Lecture 4

an introduction to differential equations

Differential equations.

- Definition: a differential equation states how a rate of change in one variable is related to other variables.
- A differential equation contains one or more terms involving derivatives of one variable (the dependent variable, *y*) with respect to another variable (the independent variable, *t*).

$$\frac{dy}{dt} = a \qquad \frac{dy}{dt} = a\,t^2 \qquad \frac{dy}{dt} = y - a\,t^2$$

- These examples have only first derivatives. They are called first order differential equations. Second order differential equations include second order derivatives

$$\frac{d^2y}{d\,t^2} = a + \frac{dy}{dt}$$

- the order of a differential equation is given by the highest order of derivative

$$\frac{d^ny}{d\,t^n} + \frac{d^2y}{d\,t^2} + \frac{dy}{d\,t} = a + \frac{dy}{dt}$$

(if $n > 2$ differential equation of order n)

In economics, decisions are rarely taken continuously. Hence differential equations are usually approximations to processes that might be more exactly represented via difference equations:

$$\frac{dy}{dt} = ay \quad or \quad y_t - y_{t-1} = ay_{t-1}$$

To solve a differential equation we also need information about y at some time t. Often we use starting point or end point.

First Order Linear Differential equations.

This kind of linear equations have no powers in y and no interaction terms between y and dy/dt.

The general form of a linear first order differential equation is:

$$\frac{dy}{dt} + u(t)y = w(t)$$

where $u(t)$ and $w(t)$ are functions of t and may be linear as well non linear.

First Order Linear Differential equations with constant term and constant coefficient

This is a special case of first order linear differential equations where $u(t)$ and $w(t)$ are constants

$$\frac{dy}{dt} + ay = b$$

Homogeneous equations are ones where b is equal to zero.

$$\frac{dy}{dt} + ay = 0$$

Non-Homogeneous equations are ones b is different from zero.

$$\frac{dy}{dt} + ay = b$$

Solution of the homogeneous case

$$\frac{dy}{dt} = ay$$

One way to approach the solution is to treat dy and dt as separate items and integrate:

$$\frac{1}{y}\ dy = a\ dt$$

$$\int \frac{1}{y}\ dy = \int a\ dt$$

$$\ln(y) = a\ t + c$$

$$y = e^c e^{at} = A\ e^{at}$$

Now $y(t) = Ae^{at}$

(commonly used to describe the evolution of GDP over time where a = rate of growth). Note that A could be any value

(in the growth literature A is used as a measure of technical progress)

$y(t) = Ae^{at}$ is said to be the ***general solution*** to the differential equation

$$\frac{dy}{dt} = ay$$

For a particular value of A this becomes a ***particular solution***

If we start off the process y(0)=A, then this gives the ***definite solution***

Note the solution ($y(t) = Ae^{at}$) is free of any derivative and is not a numerical value rather a function (or a “time path”) giving the value of y at any point in time

Solution of the non homogeneous case

$$\frac{dy}{dt} = ay + b$$

The solution to the related homogeneous equation is called the **complementary function (y_c)**

We call the **particular integral (y_p)** any particular solution to the non-homogeneous equation)

The sum of the complementary function and the particular integral constitutes the **general solution** of a 1st order linear non-homogenous differential equation

$$\frac{dy}{dt} = ay + b$$

To solve proceed in 2 steps:

Step 1

Look for ***any*** value that satisfies the equation

the simplest is to let $y(t) = k$ $(= a\ constant\ )$

then $dy/dt = 0$ and the differential equation becomes

$$-ay = b$$

and

$$y(t) = -b/a = k\ (a \neq 0\ )$$

This solution to the non-homogeneous equation is called the **particular integral**

Step 2

We consider the homogeneous related equation

$$\frac{dy}{dt} = ay$$

and we have already seen that the solution to this type of homogenous differential equation is $y(t) = Ae^{at}$

