## The Framework

Suppose that, for a particular commodity, the demand and supply functions are as follows:

$$\begin{aligned} Q_d &= \alpha - \beta P \qquad (\alpha, \beta > 0) \\ Q_s &= -\gamma + \delta P \qquad (\gamma, \delta > 0) \end{aligned} \tag{15.8}$$

Then, according to (3.4), the equilibrium price should be[†]

$$P^* = \frac{\alpha + \gamma}{\beta + \delta} \qquad (= \text{some positive constant}) \tag{15.9}$$

If it happens that the initial price $P(0)$ is precisely at the level of $P^*$, the market will clearly be in equilibrium already, and no dynamic analysis will be needed. In the more interesting case of $P(0) \neq P^*$, however, $P^*$ is attainable (if ever) only after a due process of adjustment, during which not only will price change over time but $Q_d$ and $Q_s$, being functions of $P$, must change over time as well. In this light, then, the price and quantity variables can *all* be taken to be *functions of time*.

Our dynamic question is this: Given sufficient time for the adjustment process to work itself out, does it tend to bring price to the equilibrium level $P^*$? That is, does the time path $P(t)$ tend to converge to $P^*$, as $t \to \infty$?

## The Time Path

To answer this question, we must first find the time path $P(t)$. But that, in turn, requires a specific pattern of price change to be prescribed first. In general, price changes are governed by the relative strength of the demand and supply forces in the market. Let us assume, for the sake of simplicity, that the rate of price change (with respect to time) at any moment is always directly proportional to the *excess demand* $(Q_d - Q_s)$ prevailing at that moment. Such a pattern of change can be expressed symbolically as

$$\frac{dP}{dt} = j(Q_d - Q_s) \qquad (j > 0) \tag{15.10}$$

where $j$ represents a (constant) *adjustment coefficient.* With this pattern of change, we can have $dP/dt = 0$ if and only if $Q_d = Q_s$. In this connection, it may be instructive to note two senses of the term *equilibrium price:* the intertemporal sense ($P$ being constant over time) and the market-clearing sense (the equilibrium price being one that equates $Q_d$ and $Q_s$). In the present model, the two senses happen to coincide with each other, but this may not be true of all models.

By virtue of the demand and supply functions in (15.8), we can express (15.10) specifically in the form

$$\frac{dP}{dt} = j(\alpha - \beta P + \gamma - \delta P) = j(\alpha + \gamma) - j(\beta + \delta)P$$

or

$$\frac{dP}{dt} + j(\beta + \delta)P = j(\alpha + \gamma) \tag{15.10'}$$

[†] We have switched from the symbols $(a, b, c, d)$ of (3.4) to $(\alpha, \beta, \gamma, \delta)$ here to avoid any possible confusion with the use of $a$ and $b$ as parameters in the differential equation (15.4) which we shall presently apply to the market model.

Since this is precisely in the form of the differential equation (15.4), and since the coefficient of $P$ is nonzero, we can apply the solution formula (15.5′) and write the solution—the time path of price—as

$$P(t) = \left[P(0) - \frac{\alpha+\gamma}{\beta+\delta}\right]e^{-j(\beta+\delta)t} + \frac{\alpha+\gamma}{\beta+\delta}$$
$$= [P(0) - P^*]e^{-kt} + P^* \qquad [\text{by (15.9); } k \equiv j(\beta+\delta)] \qquad \textbf{(15.11)}$$

