beta\gamma\delta}$. In this sense **gravity curves space-time**. But what are these locally inertial reference frames?

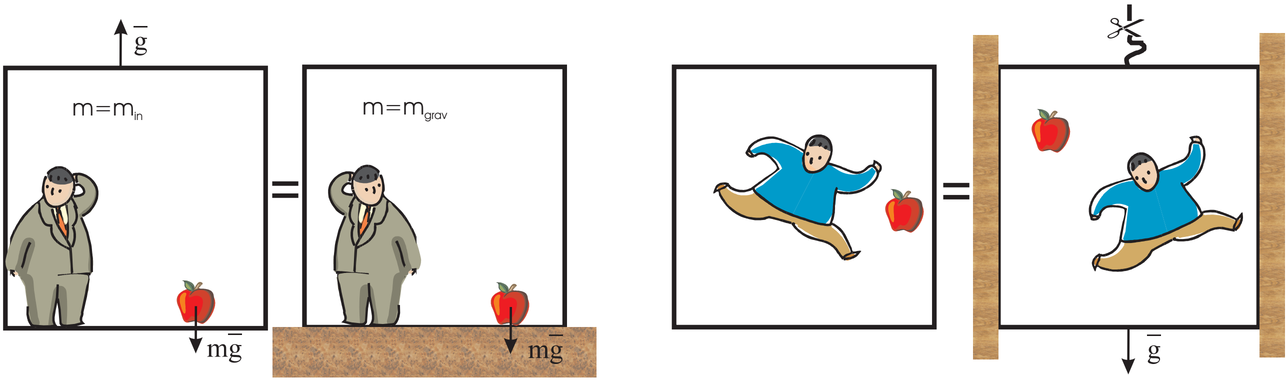
3.5 The Principle of Equivalence

The evidence that all bodies fall (in the absence of air resistance) in the same way under the effect of gravity, led to conclude, with great precision, that inertial mass m_{in} and gravitational mass m_{grav} are mutually proportional (and are, in practice, the same, by including the constant of proportionality within the gravitational constant G). Einstein assumed that, *by definition*, $m_{in} \equiv m_{grav}$. This leads to the famous thought experiment of **Einstein elevator**: an observer, equipped with scientific instruments and locked up into an elevator without the possibility to see what is happening around him, will not be able to distinguish, by his experiments in mechanics, between the two situations:

- he is at rest in a gravitational field with gravitational acceleration $\overline{g}$
- he is in empty space, and the elevator is accelerated upward with constant acceleration $\overline{g}$

Similarly, since all bodies fall in the same way in a gravitational field, the observer will not be able to distinguish between the situations of:

- uniform rectilinear motion in the vacuum
- free fall in a gravitational field



This allows us to say what are the *locally inertial frames: those in free fall.* Then, in a free falling frame, the laws of Special Relativity hold locally (and to the first order in $g_{\alpha\beta}$).

The Principle of Equivalence requires that all the laws of physics (not just those of mechanics) are the same both in a locally inertial frame and in Special Relativity.

Since the effects of gravitation disappear in a system in free fall, the phenomena occurring there are totally independent from the presence of nearby masses. However, according to the point of view of Mach, a large, nearby mass should introduce an anisotropy of the inertial mass. Effects due to the Sun or our Galaxy have been searched, but not found within $\Delta m/m \sim 10^{-20}$, for which the Principle of Equivalence seems favored over the assumptions of Mach (so they are not completely consistent with General Relativity, apart from the inspiration provided to Einstein[9]).

[9]Einstein conceived his theory of General Relativity trying to incorporate the idea of Mach according to which the inertia is due to gravitational interactions with all matter in the universe. But, as admitted by himself, he was only partially successful, since he obtained a solution of his field equations in which a single particle, immersed in a completely empty universe, had inertial properties.

3.6 The Principle of General Covariance

This principle tells us how to write the equations of physics in the presence of a gravitational field, when we know how they are made in the absence of gravity.

In order for an equation, expressing a physical law, applies in a gravitational field it is necessary that:

1. It is "covariant", i.e. does not change shape changing reference frame, and this happens when it is expressed as a relationship between tensors.
2. The equation applies in the absence of gravity, i.e. when $g_{\alpha\beta} \equiv \eta_{\alpha\beta}$ and $\Gamma^{\alpha}_{\beta\gamma} \equiv 0$.

There can be many covariant equations which are reduced, in the absence of gravity, to the same equation of Special Relativity. However, as the Principle of General Covariance and the Priciple of Equivalence, operate on small scales, we expect that only $g_{\alpha\beta}$ and its low order derivatives come into play. This also obeys a principle of simplicity.

In this way${T^{\alpha\beta}}_{,\beta} \to {T^{\alpha\beta}}_{;\beta}$ (covariant derivative) or, for instance, for the free fall equation,

$$\frac{\mathrm{d}^2 x^\alpha}{\mathrm{d}s^2} = 0 \to \frac{\mathrm{d}^2 x^\alpha}{\mathrm{d}s^2} + \Gamma^{\alpha}_{\beta\gamma}\frac{\mathrm{d}x^\beta}{\mathrm{d}s}\frac{\mathrm{d}x^\gamma}{\mathrm{d}s} = 0$$

We have seen that at each point (event) we can define a locally inertial system, and in it the second derivatives of $g_{\alpha\beta}$ are in general $\neq 0$: i*t is therefore at the level of the second derivatives of the metric tensor that the gravitational field comes into play.*

Similarly, in Newtonian physics, in a system in free fall, what can be measured is the difference in gravitational acceleration between two bodies $\Delta g/\Delta x$. This is the kind of phenomenon we call **tide**. But $\overline{g} = -\overline{\nabla}\Phi_{grav}$ and then $\partial g/\partial x \propto \partial^2 \Phi_{grav}/\partial x^2$. What can be measured are therefore the second derivatives of Φ_{grav}, as in General Relativity are the second derivatives of $g_{\alpha\beta}$. Then we see that there is an analogy between $g_{\alpha\beta}$ and Φ_{grav}: the $g_{\alpha\beta}$ take the place of the Newtonian gravitational potential.