The solution to the related homogeneous equation (**complementary function (y_c)** in this case is

$$y(t) = Ae^{at}$$

A particular solution to the non-homogeneous equation (**particular integral (y_p)**) in this case is

$$y(t) = -b/a$$

The sum of the complementary function and the particular integral constitutes the **general solution** of a 1st order linear non-homogenous differential equation, in this case:

$$y(t) = y_c + y_p = Ae^{at} - b/a$$

$$y(t) = y_c + y_p = Ae^{at} - b/a$$

to get the definite solution we need to impose an initial condition:

the value of y at $t = 0$

In this example

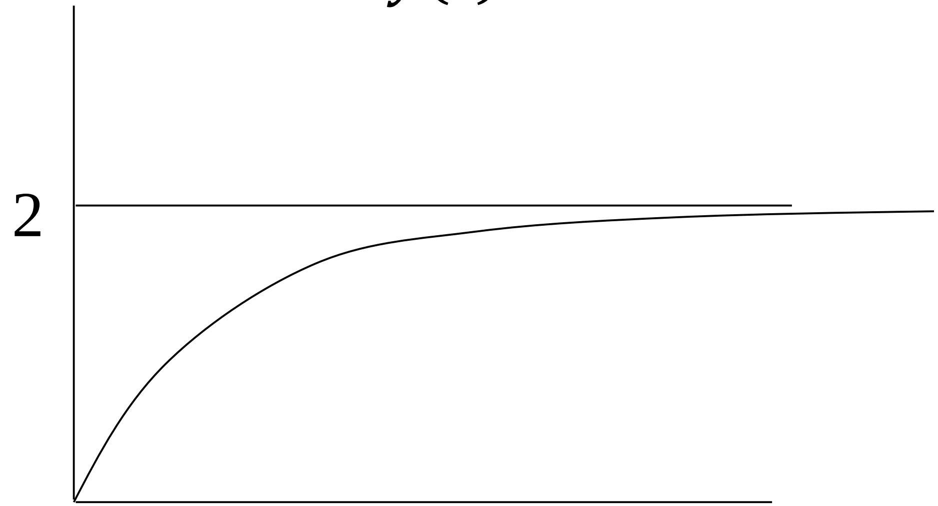
$$y(0) = Ae^0 - b/a = A - (b/a)$$

$$A = y(0) + (b/a)$$

$$y(t) = (y(0) + (b/a))e^{at} - (b/a)$$

Example 1 $\frac{dy}{dt} = ay + b$ $a = -1,\ b = 2,\ y(0) = 0$

$$y(t) = (y(0) + (b/a))e^{at} - (b/a)$$
$$y(t) = 2 - 2e^{-t}$$



Often in economics the particular integral will represent the equilibrium of a system (here $y = 2$), while the complementary function supplies the dynamics to get there (from the initial condition value)

Example 2

Solve $dy/dt + 2y(t) = 6$ with an initial condition $y(0) = 10$

The constant solution $dy/dt = 0$ gives the particular integral

$$y_p = 6/2 = 3$$

The complementary function is the solution to

$$dy/dt = -2y(t)$$

$$\frac{1}{y}dy = -2dt$$

Integrating both sides:

$$\int \frac{1}{y}dy = \int -2dt$$

$$\ln(y) + c_1 = -2t + c_2 \quad \rightarrow \quad \ln(y) = -2t + c_2 - c_1$$

$$y = e^{-2t}e^k; k = c_2 + c_1$$

$$y = Ae^{-2t}$$

So the general solution is:

$$y(t) = y_c + y_p = Ae^{-2t} + 3$$

To obtain the definite solution we use the value for the initial conditions $t = 0$, $y(0) = 10$

$y(0) = Ae^0 + 3 = A + 3$ so then $A = y(0) - 3 = 10 - 3 = 7$

So then

$$y(t) = 7e^{-2t} + 3$$

This is saying the system converges to the value 3, starting from an initial value of 10 as $t \to \infty$

Variable coefficient and variable term

$$\frac{dy}{dt} + u(t)y = w(t)$$

($u(t)$ and $w(t)$ are not constants)

