Lecture 4

Economic dynamics and Integration

Chapter 14 of the textbook

Introduction

Static models:

the problem is to find the values of endogenous variables that satisfy the equilibrium conditions.

- Market equilibrium, supply = demand
- Profit maximization, FOC
- ………

Dynamic models:

The problem is to find the time path of dome variable, knowing the pattern of change.

Dynamic models: an example

Let be H the population size

Over the time H changes at the rate: $\frac{\partial H}{\partial t} = t^{-\frac{1}{2}}$

What is the time path $H(t)$ that produces this rate of change?

We have to find the *primitive* function from a given *derived* function, i.e. a function such that its partial derivative respect to t is equal to $t^{-\frac{1}{2}}$.

The relevant method to use is the *integration.*

Note that all functions $H(t) = 2t^{\frac{1}{2}} + c$, where c is an arbitrary constant, are a possible solution of the problem.

To find the function describing the time path we need some additional information, usually *initial condition* or *boundary condition*.

If we know the population at time 0, $H(0)$, we can solve the problem.

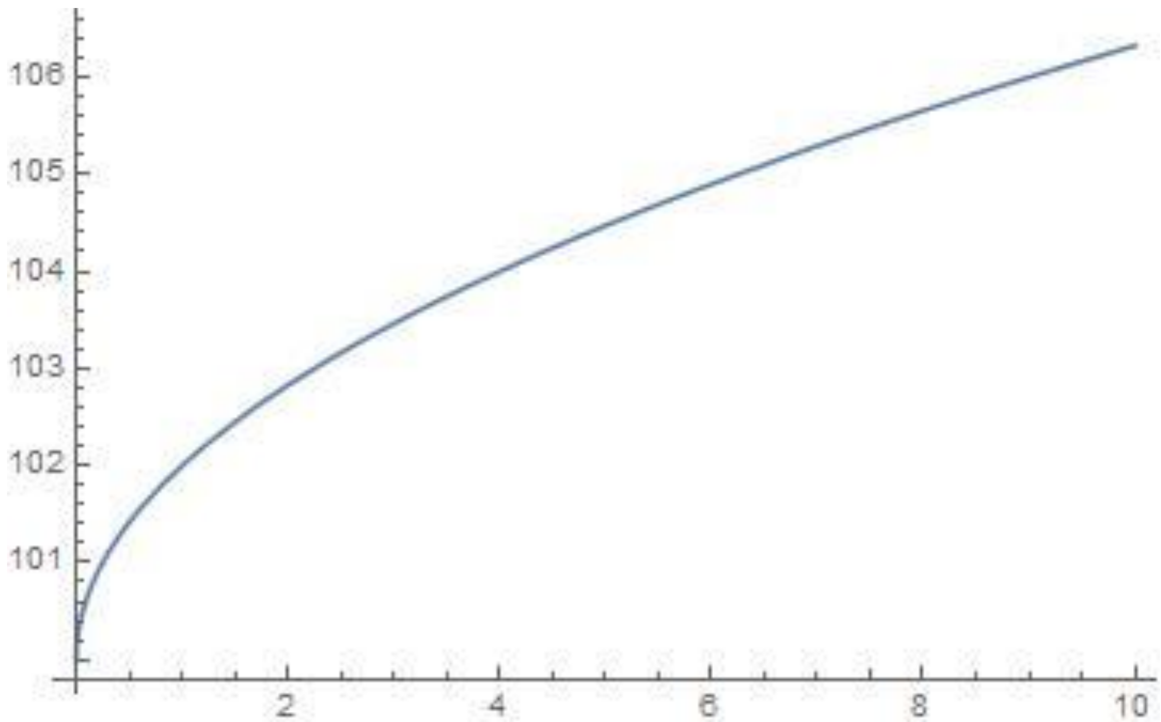
Suppose $H(0) = 100$.

If $t = 0$, $H(0) = 2(0)^{\frac{1}{2}} + c = c$

Given that $H(0) = 100$ we can conclude that $c = 100$.