3.7 The Einstein equations

In Newton's theory of gravitation the potential Φ satisfies Poisson equation: $\nabla^2\Phi = 4\pi G\rho_0$ and $\overline{g} = -\overline{\nabla}\Phi$. Special Relativity teaches us that all forms of energy are equivalent to mass, and then a relativistic theory of gravity will have as sources of the gravitational field all forms of energy, and not just ρ_0. In particular, the energy density of the gravitational field itself is proportional to $(\overline{\nabla}\Phi)^2$ in the Newtonian case [10](think, by analogy, that the energy density of the electromagnetic field is proportional to E^2).

If, therefore, we carry on the left, in Poisson equation, the term $\propto (\overline{\nabla}\Phi)^2$ which would result from the gravitational energy density, we obtain a non-linear differential equation (which will linear in the second derivative and quadratic in the first one) for the gravitational field.

Formally we will have an equation such as:

$$F(g) \sim \kappa T$$

where g is the metric tensor (corresponding to Φ), F is a differential operator (likely something linear in the second derivatives and quadratic in the first derivatives) which reduces to ∇^2 in the weak field limit, when Newton's law holds, κ is a proportionality constant that contains G, T is a quantity that describes all forms of non-gravitational energy, and that, in the non-relativistic case, should essentially be reduced to ρ_0.

A natural candidate for T is the component T^{00} of the stress-energy tensor. But keeping as a source of the field only one component of a tensor would not produce an invariant theory: we should adopt a particular reference frame to calculate T^{00}. Hence arose the idea of Einstein to use as source the entire $T^{\alpha\beta}$: pressure,stresses (if $T^{\alpha\beta}$ is not diagonal), etc. .. all acts as a source. But if T is a tensor, then the left-hand side of the equation must be a also tensor function of the metric tensor.

[10]We can see that the energy density of the gravitational field is proportional to $(\overline{\nabla}\Phi)^2$, that is to g^2, in the following way. The gravitational potential energy of a mass $M$of radius R is given by $E = -GM^2/R$. If you think that this energy is distributed in the field ($g \propto M/r^2$) created by M, between R and ∞, we see that, by calling δ_G the density of gravitational energy, for it to be $-\int_R^\infty \delta_G(r)\cdot 4\pi r^2 \mathrm{d}r \approx -M^2/R$, $\delta_G(r) \sim (M/r^2)^2 \sim g^2$ is required.

But $T^{\alpha\beta}$ is symmetrical, and has vanishing covariant divergence: $T^{\alpha\beta}{}_{;\beta} = 0$. Then the left-hand side must share these properties. Moreover, we expect it to be linear in the second derivatives of $g_{\alpha\beta}$ and quadratic in the first derivatives.

But we have already met a tensor with these properties, and we have seen that it is unique: the Einstein tensor

$$G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R$$

Einstein therefore proposed, as a possible equation of the gravitational field,

$$R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta} = \kappa T_{\alpha\beta}$$

If we think to include the derivatives of order zero in the differential operator, being $g_{\alpha\beta}$ symmetric and with vanishing covariant divergence, we can add a term proportional to $g_{\alpha\beta}$:

$$R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta} - \Lambda g_{\alpha\beta} = \kappa T_{\alpha\beta}$$

Λ and κ are constant; Λ is the so-called **cosmological constant**.

3.8 The Newtonian limit (weak field)

Once written Einstein's equations, we must check that, within the limits of validity of classical physics, they reduce to Newton's law; we must also find what is the constant κ that appears in the equations.

Let us suppose that the field is stationary (i.e. its time derivative is zero), the velocities of the particles are small ($v \ll c$) and that, at large distances from the masses that generate the field, the metric tensor is asymptotically flat: $g_{\alpha\beta} \to \eta_{\alpha\beta}$. We also assume that the field is weak: the deviations from metric $\eta_{\alpha\beta}$ are small:

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta} \qquad \text{with } |h| \ll 1$$

Since $v/c \ll 1$ we have

$$\begin{aligned} \frac{\mathrm{d}x^0}{\mathrm{d}s} &= \frac{c\mathrm{d}t}{c\mathrm{d}\tau} = \frac{\mathrm{d}t}{\mathrm{d}\tau} \\ \frac{\mathrm{d}x^i}{\mathrm{d}s} &= \frac{\mathrm{d}x^i}{c\mathrm{d}\tau} = \frac{1}{c}\frac{\mathrm{d}x^i}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}\tau} = \frac{v^i}{c}\frac{\mathrm{d}t}{\mathrm{d}\tau} \ll \frac{\mathrm{d}t}{\mathrm{d}\tau} \equiv \frac{\mathrm{d}x^0}{\mathrm{d}s} \end{aligned}$$

The geodesic equation is, as usual,

$$\frac{\mathrm{d}^2 x^\alpha}{\mathrm{d}s^2} + \Gamma^\alpha_{\beta\gamma} \frac{\mathrm{d}x^\beta}{\mathrm{d}s}\frac{\mathrm{d}x^\gamma}{\mathrm{d}s} = 0$$

but, keeping α fixed, in the sum on the indices β and γ, the terms containing the $\mathrm{d}x^i/\mathrm{d}s$ are negligible compared to the term containing $\big(\mathrm{d}x^0/\mathrm{d}s\big)\big(\mathrm{d}x^0/\mathrm{d}s\big)$, so

$$\frac{\mathrm{d}^2 x^\alpha}{\mathrm{d}s^2} + \Gamma^\alpha_{00}\left(\frac{\mathrm{d}x^0}{\mathrm{d}s}\right)^2 = \frac{\mathrm{d}^2 x^\alpha}{\mathrm{d}s^2} + \Gamma^\alpha_{00}\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 \simeq 0$$