<u>Case $u(t) = 0$</u>

$$\frac{dy}{dt} = w(t)$$

Integrate both sides wrt t

$$\int \frac{dy}{dt} dt = \int w(t)dt$$

$$\rightarrow y = \int w(t)dt + c$$

Example 3

Suppose, $w(t) = t^2 - 1 = dy/dt$

then we get $y = t^3 /3 - t + c$

So we can compute the values of y in any period

Case w$(t) = 0$ (homogeneous case)

$$\frac{dy}{dt} + u(t)y = 0$$

$$\frac{dy}{y} = -u(t)dt \rightarrow \int \frac{dy}{y} = -\int u(t)dt$$

$$\ln(y) = -\int u(t)dt - c$$

$$y = e^{-c} e^{-\int u(t)dt}$$

$$y = Ae^{-\int u(t)dt} \quad where\ A = e^{-c}$$

Example 4

Find the general solution of $\frac{dy}{dt} + 2ty = 0$

$$u(t) = 2t$$

$$\int u(t)dt = \int 2t\,dt = t^2 + k$$

Then $y = Ae^{-\int u(t)dt}$ $where\ A = e^{-c}$ is written as

$$y = Ae^{-t^2-k}$$

$$y = Be^{-t^2}\ where\ B = e^{-c-k}$$

General case

$$\frac{dy}{dt} + u(t)y = w(t)$$

The general solution is:

$$y = e^{-\int u(t)dt}(A + \int w(t)e^{\int u(t)dt}dt)$$

(after we will see the proof to get this formula)

Example 5

Find the general solution of $\frac{dy}{dt} + 2ty = t$.

$u(t) = 2t$, $w(t) = t$

$\int u(t)dt = \int 2t\,dt = t^2 + k$

Replacing in the general solution:

$$y = e^{-(t^2+k)}\left(A + \int te^{(t^2+k)}dt\right)$$

$$y = e^{-(t^2+k)}\left(A + \int te^{(t^2+k)}dt\right)$$

$$y = e^{-(t^2+k)}\left(A + \frac{1}{2}e^{(t^2+k)} + h\right)$$

$$y = Ae^{-(t^2+k)} + \frac{1}{2}e^{(t^2+k)}e^{-(t^2+k)} + he^{-(t^2+k)}$$

$$y = Ae^{-(t^2+k)} + \frac{1}{2} + he^{-(t^2+k)}$$

$$y = (Ae^{-k} + he^{-k})e^{-t^2} + \frac{1}{2}$$

$$y = Be^{-t^2} + \frac{1}{2}$$

Where $B = Ae^{-k} + he^{-k}$ is an arbitrary constant

Checking the solution

$$y = Be^{-t^2} + \frac{1}{2}$$

$$\frac{dy}{dt} = -2tBe^{-t^2}$$

We replace in $\frac{dy}{dt} + 2ty = t$ and we get:

$$-2tBe^{-t^2} + 2\mathrm{t}\left(Be^{-t^2} + \frac{1}{2}\right) = \mathrm{t}$$

$$-2tBe^{-t^2} + 2\mathrm{t}Be^{-t^2} + 2t\frac{1}{2} = \mathrm{t}$$

$$t = t$$

Then solution is right!

Exact differential equations

The total differential of $F(y,t)$ is:

$$dF(y,t) = \frac{dF}{dy}dy + \frac{dF}{dt}dt$$

If $dF(y,t) = 0$, the result is called **an exact differential equation**.

Example 6

Consider the function

$$F(y,t) = 4y^3t + k$$

the total differential is

$$\mathrm{d}F(y,t) = 12y^2t\ dy + 4y^3\ dt$$

The differential equation

$$12y^2t\ dy + 4y^3\ dt = 0$$

is exact.