The time path is:

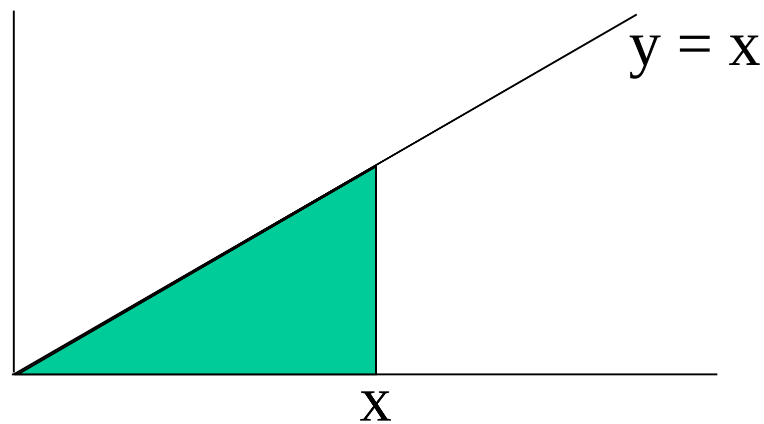
$$H(t) = 2t^{\frac{1}{2}} + 100$$



Introduction to Integration

Integration has two interpretations:

- As the inverse of differentiation
  - E.g. what function of x differentiates to become y=x?
- As a means of calculating areas under graphs.



Inverse of differentiation.

E.g. what function of x differentiates to become y=x?

- We know x^2 differentiates to 2x,
- So $0.5x^2$ must differentiate to x.
- But, $0.5x^2 + 5$ also differentiates to x and so does $0.5x^2 – 5$
- In fact $0.5x^2 + c$ differentiates to x for any value of c.
- We call this the indefinite integral of x.
- We write this as,

$$\int x\, dx = \frac{x^2}{2} + c$$

- $\int$ is the integration symbol (it is an old fashioned 's' for SUMMA which is latin for sum)
- In general we write

$$\int f(x)dx = F(x) + c \text{ where } \frac{dF(x)}{dx} = f(x)$$

Some standard indefinite integrals.

function	integrates to:
ax	$\frac{a}{2}x^2 + c$
$ax^n, n \neq -1$	$\frac{a}{n+1}x^{n+1} + c$
$\frac{a}{x}$	$a \ln x + c$
e^x	$e^x + c$
$\frac{f'(x)}{f(x)}$	$\ln(f(x)) + c$
$f'(x)e^{f(x)}$	$e^{f(x)} + c$

Integral of a sum

The integral of a sum of functions is the sum of the integrals of those functions

$$\int f(x) + g(x)\, dx = \int f(x)\, dx + \int g(x)\, dx$$

Example:

$$\int x^3 + x\, dx = \int x^3 dx + \int x\, dx =$$

$$= \frac{x^4}{4} + c_1 + \frac{x^2}{2} + c_2 =$$

$$= \frac{x^4}{4} + \frac{x^2}{2} + c$$

Integral of a multiple

$$\int kf(x)\,dx = k\int f(x)\,dx$$

Examples:

$$\int -f(x)dx = -\int f(x)dx$$

$$\int 2x^3 dx = 2\int x^3 dx = 2\left(\frac{x^4}{4} + c_1\right) = \frac{x^4}{2} + c$$

$$\int \frac{2}{x} + x^3\,dx = \int \frac{2}{x}dx + \int x^3 dx = 2\int \frac{1}{x}dx + \int x^3 dx =$$

$$= 2(\ln x + c_1) + \frac{x^4}{4} + c_2 = 2\ln x + \frac{x^4}{4} + c$$

Substitution

$$\int f(x)\,dx = \int f(u)\frac{du}{dx}\,dx = \int f(u)\,du = F(u) + c$$

This rule is the counterpart of the chain rule

Consider $F(u)$ where $u = u(x)$, by chain rule

$$\frac{dF(u)}{dx} = \frac{dF(u)}{du}\frac{d(u)}{dx} = F'(u)\frac{d(u)}{dx} = f(u)\frac{d(u)}{dx}$$