By the assumption that $g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}$ $(|h| \ll 1)$ we can evaluate $g^{\alpha\beta}$. We know that, by definition, $g_{\alpha\delta} g^{\delta\beta} \equiv \delta^\beta_\alpha$ and that $\eta_{\alpha\delta}\eta^{\delta\beta} \equiv \delta^\beta_\alpha$. We define the quantity $h^{\gamma\delta} \equiv \eta^{\gamma\alpha}\eta^{\delta\beta} h_{\alpha\beta}$ and show that

$$(\eta_{\alpha\beta} + h_{\alpha\beta})(\eta^{\beta\delta} - h^{\beta\delta}) = \delta^\delta_\alpha :$$

Expanding the left hand side, and neglecting second order terms in h,

$$(\eta_{\alpha\beta} + h_{\alpha\beta})(\eta^{\beta\delta} - h^{\beta\delta}) = \eta_{\alpha\beta}\eta^{\beta\delta} - \eta_{\alpha\beta}h^{\beta\delta} + h_{\alpha\beta}\eta^{\beta\delta} - h_{\alpha\beta}h^{\beta\delta} =$$
$$= \delta_\alpha^\delta - \eta_{\alpha\beta}\eta^{\beta\sigma}\eta^{\delta\tau}h_{\sigma\tau} + h_{\alpha\beta}\eta^{\beta\delta} = \delta_\alpha^\delta$$

In fact $\eta_{\alpha\beta}\eta^{\beta\sigma} \equiv \delta_\alpha^\sigma$, $\delta_\alpha^\sigma h_{\sigma\tau} = h_{\alpha\tau}$ and $\eta^{\delta\tau}h_{\alpha\tau} \equiv h_{\alpha\beta}\eta^{\delta\beta}$, since τ is a dummy index and we can name it β. We than see that $\eta^{\beta\delta} - h^{\beta\delta} = g^{\beta\delta}$.

Let us calculate Γ^α_{00} (remember that stationarity implies that the derivatives with respect to x^0 are zero):

$$\Gamma^\alpha_{00} = \frac{1}{2}g^{\alpha\gamma}\left[\frac{\partial g_{0\gamma}}{\partial x^0} + \frac{\partial g_{0\gamma}}{\partial x^0} - \frac{\partial g_{00}}{\partial x^\gamma}\right] = \frac{1}{2}\Big(\eta^{\alpha\gamma} - h^{\alpha\gamma}\Big)\Big(-\frac{\partial g_{00}}{\partial x^\gamma}\Big) \simeq -\frac{1}{2}\eta^{\alpha\gamma}\frac{\partial h_{00}}{\partial x^\gamma}$$

at the first order in h. Hence

$$\frac{\mathrm{d}^2x^\alpha}{\mathrm{d}s^2} \simeq \frac{1}{2}\eta^{\alpha\gamma}\frac{\partial h_{00}}{\partial x^\gamma}\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2$$

- For $\alpha = 0$ we have:

$$\frac{\mathrm{d}^2x^0}{\mathrm{d}s^2} = \frac{1}{2}\eta^{00}\frac{\partial h_{00}}{\partial x^0}\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 = 0 \qquad \Rightarrow \qquad \frac{\mathrm{d}x^0}{\mathrm{d}s} = \text{const} = \frac{\mathrm{d}t}{\mathrm{d}\tau}$$

- For $\alpha = 1, 2, 3$ instead:

$$\frac{\mathrm{d}^2x^i}{\mathrm{d}s^2} = \frac{\mathrm{d}^2x^i}{c^2\mathrm{d}\tau^2} = \frac{1}{c^2}\frac{\mathrm{d}}{\mathrm{d}\tau}\left[\frac{\mathrm{d}x^i}{\mathrm{d}\tau}\right] = \frac{1}{c^2}\frac{\mathrm{d}t}{\mathrm{d}\tau}\cdot\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\mathrm{d}t}{\mathrm{d}\tau}\frac{\mathrm{d}x^i}{\mathrm{d}t}\right] = \frac{1}{c^2}\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2\frac{\mathrm{d}^2x^i}{\mathrm{d}t^2}$$

so that

$$\frac{\mathrm{d}^2x^i}{\mathrm{d}s^2} = \frac{1}{c^2}\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2\frac{\mathrm{d}^2x^i}{\mathrm{d}t^2} \simeq \frac{1}{2}\eta^{i\gamma}\frac{\partial h_{00}}{\partial x^\gamma}\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 \qquad (\eta^{i\gamma} = -1 \;\text{ if }\; i = \gamma)$$

which means that

$$\frac{1}{c^2}\frac{\mathrm{d}^2x^i}{\mathrm{d}t^2} \simeq -\frac{1}{2}\frac{\partial h_{00}}{\partial x^i} \qquad \text{and, by using vector notation,} \qquad \frac{1}{c^2}\frac{\mathrm{d}^2\overline{x}}{\mathrm{d}t^2} \simeq -\frac{1}{2}\overline{\nabla}h_{00}$$

But, according to Newton's gravity law, using Φ for the potential,

$$\frac{\mathrm{d}^2\overline{x}}{\mathrm{d}t^2} = -\overline{\nabla}\Phi$$

and, by comparing the two results:

$$-\overline{\nabla}\Phi \simeq -\frac{c^2}{2}\overline{\nabla}h_{00} \qquad \rightarrow \qquad h_{00} \simeq \frac{2\Phi}{c^2} + \text{const.}$$

If far from the field sources (masses), $\Phi \to 0$ and also $h_{00} \to 0$ since we assume that $g_{\alpha\beta} \to \eta_{\alpha\beta}$, the constant has to be set to zero, and then

$$h_{00} \simeq \frac{2\Phi}{c^2} \qquad \rightarrow \qquad g_{00} \simeq 1 + \frac{2\Phi}{c^2}$$

The weak field hypothesis, $|h| \ll 1$, implies that $2\Phi/c^2 \ll 1$.