In general a differential equation

$$M\ dy + N\ dt = 0$$

is exact **if and only if** there exists a function $F(y,t)$ such that:

$$M = \frac{dF(y,t)}{dy} \qquad N = \frac{dF(y,t)}{dt}$$

A simple test to check if the differential equation is exact is given by the following statement:

The differential equation $M\ dy + N\ dt = 0$ is exact if and only if

$$\frac{dN}{dy} = \frac{dM}{dt}$$

The proof is given by Young's theorem which states that:

$$\frac{d^2F}{dt\ dy} = \frac{d^2F}{dy\ dt}$$

From previous example 6 we know that

$$12y^2t\,dy + 4y^3\,dt = 0$$

is exact

In this example

$$M = 12y^2t \;\; N = 4y^3$$

Then

$$\frac{dM}{dt} = 12y^2$$

$$\frac{dN}{dy} = 12y^2$$

So $\frac{dN}{dy} = \frac{dM}{dt}$ and the exactness is verified

Solving exact differential equations

Note as the differential equation says that $dF(y,t) = 0$.

So the solution will be given by

$$F(y,t) = c$$

where c is an arbitrary constant

Then to solve an exact differential equation we have to get the function $F(y,t)$ and set it equal to an arbitrary constant c.

Method of solution

Check the condition $\frac{dN}{dy} = \frac{dM}{dt}$. If it is satisfied then we have an exact equation. If it is exact then follow the next steps

<u>Step 1</u>. Integrate M partially with respect to y. The result is:

$$F(y,t) = g(t) + \int M\,dy$$

Note the term $g(t)$ which we do not know yet.

Step 2. Partially differentiate the result in step 1 with respect to t to get N:

$$N = \frac{dF}{dt} = \frac{d\int M\,dy}{dt} + g'(t)$$

Note that $g'(t)$ is the derivative of $g(t)$ with respect to t.

Step 3. We know N so we can solve this equation to find $g'(t)$. From that we can integrate $g'(t)$ to find $g(t)$

Step 4.

We replace $g(t)$ in the solution in step 1 and we get the expression of $F(y,t)$

Step 5. We solve by y the expression

$$F(y,t) = c$$

Example 7. Solve $\frac{dy}{dt} = 4t$

Rewrite the differential equation as $0 = dy - 4t\, dt$

$M = 1$ and $N = -4t$. → $\frac{dN}{dy} = \frac{dM}{dt} = 0$

Step 1: Integrate M partially with respect to y. The result is:

$$F(y,t) = \int M\, dy + g(t) = \int 1\, dy + g(t) = y + g(t)$$

Step 2: Partially differentiate this result with respect to t to get N:

$$N = \frac{dF}{dt} = \frac{d(y + g(t))}{dt} = 0 + g'(t)$$

Step 3: we get $g'(t) = -4t$, then we integrate $g'(t)$ to find $g(t)$:

$$g(t) = \int g'(t)\, dt = \int -4t\, dt = -2t^2 + k$$

Step 4: $F(y,t) = y - 2t^2 + k$

Step 5: $F(y,t) = c \rightarrow y - 2t^2 + k = c \rightarrow y = 2t^2 + k$

Example 8. Solve $2yt\,dy + y^2dt = 0$

$M = 2yt$ and $N = y^2$. → $\frac{dN}{dy} = \frac{dM}{dt} = 2y$

Step 1: Integrate M partially with respect to y. The result is:

$$F(y,t) = \int M\,dy + g(t) = \int 2yt\,dy + g(t) = y^2t + g(t)$$

Step 2: Partially differentiate this result with respect to t to get N:

$$N = \frac{dF}{dt} = \frac{d(y^2t + g(t))}{dt} = y^2 + g'(t)$$

Step 3: $y^2 + g'(t) = y^2$ → $g'(t) = 0$

We integrate $g'(t)$ to find $g(t)$:

$$g(t) = \int g'(t)\,dt = \int 0\,dt = k$$

Step 4: $F(y,t) = y^2t + k$

Step 5: $F(y,t) = c \rightarrow y^2t + k = c \rightarrow y = \frac{k}{\sqrt{t}}$

Example 9. Solve $(t + 2y)\, dy + (y + 3t^2)\, dt = 0$

$M = (t + 2y)$ and $N = (y + 3t^2)$. → $\frac{dN}{dy} = \frac{dM}{dt} = 1$

Step 1: Integrate M partially with respect to y. The result is:

$$F(y,t) = \int M\, dy + g(t) = \int (t + 2y) dy + g(t) = ty + y^2 + g(t)$$

Step 2: Partially differentiate this result with respect to t to get N:

$$N = \frac{dF}{dt} = \frac{d(ty + y^2 + g(t))}{dt} = y + g'(t)$$

Step 3: $y + g'(t) = y + 3t^2 \rightarrow \quad g'(t) = 3t^2$

We integrate $g'(t)$ to find $g(t)$:

$$g(t) = \int g'(t)\, dt = \int 3t^2 dt = t^3 + k$$

Step 4: $F(y, t) = ty + y^2 + t^3 + k$

Step 5: the solution of the differential equation is given by the solution of the following equation

$$ty + y^2 + t^3 = c$$

Integrating factors

Sometimes M and N mean that the differential equation is inexact, but we can still find an equivalent function of t and y which we can multiply through by to get an exact equation.

Example 10.

$0 = t\,dy + 2y\,dt \quad \rightarrow \quad \frac{dM}{dt} = 1 \neq \frac{dN}{dy} = 2$

Multiply through by t to get: $0 = t^2\,dy + 2yt\,dt$

Now $\frac{dM}{dt} = \frac{dN}{dy} = 2t$

In this example t is an integrating factor for this differential equation

Now you can solve the exact differential equation

Now you can solve the exact differential equation

Integrate M partially with respect to y. The result is:

$$F(y,t) = \int M\,dy + g(t) = \int t^2 dy + g(t) = yt^2 + g(t)$$

Partially differentiate this result with respect to t to get N:

$$\frac{dF(y,t)}{t} = \frac{d(yt^2 + g(t))}{t} = 2ty + g'(t) = 2yt = N \;\rightarrow g'(t) = 0$$

Then, $g(t) = \int g'(t)dt = \int 0\,dt = k$

$$F(y,t) = yt^2 + k \;\;\rightarrow yt^2 + k = c$$

$$y = \frac{c}{t^2}$$

Therefore, when you have to solve a differential equation, if it is not exact, you have to explore if an integrating factor exists

Proof of the solution of the general case

Now we prove that the general solution of:

$$\frac{dy}{dt} + u(t)y = w(t)$$

is

$$y = e^{-\int u(t)dt}(A + \int w(t)e^{\int u(t)dt}dt)$$

Proof: we rewrite the differential equation as:

$$dy + \big(u(t)y - w(t)\big)dt = 0$$

Then $M = 1$ and $N = u(t)y - w(t)$

We look for an integrating factor I such that:

$$\frac{\partial(I \cdot M)}{\partial t} = \frac{\partial(I \cdot N)}{\partial y}$$

Replacing M and N we get:

$$\frac{\partial I}{\partial t} = \frac{\partial I \cdot (u(t)y - w(t))}{\partial y}$$

$$\frac{dI}{dt} = \frac{dI \cdot (u(t)y - w(t))}{dy}$$

It becomes:

$$\frac{dI}{dt} = I \cdot u(t) \quad or \quad \frac{dI}{I} = u(t)dt$$

Integrating both sides:

$$\int \frac{1}{I} dI = \int u(t)dt$$

$$\ln I + c = \int u(t)dt$$

$$I = Ae^{\int u(t)dt}$$

Applying this integrating factor to the differential equation we get:

$$e^{\int u(t)dt} \cdot dy + e^{\int u(t)dt} \cdot \big(u(t)y - w(t)\big)dt = 0$$