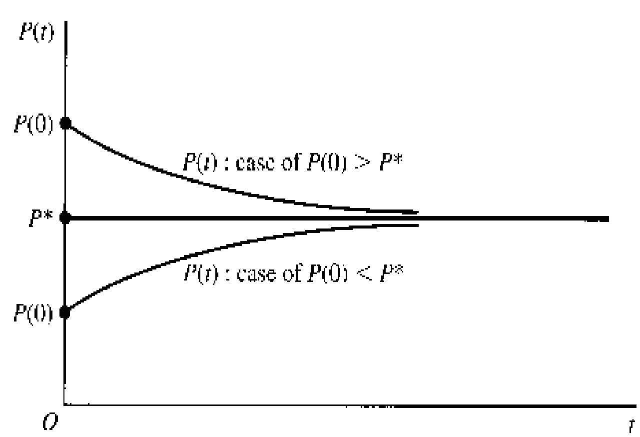
## The Dynamic Stability of Equilibrium

In the end, the question originally posed, namely, whether $P(t) \to P^*$ as $t \to \infty$, amounts to the question of whether the first term on the right of (15.11) will tend to zero as $t \to \infty$. Since $P(0)$ and $P^*$ are both constant, the key factor will be the exponential expression $e^{-kt}$. In view of the fact that $k > 0$, that expression does tend to zero as $t \to \infty$. Consequently, on the assumptions of our model, the time path will indeed lead the price toward the equilibrium position. In a situation of this sort, where the time path of the relevant variable $P(t)$ *converges* to the level $P^*$ interpreted here in its role as the intertemporal (rather than market-clearing) equilibrium—the equilibrium is said to be *dynamically stable*.

The concept of dynamic stability is an important one. Let us examine it further by a more detailed analysis of (15.11). Depending on the relative magnitudes of $P(0)$ and $P^*$, the solution (15.11) really encompasses three possible cases. The first is $P(0) = P^*$, which implies $P(t) = P^*$. In that event, the time path of price can be drawn as the horizontal straight line in Fig. 15.1. As mentioned earlier, the attainment of equilibrium is in this case a fait accompli. Second, we may have $P(0) > P^*$. In this case, the first term on the right of (15.11) is positive, but it will decrease as the increase in $t$ lowers the value of $e^{-kt}$. Thus the time path will approach the equilibrium level $P^*$ from above, as illustrated by the top curve in Fig. 15.1. Third, in the opposite case of $P(0) < P^*$, the equilibrium level $P^*$ will be approached from below, as illustrated by the bottom curve in the same figure. In general, to have dynamic stability, the *deviation* of the time path from equilibrium must either be identically zero (as in case 1) or steadily decrease with time (as in cases 2 and 3).

A comparison of (15.11) with (15.5′) tells us that the $P^*$ term, the counterpart of $b/a$, is nothing but the particular integral $y_p$, whereas the exponential term is the (definitized) complementary function $y_c$. Thus, we now have an economic interpretation for $y_c$ and $y_p$; $y_p$ represents the *intertemporal equilibrium level* of the relevant variable, and $y_c$ is the *deviation from equilibrium*. Dynamic stability requires the asymptotic vanishing of the complementary function as $t$ becomes infinite.

**FIGURE 15.1**



In this model, the particular integral is a constant, so we have a *stationary equilibrium* in the intertemporal sense, represented by $P^*$. If the particular integral is nonconstant, as in (15.7′), on the other hand, we may interpret it as a *moving equilibrium*.

## An Alternative Use of the Model

What we have done in the preceding is to analyze the dynamic stability of equilibrium (the convergence of the time path), given certain sign specifications for the parameters. An alternative type of inquiry is: In order to ensure dynamic stability, what specific restrictions must be imposed upon the parameters?

The answer to that is contained in the solution (15.11). If we allow $P(0) \neq P^*$, we see that the first ($y_c$) term in (15.11) will tend to zero as $t \to \infty$ if and only if $k > 0$—that is, if and only if

$$j(\beta + \delta) > 0$$

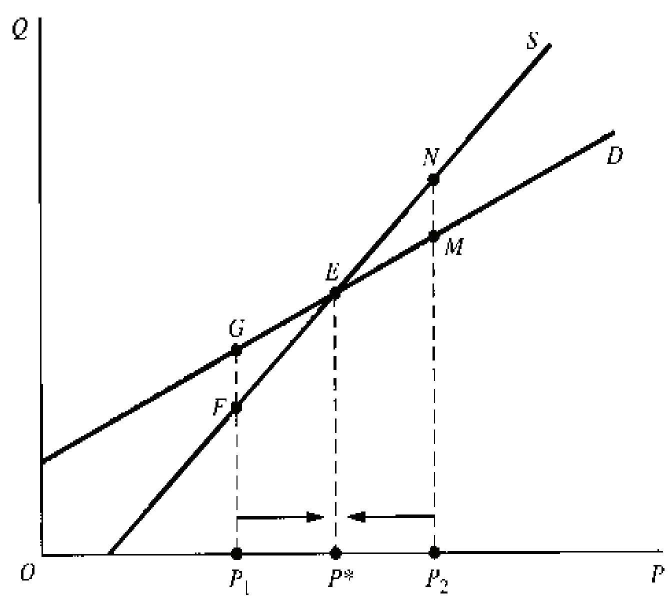
Thus, we can take this last inequality as the required restriction on the parameters $j$ (the adjustment coefficient of price), $\beta$ (the negative of the slope of the demand curve, plotted with $Q$ on the *vertical* axis), and $\delta$ (the slope of the supply curve, plotted similarly).