$$\frac{dF(u)}{dx} = f(u)\frac{d(u)}{dx}$$

Therefore

$$\int f(u)\frac{du}{dx}\,dx = F(u) + c$$

$$\int f(u)\frac{du}{dx}\, dx = \int f(u)\, du = F(u) + c$$

Example

$$\int 2x(x^2+1)\, dx \;= \int (2x^3+2x)dx = \frac{x^4}{2} + x^2 + c$$

Let $u = x^2 + 1$,

then $\frac{du}{dx} = 2x$ or $dx = \frac{du}{2x}$

We replace u and dx in the integral and we get:

$$\int 2xu\frac{du}{2x} = \int u\, du = \frac{u^2}{2} + c_1 =$$

$$= \frac{(x^2+1)^2}{2} + c_1 = \frac{x^4+2x^2+1}{2} + c_1 = \frac{x^4}{2} + x^2 + c$$

Integration by parts.

Example: how do we integrate $y = \ln(x)$? (answer is $x\, ln(x)$ – x)

To find this we use the product rule for differentiation:

If $f(x) = u(x)v(x)$ then

$$\frac{df}{dx} = \frac{du}{dx}v(x) + u(x)\frac{dv}{dx}$$

It follows that:

$$f(x) = \int \frac{df}{dx}\, dx$$

$$u(x)v(x) = \int \frac{du}{dx}v(x) + u(x)\frac{dv}{dx}dx = \int \frac{du}{dx}v(x)dx + \int u(x)\frac{dv}{dx}dx$$

Rearranging:

$$\int \frac{du}{dx}v(x)dx = u(x)v(x) - \int u(x)\frac{dv}{dx}dx$$

$$\int \frac{du}{dx} v(x) dx = u(x)v(x) - \int u(x) \frac{dv}{dx} dx$$

To use this method you have to separate your function into two components:

- $\frac{du}{dx}$
- $v(x)$

You need to choose these carefully so that you can integrate $\frac{du}{dx}$ and $u(x)\frac{dv}{dx}$.

Integration by parts - example.

How do we integrate $y = \log(x)$?

We will let $v(x) = \log(x)$ and set $\frac{du}{dx} = 1$

So that $u(x) = x$ and $\frac{dv}{dx} = \frac{1}{x}$.

It follows that:

$$\int \frac{du}{dx} v(x) dx = u(x)v(x) - \int u(x) \frac{dv}{dx} dx =$$

$$= xlog(x) - \int x \frac{1}{x} dx =$$

$$= xlog(x) - \int 1 \cdot dx =$$

$$= xlog(x) - x$$

Definite integrals

Given an indefinite integral

$$\int f(x)dx = F(x) + c$$

If we choose two values of x in the domain, say a and b ($a < b$) and we substitute in the RHS and form the difference we get:

$$[F(b) + c] - [F(a) + c] = F(b) - F(a)$$

We get a specific value (free of variable x)

It is called the *definite integral of f(x) from a to b*

a is the lower limit of integration

b is the upper limit of integration

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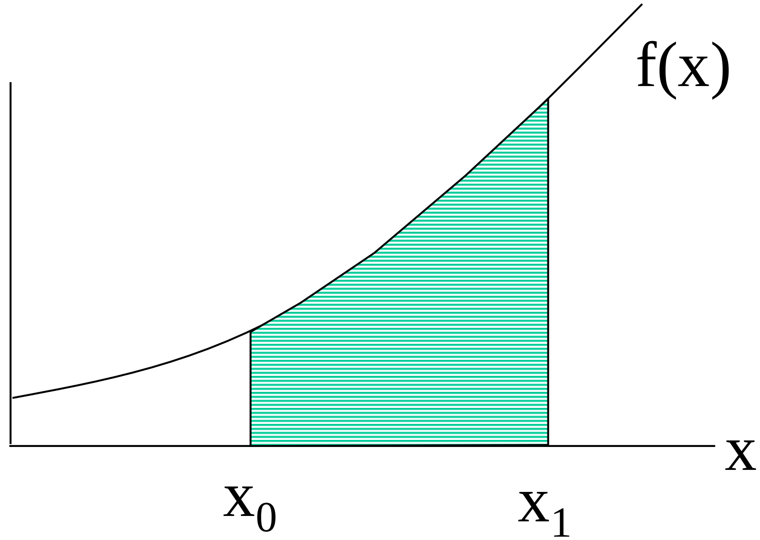
We denote the definite integral in the following way:

$$\int_a^b f(x)dx = F(x)]_a^b = F(b) - F(a)$$

Areas under graphs.