In the case of a mass M in which the density is distributed with spherical symmetry, the external potential is given by $\Phi = -GM/r$, according to Newton. The assumption that the field is weak implies that $|2\Phi/c^2| \ll 1$, i.e.

$$\frac{2GM}{rc^2} \ll 1$$

For a black hole or a generic spherical body, $R_S \equiv 2GM/c^2$ is the so-called *Schwarzschild radius*, corresponding, for a non-rotating and electrically neutral black hole, to the *event horizon*, the zone from which nothing can come out (apart from quantum effects of evaporation). In this case we see that the condition of weak field is

$$\frac{R_S}{r} \ll 1 \qquad \Rightarrow \qquad r \gg R_S$$

For our Sun, $R_S \sim 3\,\mathrm{km}$.

Let's see now, with the same assumptions made above, that the Einstein equations reduce to Poisson equation $\nabla^2\Phi = 4\pi G\rho_0$. Then we will determine the value of the constant κ. The curvature tensor is:

$$R^{\alpha}_{\ \beta\gamma\delta} = \frac{\partial\Gamma^{\alpha}_{\beta\delta}}{\partial x^{\gamma}} - \frac{\partial\Gamma^{\alpha}_{\beta\gamma}}{\partial x^{\delta}} + \Gamma^{\sigma}_{\beta\delta}\Gamma^{\alpha}_{\sigma\gamma} - \Gamma^{\sigma}_{\beta\gamma}\Gamma^{\alpha}_{\sigma\delta} \simeq \frac{\partial\Gamma^{\alpha}_{\beta\delta}}{\partial x^{\gamma}} - \frac{\partial\Gamma^{\alpha}_{\beta\gamma}}{\partial x^{\delta}} \qquad \text{(the other terms are of the second order, } \mathcal{O}(h^2)\text{)}$$

The Christoffel symbols are:

$$\Gamma^{\alpha}_{\beta\gamma} = \frac{1}{2}g^{\alpha\sigma}\left(\frac{\partial g_{\beta\sigma}}{\partial x^{\gamma}} + \frac{\partial g_{\gamma\sigma}}{\partial x^{\beta}} - \frac{\partial g_{\beta\gamma}}{\partial x^{\sigma}}\right) \simeq \frac{1}{2}\eta^{\alpha\sigma}\left(\frac{\partial h_{\beta\sigma}}{\partial x^{\gamma}} + \frac{\partial h_{\gamma\sigma}}{\partial x^{\beta}} - \frac{\partial h_{\beta\gamma}}{\partial x^{\sigma}}\right) \qquad \text{(at the order } \mathcal{O}(h)\text{)}$$

The Ricci tensor is obtained from $R^{\alpha}_{\ \beta\gamma\delta}$ by contracting the first and third index:

$$\begin{aligned}
R_{\beta\delta} = R^{\alpha}_{\ \beta(\gamma\equiv\alpha)\delta} &= \frac{\partial\Gamma^{\alpha}_{\beta\delta}}{\partial x^{\alpha}} - \frac{\partial\Gamma^{\alpha}_{\beta\alpha}}{\partial x^{\delta}} = \\
&= \frac{1}{2}\eta^{\alpha\sigma}\frac{\partial}{\partial x^{\alpha}}\left(\frac{\partial h_{\beta\sigma}}{\partial x^{\delta}} + \frac{\partial h_{\delta\sigma}}{\partial x^{\beta}} - \frac{\partial h_{\beta\delta}}{\partial x^{\sigma}}\right) - \frac{1}{2}\eta^{\alpha\sigma}\frac{\partial}{\partial x^{\delta}}\left(\frac{\partial h_{\beta\sigma}}{\partial x^{\alpha}} + \frac{\partial h_{\alpha\sigma}}{\partial x^{\beta}} - \frac{\partial h_{\beta\alpha}}{\partial x^{\sigma}}\right) = \\
&= \frac{1}{2}\eta^{\alpha\sigma}\left[\frac{\partial^2 h_{\beta\sigma}}{\partial x^{\alpha}\partial x^{\delta}} + \frac{\partial^2 h_{\delta\sigma}}{\partial x^{\alpha}\partial x^{\beta}} - \frac{\partial^2 h_{\beta\delta}}{\partial x^{\alpha}\partial x^{\sigma}} - \frac{\partial^2 h_{\beta\sigma}}{\partial x^{\delta}\partial x^{\alpha}} - \frac{\partial^2 h_{\alpha\sigma}}{\partial x^{\delta}\partial x^{\beta}} + \frac{\partial^2 h_{\beta\alpha}}{\partial x^{\delta}\partial x^{\sigma}}\right] = \\
&= \frac{1}{2}\eta^{\alpha\sigma}\left[\frac{\partial^2 h_{\delta\sigma}}{\partial x^{\alpha}\partial x^{\beta}} + \frac{\partial^2 h_{\beta\alpha}}{\partial x^{\delta}\partial x^{\sigma}} - \frac{\partial^2 h_{\beta\delta}}{\partial x^{\alpha}\partial x^{\sigma}} - \frac{\partial^2 h_{\alpha\sigma}}{\partial x^{\delta}\partial x^{\beta}}\right]
\end{aligned}$$

Let us take Einstein equation with the term containing Λ brought to the right:

$$R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R = \kappa T_{\alpha\beta} + \Lambda g_{\alpha\beta}$$

If we multiply it by $g^{\alpha\gamma}$ we get:

$$R^{\gamma}_{\ \beta} - \frac{1}{2}R\delta^{\gamma}_{\beta} = \kappa T^{\gamma}_{\ \beta} + \Lambda\delta^{\gamma}_{\beta}$$