That can be solved with the standard procedure for exact differential equations

$$e^{\int u(t)dt} \cdot dy + e^{\int u(t)dt} \cdot \big(u(t)y - w(t)\big)dt = 0$$

$$M = e^{\int u(t)dt} \quad \mathrm{N} = e^{\int u(t)dt} \cdot \big(u(t)y - w(t)\big)$$

<u>Step 1:</u>

Integrate M partially with respect to y. The result is:

$$F(y,t) = \int M\, dy \ + g(t) = \int e^{\int u(t)dt} dy \ + g(t)$$

$$= ye^{\int u(t)dt} + g(t)$$

<u>Step 2:</u>

Partially differentiate this result with respect to t to get N:

$$N = \frac{dF}{dt} = \frac{dye^{\int u(t)dt} + g(t)}{dt} = y \cdot u(t)e^{\int u(t)dt} + g'(t)$$

<u>Step 3:</u>

$$y \cdot u(t)e^{\int u(t)dt} + g'(t) = e^{\int u(t)dt} \cdot \big(u(t)y - w(t)\big) \rightarrow \quad g'(t) = 0$$

$$\rightarrow \quad g'(t) = -e^{\int u(t)dt} \cdot w(t)$$

$$g'(t) = -e^{\int u(t)dt} \cdot w(t)$$

we integrate $g'(t)$ to find $g(t)$:

$$g(t) = \int g'(t)\, dt = -\int w(t) \cdot e^{\int u(t)dt}\, dt$$

<u>Step 4:</u>

$$F(y,t) = ye^{\int u(t)dt} - \int w(t) \cdot e^{\int u(t)dt}\, dt$$

<u>Step 5:</u>

$$F(y,t) = c$$

$$ye^{\int u(t)dt} - \int w(t) \cdot e^{\int u(t)dt}\, dt = c$$

We get

$$y = e^{-\int u(t)dt}(A + \int w(t)e^{\int u(t)dt}dt)$$

Non linear Differential equations of first oder and first degree

The **degree** of a **differential equation** is the power of the highest **order** derivative in the **equation**.

When y appears in a power higher than one the differential equation is not linear.

$$f(y,t)dy + g(y,t)dt = 0$$

or

$$\frac{dy}{dt} = h(y,t)$$

where no there is no restriction on the power of y and t is a *first-order first-degree nonlinear differential equation.*

Three cases

1) Exact differential equations
2) Separable variables
3) Equation reducible to the linear form

1) Exact differential equations

If the non linear differential equation is exact, you can apply the procedure

Example 10: the exact differential equation is not linear.

2) Separable variables

$$f(y,t)dy + g(y,t)dt = 0$$

If the argument of f is only y and the argument of g is only t the differential equation is:

$$f(y)dy + g(t)dt = 0$$

In such an event the variable are said to be *separable.*

To solve it it is enough a simple integration

Example 11

$$3\mathrm{y}^2\mathrm{dy} - \mathrm{tdt} = 0$$

we integrate each side

$$\int 3\mathrm{y}^2\mathrm{dy} = \int tdt$$

we get

$$y^3 + c_1 = \frac{1}{2}t^2 + c_2$$

………

……..

3) Equation reducible to the linear form

If the differential equation

$$\frac{dy}{dt} = h(y,t)$$

is of the form

$$\frac{dy}{dt} + Ry = Ty^m$$

Where R and T are functions of t and m is any number different from 1 and 0, then the equation can be reduced to a linear one

Prrocedure:

$$y^{-m}\frac{dy}{dt} + Ry^{1-m} = T$$

Consider a new variable $z = y^{1-m}$

$$\frac{dz}{dt} = (1-m)y^{-m}\frac{dy}{dt}$$

Then

$$\frac{1}{1-m}\frac{dz}{dt} + Rz = T$$

$$\frac{1}{1-m}\frac{dz}{dt} + Rz = T$$

Rearranging we get

$$dz + [(1-m)Rz - (1-m)T]dt = 0$$

This is a first order linear equation where z has replaced y

We can solve it by the usual method to find its solution $z(t)$

Finally we have to go back from z to y