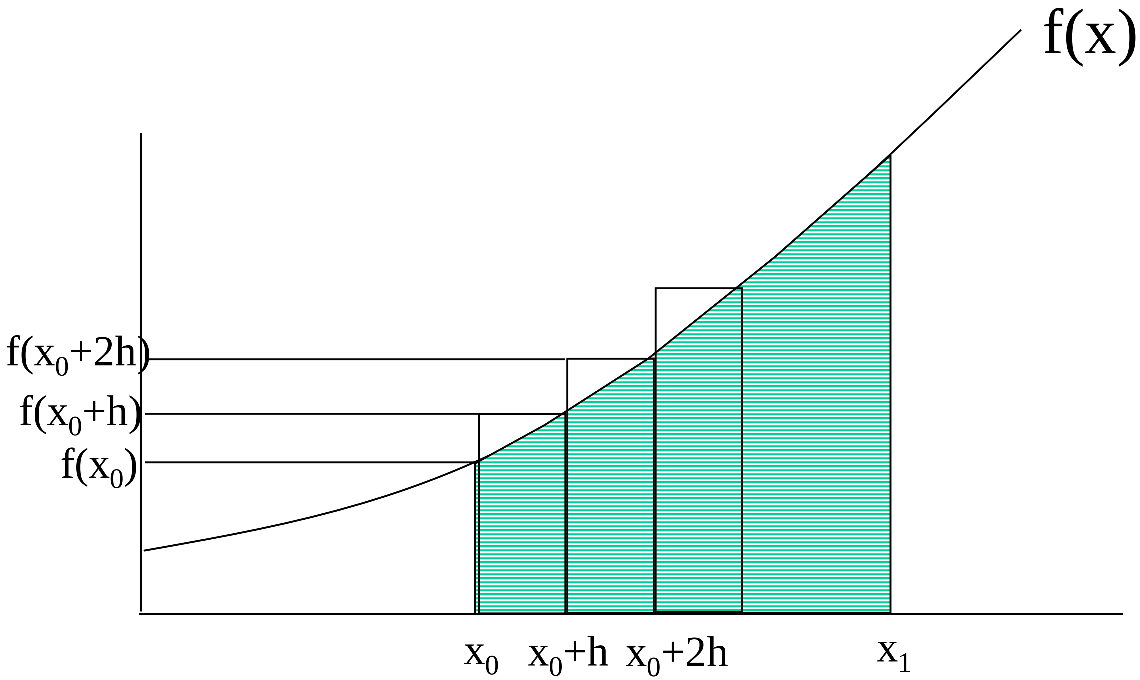
Suppose we wish to find the area under the function f(x) between x_0 and x_1

We write this as $F(x_1, x_0)$ or

$$F(x_1, x_0) = \int_{x_0}^{x_1} f(x)dx$$

One way to find an approximate answer is to divide x_1-x_0 into n intervals of width h (so that h = $(x_1 - x_0)/n$).

Areas under graphs.