Let us put$\gamma = \beta$ (that is, we add on $\beta = \gamma = 0, 1, 2, 3$, $\delta^{\beta}_{\beta} = \delta^0_0 + \delta^1_1 + \delta^2_2 + \delta^3_3 = 1 + 1 + 1 + 1 = 4$) and contract tensors; since $R = R^{\gamma}_{\ \gamma}$ we get:

$$R - \frac{1}{2}R \cdot 4 = \kappa T^{\gamma}_{\ \gamma} + 4\Lambda \qquad \rightarrow \qquad R = -\kappa T^{\gamma}_{\ \gamma} - 4\Lambda$$

Substituting this result into the starting equation, it becomes:

$$R_{\alpha\beta} = \kappa T_{\alpha\beta} + \Lambda g_{\alpha\beta} + \frac{1}{2}g_{\alpha\beta}\big(-\kappa T^{\gamma}_{\ \gamma} - 4\Lambda\big) = \kappa\big(T_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}T^{\gamma}_{\ \gamma}\big) - \Lambda g_{\alpha\beta}$$

We evaluate now, always with the assumed approximations, the component00 of $R_{\alpha\beta}$:

$$R_{00} \simeq \frac{1}{2}\eta^{\alpha\sigma}\left[\frac{\partial^2 h_{0\sigma}}{\partial x^{\alpha}\partial x^{0}} + \frac{\partial^2 h_{0\alpha}}{\partial x^{0}\partial x^{\sigma}} - \frac{\partial^2 h_{00}}{\partial x^{\alpha}\partial x^{\sigma}} - \frac{\partial^2 h_{\alpha\sigma}}{\partial x^{0}\partial x^{0}}\right] \simeq -\frac{1}{2}\eta^{\alpha\sigma}\frac{\partial^2 h_{00}}{\partial x^{\alpha}\partial x^{\sigma}} \qquad \text{(for stationarity } \partial/\partial x^0 = 0\text{)}$$

but if $\alpha = 0$ and/or $\sigma = 0$ the derivative is zero; then remain only the terms with indices $1, 2, 3$ (and $\eta^{11} = \eta^{22} = \eta^{33} = -1$):

$$R_{00} \simeq \frac{1}{2}\left[\frac{\partial^2 h_{00}}{\partial x^1\partial x^1} + \frac{\partial^2 h_{00}}{\partial x^2\partial x^2} + \frac{\partial^2 h_{00}}{\partial x^3\partial x^3}\right] \simeq \frac{1}{2}\nabla^2 h_{00}$$

On the other hand

$$R_{00} = \kappa \left[T_{00} - \frac{1}{2} g_{00} T^{\gamma}_{\gamma} \right] - \Lambda g_{00}$$

For $T^{\alpha\beta}$ in the *dust* case: $T^{\alpha\beta} = \rho_0 c^2 u^\alpha u^\beta$, neglecting, with respect to $\rho_0 c^2$, terms containing pressure. We get

$$T_{\mu\nu} = g_{\mu\alpha} g_{\nu\beta} \rho_0 c^2 u^\alpha u^\beta \qquad (u^i \ll u^0 \simeq 1)$$

$$T_{00} = g_{0\alpha} g_{0\beta} \rho_0 c^2 u^\alpha u^\beta \simeq g_{00} g_{00} \rho_0 c^2 u^0 u^0 \simeq g_{00}^2 \rho_0 c^2$$

But $g_{00} = 1 + h_{00} \simeq 1$ and so

$$T_{00} \simeq \rho_0 c^2$$

$$T^{\alpha}_{\gamma} = g_{\gamma\beta} T^{\alpha\beta} = g_{\gamma\beta} \rho_0 c^2 u^\alpha u^\beta$$

$$T^{\gamma}_{\gamma} = g_{\gamma\beta} \rho_0 c^2 u^\gamma u^\beta \simeq g_{00} \rho_0 c^2 u^0 u^0 (\text{+negligible terms}) \simeq g_{00} \rho_0 c^2 \simeq \rho_0 c^2$$

We then have

$$\frac{1}{2} \nabla^2 h_{00} \simeq \kappa \big(\rho_0 c^2 - \frac{1}{2} \rho_0 c^2 \big) - \Lambda \qquad \rightarrow \qquad \nabla^2 h_{00} \simeq \kappa \rho_0 c^2 - 2\Lambda$$

but $h_{00} = 2\Phi/c^2$ and then:

$$\nabla^2 \Phi \simeq \frac{\kappa \rho_0 c^4}{2} - \Lambda c^2 \simeq \kappa \left(\frac{\rho_0 c^4}{2} - \frac{c^2 \Lambda}{\kappa} \right)$$

Poisson equations tells that $\nabla^2 \Phi = 4\pi g \rho_0$; the two relations coicide if

$$(A) \qquad \frac{\rho_0 c^4}{2} \gg \left| \frac{c^2 \Lambda}{\kappa} \right|$$

$$(B) \qquad 4\pi G \rho_0 = \frac{\kappa \rho_0 c^4}{2} \quad \rightarrow \quad \kappa = \frac{8\pi G}{c^4}$$

Finally we arrive to the complete Einstein equation

$$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R - \Lambda g_{\alpha\beta} = \frac{8\pi G}{c^4} T_{\alpha\beta}$$

The above conditio on Λ becomes:

$$|\Lambda| \ll \frac{4\pi G \rho_0}{c^2} = \Lambda_E$$

In 1916, when Einstein wrote the equations of General Relativity, he was not aware of cosmic expansion, and sought a static solution for his model of universe. We see, from the "classical" point of view, that if $\Lambda = \Lambda_E$ and ρ_0 is the density of the universe, we have $\nabla^2 \Phi = 0$, $\Phi = const$, $\overline{g} = -\overline{\nabla} \Phi = 0$.

A similar result comes from the equations of General Relativity. This static model, however, is unstable: just a small density fluctuation and locally we have expansion or contraction.