In case the price adjustment is of the "normal" type, with $j > 0$, so that excess demand drives price up rather than down, then this restriction becomes merely $(\beta + \delta) > 0$ or, equivalently,

$$\delta > -\beta$$

To have dynamic stability in that event, the slope of the supply must exceed the slope of the demand. When both demand and supply are normally sloped ($-\beta < 0$, $\delta > 0$), as in (15.8), this requirement is obviously met. But even if one of the curves is sloped "perversely," the condition may still be fulfilled, such as when $\delta = 1$ and $-\beta = 1/2$ (positively sloped demand). The latter situation is illustrated in Fig. 15.2, where the equilibrium price $P^*$ is, as usual, determined by the point of intersection of the two curves. If the initial price happens to be at $P_1$, then $Q_d$ (distance $P_1G$) will exceed $Q_s$ (distance $P_1F$), and the excess demand ($FG$) will drive price up. On the other hand, if price is initially at $P_2$, then

**FIGURE 15.2**



there will be a *negative* excess demand $MN$, which will drive the price down. As the two arrows in the figure show, therefore, the price adjustment in this case will be *toward* the equilibrium, no matter which side of $P^*$ we start from. We should emphasize, however, that while these arrows can display the direction, they are incapable of indicating the magnitude of change. Thus Fig. 15.2 is basically static, not dynamic, in nature, and can serve only to illustrate, not to replace, the dynamic analysis presented.

## EXERCISE 15.2

1. If both the demand and supply in Fig. 15.2 are negatively sloped instead, which curve should be steeper in order to have dynamic stability? Does your answer conform to the criterion $\delta > -\beta$?
2. Show that (15.10′) can be rewritten as $dP/dt + k(P - P^*) = 0$. If we let $P - P^* \equiv \Delta$ (signifying deviation), so that $d\Delta/dt = dP/dt$, the differential equation can be further rewritten as
$$\frac{d\Delta}{dt} + k\Delta = 0$$
Find the time path $\Delta(t)$, and discuss the condition for dynamic stability.
3. The dynamic market model discussed in this section is closely patterned after the static one in Sec. 3.2. What specific new feature is responsible for transforming the static model into a dynamic one?
4. Let the demand and supply be
$$Q_d = \alpha - \beta P + \sigma\frac{dP}{dt} \qquad Q_s = -\gamma + \delta P \qquad (\alpha, \beta, \gamma, \delta > 0)$$
(*a*) Assuming that the rate of change of price over time is directly proportional to the excess demand, find the time path $P(t)$ (general solution).
(*b*) What is the intertemporal equilibrium price? What is the market-clearing equilibrium price?
(*c*) What restriction on the parameter $\sigma$ would ensure dynamic stability?
5. Let the demand and supply be
$$Q_d = \alpha - \beta P - \eta\frac{dP}{dt} \qquad Q_s = \delta P \qquad (\alpha, \beta, \eta, \delta > 0)$$
(*a*) Assuming that the market is cleared at every point of time, find the time path $P(t)$ (general solution).
(*b*) Does this market have a dynamically stable intertemporal equilibrium price?
(*c*) The assumption of the present model that $Q_d = Q_s$ for all $t$ is identical with that of the static market model in Sec. 3.2. Nevertheless, we still have a dynamic model here. How come?

# 15.3 Variable Coefficient and Variable Term

In the more general case of a first-order linear differential equation

$$\frac{dy}{dt} + u(t)y = w(t) \tag{15.12}$$