$$Approximate\ area\ within\ interval\ i\ =\ h[f(x_0+(i-1)h)+f(x_0+ih)]/2$$

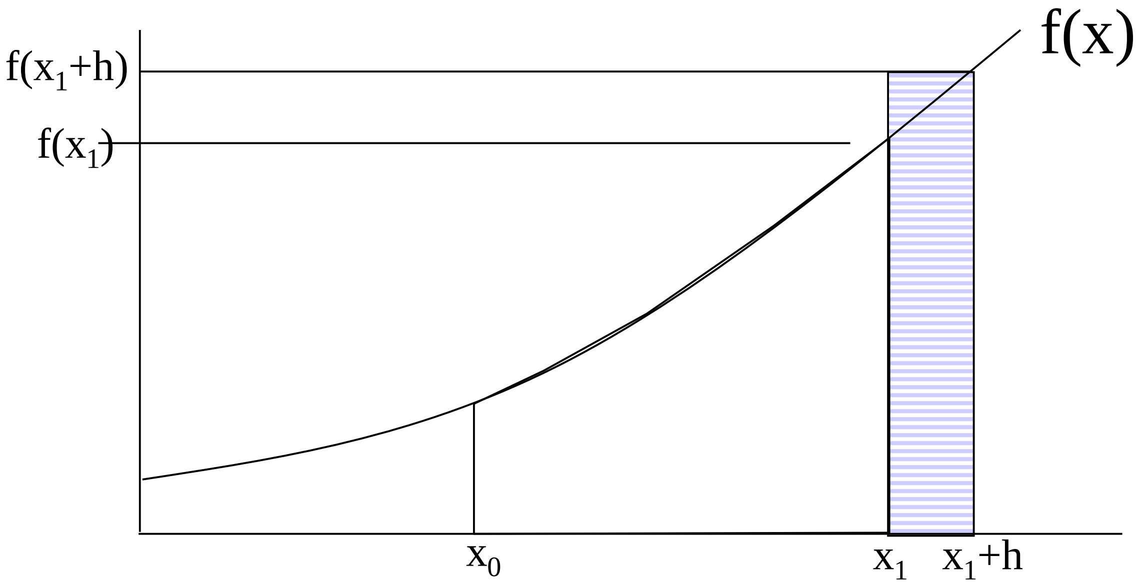
$$Total\ approximation\ =\sum_{i=1}^{n} h[f(x_0+(i-1)h)+f(x_0+ih)]/2$$

The integral of f between x_0 and x_1 is then the limit of this sum as $h \rightarrow 0$

Note that one implication of this definition is that integrals are additive so that

$F(x_1,0) = F(x_1,x_0)\ +\ F(x_0,0)$ or $F(x_1,x_0) =\ F(x_1,0)\ -\ F(x_0,0)$.

Usually we drop the zero part of this and just write $F(x_1)$ to mean the integral of x between 0 and x_1, then $F(x_1,x_0) =\ F(x_1)\ -\ F(x_0)$.



The area under the graph between x_0 and x_1 is then just $F(x_1) - F(x_0)$

Now consider the area between x_1 and x_1+h as $h \to 0$

This is $F(x_1 + h) - F(x_1)$.

We have $$F(x_1+h) - F(x_1) \approx h[f(x_1+h) + f(x_1)]/2 \approx hf(x_1)$$

or, $\frac{F(x_1+h)-F(x_1)}{h} \approx f(x_1)$ in the limit, as $h \to 0$, the left hand side of this expression is the derivative of F at x_1.

In other words, f is just the derivative of F.

Properties of definite integrals

1) The interchange of the limits of integration changes the sign of the definite integral

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

2) A definite integral has value 0 when the two limit of integration are identical

$$\int_a^a f(x)dx = 0$$

3) A definite integral can be expressed as a sum of a finite number of sub integrals as follows

$$\int_a^d f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx + \int_c^d f(x)dx$$

where $a < b < c < d$

4)

$$\int_a^b -f(x)dx = -\int_a^b f(x)dx$$

5)

$$\int_a^b kf(x)dx = k\int_a^b f(x)dx$$

6)

$$\int_a^b [f(x) + g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$$

7) Integration by parts

$$\int_{x=a}^{x=b} v\,du = uv]_{x=a}^{x=b} - \int_a^b u\,dv$$

8) Substitution: we have to replace the integration limits, a, b by $u(a)$ and $u(b)$

Example

$$\int_1^2 2x(x^2+1)\,dx = \int_1^2 (2x^3+2x)dx = \frac{x^4}{2} + x^2\Big|_1^2 = 10.5$$

Let $u = x^2 + 1$,

then $\frac{du}{dx} = 2x$ or $dx = \frac{du}{2x}$

We replace u and dx in the integral and we get:

$$\int_2^5 2xu\frac{du}{2x} = \int_2^5 u\,du = \frac{u^2}{2}\Big|_2^5 = \frac{25}{2} - \frac{4}{2} = 10.5$$

Improper integrals.