According to dimensional analysis $[c^2 \Lambda] = [4\pi g \rho_0] = [\nabla^2 \Phi]$ that is $[\Lambda] = [\nabla^2 (\Phi/c^2)] = L^{-2}$(remember that Φ/c^2 is adimensional).

From the relation $\nabla^2 \Phi = 4\pi G [\rho_0 - c^2 \Lambda / 4\pi G]$ we can think that Λ corresponds to the mass-energy of vacuum.

To estimate an upper limit of Λ we can assume for ρ_0 the average density of a gravitating systems for which Newton's laws are good and therefore requires $\Lambda \sim 0$. If we take as gravitating system the solar system (mass $= M_\odot = 2 \cdot 10^{33} g$, radius of the orbit of Pluto $\sim 6 \cdot 10^9 km$) we obtain

$$|\Lambda_{SS}| \ll 2 \cdot 10^{-39} cm^{-2}$$

If we use a cluster of galaxies as self gravitating system (but in this case the confidence in Newton's laws is lower), with a mass equal to $\sim 10^{15} h^{-1} M_\odot$ and radius $\sim 3 h^{-1} Mpc$, we obtain

$$|\Lambda_{SS}| \ll 10^{-54} h^2 cm^{-2}$$

After the discovery, by Hubble, that the universe expands, Einstein described the introduction of Λ as the biggest mistake of his life, but, as we shall see, it has come back strongly in vogue in recent years.

Recent observations (1997) based on Type Ia supernovae in distant galaxies, and the study of the cosmic microwave background, we have obtained no longer an upper limit, but a possible estimate of$\Lambda \sim 2 \cdot 10^{-56} h^2 cm^{-2}$.

3.9 Weak field metric, gauge transformations and gravitational waves

We have seen, treating the weak field, that Ricci tensor can be written

$$R_{\beta\delta} \simeq \frac{1}{2}\eta^{\alpha\sigma}\left[\frac{\partial^2 h_{\delta\sigma}}{\partial x^\alpha \partial x^\beta} + \frac{\partial^2 h_{\beta\alpha}}{\partial x^\delta \partial x^\sigma} - \frac{\partial^2 h_{\beta\delta}}{\partial x^\alpha \partial x^\sigma} - \frac{\partial^2 h_{\alpha\sigma}}{\partial x^\delta \partial x^\beta}\right]$$

Bringing $\eta^{\alpha\sigma}$ (constant) within the partial derivation operators, and calling h the trace of $h_{\alpha\sigma}$

$$h = h^\alpha_{\ \alpha} = \eta^{\alpha\sigma}h_{\alpha\sigma} = h_{00} - (h_{11} + h_{22} + h_{33})$$

we can rewrite the above relationship as

$$2R_{\beta\delta} \simeq \frac{\partial^2 h^\alpha_{\ \delta}}{\partial x^\alpha \partial x^\beta} + \frac{\partial^2 h^\sigma_{\ \beta}}{\partial x^\delta \partial x^\sigma} - \eta^{\alpha\sigma}\frac{\partial^2 h_{\beta\delta}}{\partial x^\sigma \partial x^\alpha} - \frac{\partial^2 h}{\partial x^\delta \partial x^\beta}$$

We now define an auxiliary field $\overline{h}_{\beta\delta}$, definmed in such a way that $\overline{h}_{\beta\delta} \equiv h_{\beta\delta} - \frac{1}{2}\eta_{\beta\delta} \cdot h$ and we have, by multiplying by $\eta^{\beta\delta}$:

$$\overline{h} = h - \frac{1}{2}h \cdot 4 = -h \qquad \text{since} \quad \eta_{\beta\delta} \cdot \eta^{\beta\delta} = \delta^\beta_\beta = 4$$

$$h_{\beta\delta} = \overline{h}_{\beta\delta} - \frac{1}{2}\eta_{\beta\delta}\overline{h}$$

We also observe that, by using $\overline{h}_{\beta\delta}$, we have, for the mixed terms of $h_{\beta\delta}$,

$$h^\alpha_{\ \delta} = \eta^{\alpha\beta}h_{\beta\delta} = \eta^{\alpha\beta}\overline{h}_{\beta\delta} - \frac{1}{2}\eta^{\alpha\beta}\eta_{\beta\delta}\overline{h} = \overline{h}^\alpha_{\ \delta} - \frac{1}{2}\delta^\alpha_\delta\overline{h}$$

Substituting in the expression of Ricci tensor we have:

$$\begin{aligned} 2R_{\beta\delta} &\simeq \frac{\partial^2 \overline{h}^\alpha_{\ \delta}}{\partial x^\alpha \partial x^\beta} - \frac{1}{2}\delta^\alpha_\delta \frac{\partial^2 \overline{h}}{\partial x^\alpha \partial x^\beta} + \frac{\partial^2 \overline{h}^\sigma_{\ \beta}}{\partial x^\delta \partial x^\sigma} - \frac{1}{2}\delta^\sigma_\beta \frac{\partial^2 \overline{h}}{\partial x^\delta \partial x^\sigma} - \eta^{\alpha\sigma}\frac{\partial^2 h_{\beta\delta}}{\partial x^\alpha \partial x^\sigma} + \frac{\partial^2 \overline{h}}{\partial x^\delta \partial x^\beta} \\ &= \frac{\partial^2 \overline{h}^\alpha_{\ \delta}}{\partial x^\alpha \partial x^\beta} + \frac{\partial^2 \overline{h}^\sigma_{\ \beta}}{\partial x^\delta \partial x^\sigma} - \eta^{\alpha\sigma}\frac{\partial^2 h_{\beta\delta}}{\partial x^\alpha \partial x^\sigma} \end{aligned}$$

since the second and fourth term are equal, but of opposite sign, to the half of the sixth one and vanish with this.