A definite integral is improper in one of two cases:

1) One or both of the limits of the integral is infinite.

An improper integral can be defined as

$$\int_a^{\infty} f(x)dx = \lim_{b\to\infty} \int_a^b f(x)dx$$

If this limit exists, then the integral is said to converge

2) When the integrand becomes infinite somewhere in the interval of integration $[a, b]$

$f(x) \to \infty \; as \; x \to p$ and $a < p < b$

$$\int_a^b f(x)dx = \int_a^p f(x)dx + \int_p^b f(x)dx = \lim_{y \to p^-} \int_a^y f(x)dx + \lim_{y \to p^+} \int_y^b f(x)dx$$

If these limits exist, then the integral is said to converge

Examples

$$\lim_{b\to\infty}\int_2^b \frac{1}{x^2}\,dx = \lim_{b\to\infty} -\frac{1}{x}\Big]_2^b = \lim_{b\to\infty} -\frac{1}{b}+\frac{1}{2} = \frac{1}{2}$$

$$\lim_{b\to\infty}\int_2^b \frac{1}{x}\,dx = \lim_{b\to\infty} \ln(x)\Big]_2^b = \lim_{b\to\infty} \ln(b)-\ln(2) = \infty$$

$$\int_a^3 \frac{1}{x}\,dx = \lim_{a\to 0}\int_a^3 \frac{1}{x}\,dx = \lim_{a\to 0} \ln(x)\Big]_a^3 = \lim_{a\to 0} \ln(3)-\ln(a) = \infty$$

$$\int_{-1}^{1} \frac{1}{x^3}\,dx = \int_{-1}^{0} \frac{1}{x^3}\,dx + \int_{0}^{1} \frac{1}{x^3}\,dx$$

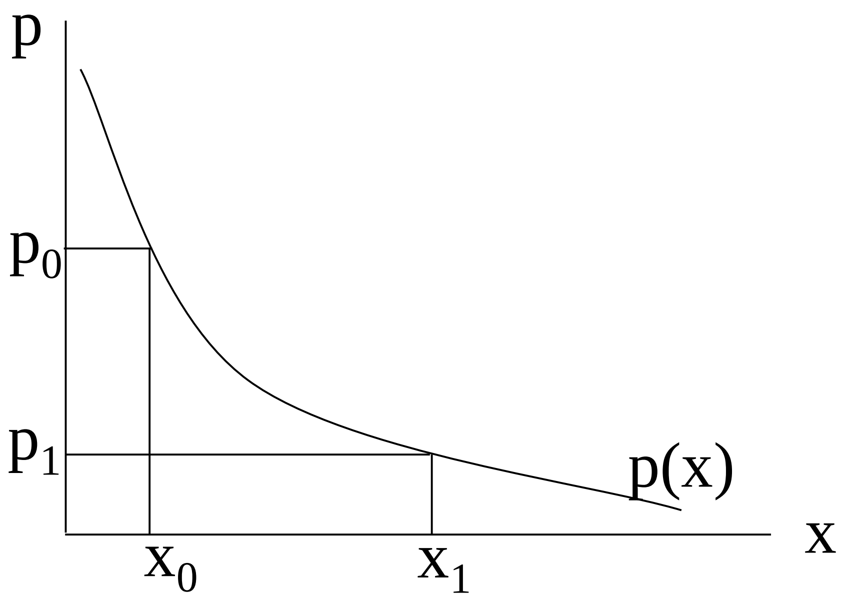
The integral is divergent because

$$\lim_{b\to 0^-}\int_{-1}^{b} \frac{1}{x^3}\,dx = \lim_{b\to 0^-}\left[-\frac{1}{2x^2}\right]_{-1}^{b} =$$