We observe that the Einstein equations for $G_{\mu\nu}$ are 10, being $G_{\mu\nu}$ symmetrical. But $G_{\mu\nu}$ also satisfies the four conditions $G^\mu_{\nu;\mu} = 0$, so we are left with $10 - 4 = 6$ independent equations. The unknowns are the 10 components of the metric tensor, in this case $h_{\beta\delta}$. Thus we see that the Einstein equations can not uniquely define the solution. To do this it is necessary to impose other four conditions on $h_{\beta\delta}$, that is we have to choose a particular gauge. In this case, the so-called Lorentz gauge is particularly suitable (we will see later that this can be done and what is its meaning):

$$\frac{\partial \overline{h}^\alpha_{\ \delta}}{\partial x^\alpha} \equiv 0 \qquad \text{(4 conditions: } \delta = 0, 1, 2, 3)$$

With this choice, the two terms containing $\overline{h}^\alpha_{\ \delta}$ in the equation above are zero and we are left with:

$$R_{\beta\delta} \simeq -\frac{1}{2}\eta^{\alpha\sigma}\frac{\partial^2 h_{\beta\delta}}{\partial x^\alpha \partial x^\sigma} = -\frac{1}{2}\left[\frac{1}{c^2}\frac{\partial h^2_{\beta\delta}}{\partial t^2} - \left(\frac{\partial^2 h_{\beta\delta}}{\partial x^2} + \frac{\partial^2 h_{\beta\delta}}{\partial y^2} + \frac{\partial^2 h_{\beta\delta}}{\partial z^2}\right)\right] \equiv -\frac{1}{2}\Box^2 h_{\beta\delta}$$

where $\Box^2$ is the d'Alembert operator or d'Alembertian.

$\Longrightarrow$*Metric of the weak field (stationary)*

$\Longrightarrow$*Gauge transformations*

We have seen that the Einstein equations can be written also in the form $R_{\beta\delta} =$ a function of $T_{\beta\delta}$ and Λ. If we are in vacuum, and we neglect Λ, they become:

$$R_{\beta\delta} \equiv 0 \qquad \Rightarrow \qquad \Box^2 h_{\beta\delta} \equiv 0$$

which is the equation of **a wave propagating at the speed of light**.

We must add to this equation the **gauge conditions**

$$\frac{\partial \overline{h}^{\alpha}_{\ \delta}}{\partial x^{\alpha}} = 0 \qquad \text{but since} \qquad \overline{h}^{\alpha}_{\ \delta} = \eta^{\alpha\beta}\overline{h}_{\beta\delta} = \eta^{\alpha\beta}h_{\beta\delta} - \frac{1}{2}\eta^{\alpha\beta}\eta_{\beta\delta}h = h^{\alpha}_{\delta} - \frac{1}{2}\delta^{\alpha}_{\delta}h$$

$$\Rightarrow \qquad \frac{\partial h^{\alpha}_{\delta}}{\partial x^{\alpha}} - \frac{1}{2}\delta^{\alpha}_{\delta}\frac{\partial h}{\partial x^{\alpha}} = \frac{\partial h^{\alpha}_{\delta}}{\partial x^{\alpha}} - \frac{1}{2}\frac{\partial h}{\partial x^{\delta}} = 0$$

We look for a solution represented by a **plane wave**: $h_{\beta\delta} = A_{\beta\delta}\, e^{i\ k_{\gamma}\cdot x^{\gamma}}$ with $A_{\beta\delta} = const.$

As $h_{\beta\delta}$ is symmetric (10 independent components), after imposing the 4 gauge conditions, 6 degrees of freedom remain. But of these six, four are actually fictitious, related to the arbitrariness of the reference system (see the discussion on gauge transformations), so at the end we are left with only 2 true degrees of freedom. Let's see how.

We choose, as 4 conditions which fix the reference system, the following:

$$h = 0 \qquad\qquad h_{0i} = 0$$

From the first it follows that $\overline{h}_{\alpha\beta} \equiv h_{\alpha\beta}$.

We substitute now the plane wave in the $\Box^2 h_{\beta\delta} = 0$, that we write in the form

$$\eta^{\alpha\sigma}\frac{\partial^2 h_{\beta\delta}}{\partial x^{\alpha}\partial x^{\sigma}} = 0$$

and we get

$$A_{\beta\delta}\,\eta^{\alpha\sigma}\frac{\partial}{\partial x^{\alpha}}\left(\frac{\partial}{\partial x^{\sigma}}e^{i\ k_{\gamma}\cdot x^{\gamma}}\right) = A_{\beta\delta}\,\eta^{\alpha\sigma}\frac{\partial}{\partial x^{\alpha}}\left(i\ k_{\sigma}e^{i\ k_{\gamma}\cdot x^{\gamma}}\right) = i\ k_{\sigma}A_{\beta\delta}\,\eta^{\alpha\sigma}\cdot e^{i\ k_{\gamma}\cdot x^{\gamma}}\cdot i\ k_{\alpha} =$$
$$= -A_{\beta\delta}\,e^{i\ k_{\gamma}\cdot x^{\gamma}}\cdot k_{\sigma}k_{\alpha}\eta^{\alpha\sigma} = -A_{\beta\delta}\ \ e^{i\ k_{\gamma}\cdot x^{\gamma}}\cdot k_{\sigma}k^{\sigma} = 0 \quad \Rightarrow \quad k_{\sigma}k^{\sigma} = 0$$

We write the four vector k^{σ} as $k^{\sigma} \equiv (\frac{\omega}{c}, \overline{k})$. Since $k_{\sigma} = \eta_{\sigma\alpha}k^{\alpha}$ we have:

$$k_{\sigma}k^{\sigma} = \eta_{\sigma\alpha}k^{\alpha}k^{\sigma} = k^0k^0 - (k^1k^1 + k^2k^2 + k^3k^3) = \frac{\omega^2}{c^2} - |\overline{k}|^2 \equiv 0$$

So we have $\quad \omega = kc \quad \rightarrow \quad \hbar\omega = \hbar k\cdot c \quad \Rightarrow \quad E = P\cdot c \quad$ as for photons, with zero rest mass: the quantum mediating the gravitational interaction, the *graviton*, has zero mass.