$$\lim_{b\to 0^-} -\frac{1}{2b^2} + 0.5 = -\infty$$

Example 1: Consumer Surplus

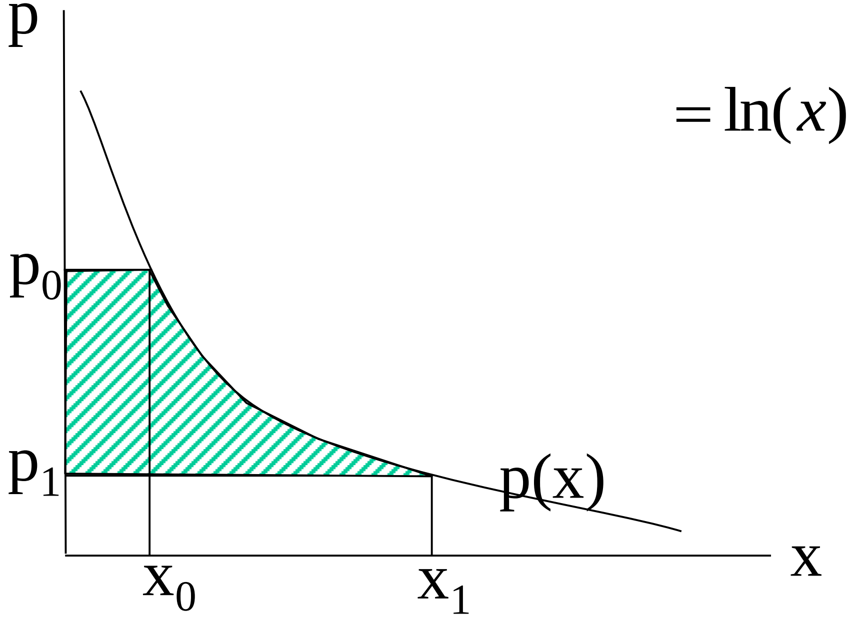
- Possibly the commonest application of integration in economics is in the calculation of consumer surplus. Mathematically this is straightforward, but it is confused by the way we put the ‘x’ variable on the vertical:



- Suppose demand function is p = 1/x. The price falls from p=2 to p=1. Find the increase in consumer surplus.
- Note that x rises from 0.5 to 1. It is tempting to find the surplus change by integrating the inverse demand function from 0.5 to 1.
- But this is wrong!

Example 1: Consumer Surplus

- The change in cs is the shaded area.
- i.e. write x = 1/p and integrate between p=1 and p=2: $\Delta CS = \int_1^2 \frac{1}{x} dx$

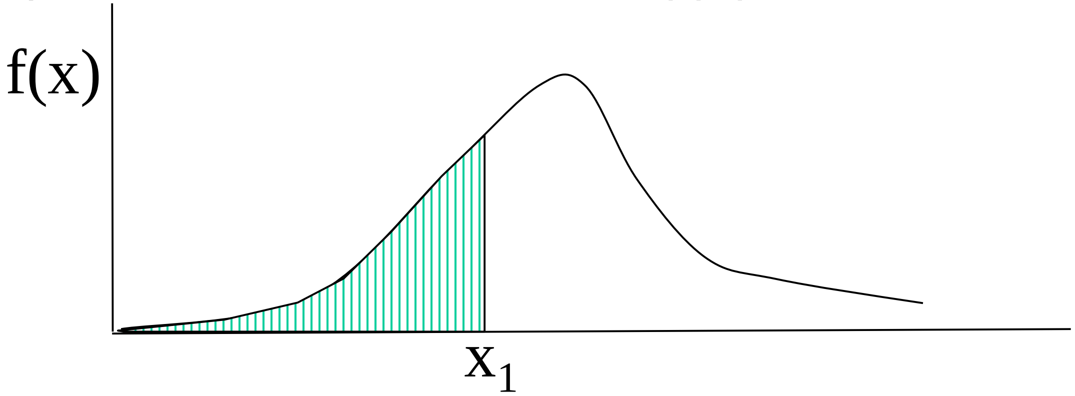


$$= \ln(x)]_1^2 = \ln(2) \approx 0.693$$

- Note that this is an example where CS is undefined (the relevant integral is improper and does not converge), but changes in CS are meaningful.

Example 2: Probability

- Integration is also useful in probability theory and statistics.
- Consider a continuous random variable– e.g. height or (roughly) income.
- The probability that the variable is less than or equal to x is called the cumulative density function, F(x).
- The probability density function (**pdf**), f(x) is the probability density that the variable equals x. It is the derivative of F(x) (where the derivative exists).



- That is,

$$F(x)=\int_{-\infty}^{x} f(x)dx$$

- Note that: $$\int_{-\infty}^{x} f(x)dx \geq 0; \int_{-\infty}^{\infty} f(x)dx = 1;$$

- Note also that in some contexts x may not be defined everywhere on [-∞,∞]

Example 2: Probability

- The expected value of a random variable is its mean. It is calculated as:

$$E(x) = \int_{-\infty}^{\infty} x f(x) dx$$

- The variance is:

$$E\left[\left(x - E(x)\right)^2\right] = \int_{-\infty}^{\infty} \left(x - E(x)\right)^2 f(x) dx$$

- Example: at Egham Station the probability of queuing at least t minutes for a ticket is $e^{-0.3t}$. What is the expected waiting time?
- The probability of queuing less than t minutes = 1 – $e^{-0.3t}$
- The pdf is the derivative of this function, and is $0.3e^{-0.3t}$

$$E(t) = \int_{0}^{\infty} t f(t) dt = \int_{0}^{\infty} t 0.3 e^{-0.3t} dt$$

- Integrate by parts

Example 2: Probability

$$\int \frac{du}{dt} v(t) dt = u(t)v(t) - \int u(t) \frac{dv}{dt} dt$$

Let $v(t) = t$ and $\frac{du}{dt} = 0.3e^{-0.3t}$

Then $\frac{dv}{dt} = 1$ and $u(t) = -e^{-0.3t}$

$$\int_0^\infty t0.3e^{-0.3t}\, dt =$$

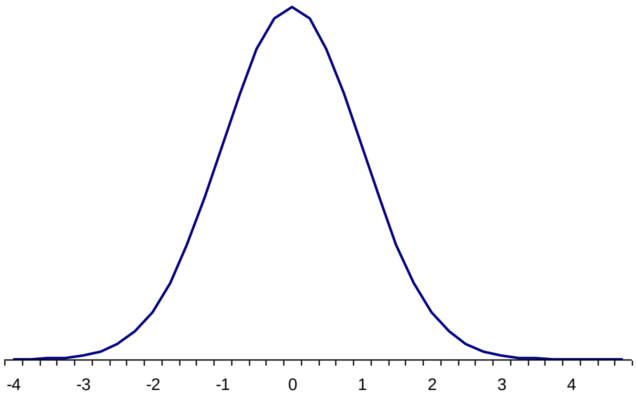
$$= -te^{-0.3t}\Big|_0^\infty - \int_0^\infty -e^{-0.3t} dt =$$

$$= 0 - \frac{e^{-0.3t}}{0.3}\Bigg|_0^\infty \approx 3.33$$

Example 2: Probability II

- The probability density function for a normal distribution with mean 0 and variance 1 is:

$$f(x) = \frac{e^{-(1/2)\left[x^2\right]}}{\sqrt{2\pi}}$$



- This is known as a standard normal distribution. The expected value is:

$$\int_{-\infty}^{\infty} xf(x)dx = \int_{-\infty}^{\infty} \frac{xe^{-(1/2)\left[x^2\right]}}{\sqrt{2\pi}}dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} xe^{-(1/2)\left[x^2\right]}dx$$

- We can integrate:

$$\int_{-\infty}^{\infty} xf(x)dx = \int_{-\infty}^{\infty} \frac{xe^{-(1/2)\left[x^2\right]}}{\sqrt{2\pi}}dx = \left.\frac{-e^{-(1/2)\left[x^2\right]}}{\sqrt{2\pi}}\right|_{-\infty}^{\infty} = \left.\frac{-e^{-(1/2)\left[x^2\right]}}{\sqrt{2\pi}}\right|_{-\infty}^{0} + \left.\frac{-e^{-(1/2)\left[x^2\right]}}{\sqrt{2\pi}}\right|_{0}^{\infty}$$

$$= \frac{-1}{\sqrt{2\pi}} + \frac{--1}{\sqrt{2\pi}} = 0$$

All example of Section 14.5

Domar model, section 14.6

Advice:

Try to do all exercises in Chapter 14