We also observe that

$$k_{\gamma}\cdot x^{\gamma} = \eta_{\gamma\sigma}k^{\sigma}\cdot x^{\gamma} = k^0x^0 - |\overline{k}\cdot\overline{x}| = \omega t - \overline{k}\cdot\overline{x}$$

From the gauge condition:

$$\frac{\partial h^{\alpha}_{\delta}}{\partial x^{\alpha}} = 0 \qquad \text{since} \qquad h^{\alpha}_{\delta} = \eta^{\alpha\sigma}h_{\sigma\delta} = \eta^{\alpha\sigma}A_{\sigma\delta}\,e^{i\ k_{\gamma}\cdot x^{\gamma}}$$

$$\frac{\partial}{\partial x^{\alpha}}\left(A_{\sigma\delta}\,\eta^{\alpha\sigma}e^{i\ k_{\gamma}\cdot x^{\gamma}}\right) = A_{\sigma\delta}\,\eta^{\alpha\sigma}e^{i\ k_{\gamma}\cdot x^{\gamma}}\cdot i\ k_{\alpha} \equiv 0$$

that is

$$i\ k_{\alpha}A^{\alpha}_{\delta}\,e^{i\ k_{\gamma}\cdot x^{\gamma}} = i\ k_{\alpha}h^{\alpha}_{\delta} = 0 \qquad \Rightarrow \qquad h^{\alpha}_{\delta}\cdot k_{\alpha} = 0$$

which is called *transversality condition*. Let's see why.

We choose the direction of propagation along the x-axis: thus $\overline{k} \equiv (k, 0, 0)$ and $h^{\alpha}_{\sigma}\cdot k_{\alpha} = \eta^{\alpha\delta}h_{\delta\sigma}k_{\alpha} = h_{\delta\sigma}k^{\delta} = 0$. We remember the conditions of choice of the reference system ($h = 0$ e $h_{0i} = 0$).

$$\begin{array}{llll}
\sigma = 0 & \rightarrow & h_{00}k^0 + h_{10}k^1 + h_{20}k^2 + h_{30}k^3 = 0 & \rightarrow \qquad h_{00} = 0 \\
\sigma = 1 & \rightarrow & h_{01}k^0 + h_{11}k^1 + h_{21}k^2 + h_{31}k^3 = 0 & \rightarrow \qquad h_{11} = 0 \\
\sigma = 2 & \rightarrow & h_{02}k^0 + h_{12}k^1 + h_{22}k^2 + h_{32}k^3 = 0 & \rightarrow \qquad h_{12} = h_{21} = 0 \\
\sigma = 3 & \rightarrow & h_{03}k^0 + h_{13}k^1 + h_{23}k^2 + h_{33}k^3 = 0 & \rightarrow \qquad h_{13} = h_{31} = 0
\end{array}$$

Summarizing all in matrix form

$$h_{\beta\delta} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & h_{22} & h_{23} \\ 0 & 0 & h_{32} & h_{33} \end{pmatrix} \qquad \begin{array}{l} h = 0 \Rightarrow h_{22} + h_{33} = 0 \Rightarrow h_{22} = -h_{33} \equiv h_+ \\ \text{and for symmetry } h_{23} = h_{32} \equiv h_\times \end{array} \qquad \Rightarrow \qquad h_{\beta\delta} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & h_+ & h_\times \\ 0 & 0 & h_\times & -h_+ \end{pmatrix}$$

We see that if $\overline{k}$ is along the x axis, the non null components of the wave are perpendicular to the x axis. They are transverse waves with two components (polarization):

$$\begin{cases} h_+ = A_+ \, e^{i(\omega t - \overline{k}\cdot\overline{x})} \\ h_\times = A_\times \, e^{i(\omega t - \overline{k}\cdot\overline{x})} \end{cases}$$

3.10 Gravitational lenses

We have seen that the metric of the weak, stationary field can be written in the form

$$\mathrm{d}s^2 = \left(1 + \frac{2\Phi}{c^2}\right) c^2 \mathrm{d}t^2 - \left(1 - \frac{2\Phi}{c^2}\right)\left(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\right)$$

This allows us to obtain another very interesting result. For a light ray $\mathrm{d}s^2 = 0$ and, assuming $\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 \equiv \mathrm{d}l^2$, we have

$$\left(1 + \frac{2\Phi}{c^2}\right) c^2 \mathrm{d}t^2 = \left(1 - \frac{2\Phi}{c^2}\right) \mathrm{d}l^2$$

from which

$$\left(\frac{\mathrm{d}l}{\mathrm{d}t}\right)^2 = c^2 \frac{\left(1 + \frac{2\Phi}{c^2}\right)}{\left(1 - \frac{2\Phi}{c^2}\right)} \equiv v_{eff}^2 \equiv \frac{c^2}{n_g^2}$$

where v_{eff} is the *effective* speed of propagation of the luminous wave and n_g can be thought as an index of refraction of gravity. it is

$$n_g = \sqrt{\frac{\left(1 - \frac{2\Phi}{c^2}\right)}{\left(1 + \frac{2\Phi}{c^2}\right)}} \sim \sqrt{\left(1 - \frac{2\Phi}{c^2}\right)\left(1 - \frac{2\Phi}{c^2}\right)} \qquad \rightarrow \qquad n_g \simeq 1 - \frac{2\Phi}{c^2}$$

If $\Phi = 0$ to the infinity and is negative near a mass, $n_g > 1$ and $v_{eff} < c$. This relation show us that space, as a consequence of gravitation, behaves as a refractive medium: this is the basis of those phenomena known as **gravitational lenses**.