An introduction to dynamic optimization

We divide these problems in according to dimensions:

1. Time can be modelled as discrete or continuous
2. Horizon can be finite or infinite

The solutions take the form of a complete time path

Dynamic maximization in continuous time

Example: profit maximization

Initial time $t = 0$

Terminal time t $= T$

At any time we have to choose a control variable u_t affecting a state variable y_t through an *equation of motion.*

State variable $y_t$affects the profit π_t

The problem is to maximize the profit over the whole period

Then the objective function is

$$\int_0^T \pi_t \, dt$$

y_0 and y_T are given

The simplest problem takes the form of:

$$\max \int_0^T F(t, y, u)\ dt$$

$$subject\ to$$

$$\frac{dy}{dt} \equiv y' = f(t, y, u)$$

$$y_0 = A \quad y_T\ free$$

$$u_t \in U \quad \forall t \in [0, T]$$

We require that $F(t, y, u)$ and $f(t, y, u)$

- are continuous in all arguments,
- have continuous first order partial derivatives w.r.t. y and t

Example

An economy produces Y using capita K and labor L using the production function:

$$Y = Y(K, L)$$

Further $Y = C + I$ where I is the capital.

Capital does not depreciate then:

$$I = \frac{dK}{dt}$$

$$I = Y - C = Y(K, L) - C = \frac{dK}{dt}$$

If we want maximize the utility over the period from 0 to T the problem becomes:

$$\max \int_0^T U(C)\ dt$$

$$subject\ to$$

$$\frac{dK}{dt} = Y(K,L) - C$$

$$K_0,\ K_T\ given$$

C is the control variable

K is the state variable

Hamiltonian function and costate variable

Costate variable is denoted by $\lambda(t)$

It measures the *shadow price* of the state variable

Hamiltonian function is:

$$H(t, y, u, \lambda) = F(t, y, u) + \lambda(t) f(t, y, u)$$

Then the Hamiltonian function as four arguments: $t, \; y, \; u, \; \lambda$

Maximum principle

(i) $H(t, y, u^*, \lambda) \geq H(t, y, u, \lambda) \; \forall \; t \in [0, T]$

(ii) $y' = \frac{dH}{d\lambda} = f(t, y, u)$ state equation

(iii) $\lambda' = -\frac{dH}{dy}$ costate equation

Hamiltonian System

(iv) $\lambda(T) = 0$ transversality condition

In the case the Hamiltonian is differentiable w.r.t. u and yields an interior solution, condition (i) can be replaced by:

$$\frac{dH}{du} = 0$$

Condition (iv) is appropriate only for the free-terminal state problem only

Example 1

$$\max \int_0^T -\sqrt{1+u^2}\ dt$$

$$subject\ to$$

$$y' = u$$

$$y_0 = A \quad y_T\ free$$

The Hamiltonian function is

$$H = -\sqrt{1+u^2} + \lambda u$$

$$H = -\sqrt{1+u^2} + \lambda u$$

(i) $H(t, y, u^*, \lambda) \geq H(t, y, u, \lambda) \; \forall \; t \in [0, T]$

(ii) $y' = \frac{dH}{d\lambda}$ state equation

(iii) $\lambda' = -\frac{dH}{dy}$ costate equation

(iv) $\lambda(T) = 0$ transversality condition

(i) $\frac{dH}{du} = \frac{-u}{\sqrt{1+u^2}} + \lambda = 0$

(ii) $y' = u$

(iii) $\lambda' = -\frac{dH}{dy} = 0$

(iv) $\lambda(T) = 0$

From (i) we get $u(t) = \frac{\lambda}{\sqrt{1-\lambda^2}}$

(i) $\frac{dH}{du} = \frac{-u}{\sqrt{1+u^2}} + \lambda = 0$

(ii) $y' = u$

(iii) $\lambda' = -\frac{dH}{dy} = 0$

(iv) $\lambda(T) = 0$

From (i) we get $u(t) = \frac{\lambda}{\sqrt{1-\lambda^2}}$

From (iii) we get that λ is constant over time

Then using condition (iv) we can conclude that $\lambda^*(t) = 0 \;\; \forall t \in [0, T]$

Then replacing in (i) we get $u^*(t) = 0$

Using (ii) we have $y' = 0 \rightarrow y^* = c$ *(a constant)*

Using the initial condition $y(0) = A \rightarrow c = A$

$$y^*(t) = A \quad \forall t \in [0, T]$$

Example 2

$$\max \int_0^1 (y - u^2)\ dt$$

$$subject\ to$$

$$y' = u$$

$$y(0) = 5 \quad y(1)\ free$$

The Hamiltonian function is

$$H = y - u^2 + \lambda u$$

$$H = y - u^2 + \lambda u$$

(i) $H(t, y, u^*, \lambda) \geq H(t, y, u, \lambda)\ \forall\ t \in [0, T]$

(ii) $y' = \frac{dH}{d\lambda}$ state equation

(iii) $\lambda' = -\frac{dH}{dy}$ costate equation

(iv) $\lambda(T)=0$ transversality condition

(i) $\frac{dH}{du} = -2\,u + \lambda = 0$

(ii) $y' = u$

(iii) $\lambda' = -\frac{dH}{dy} = -1$

(iv) $\lambda(T)=0$

(i) $\frac{dH}{du} = -2\,u + \lambda = 0$

(ii) $y' = u$

(iii) $\lambda' = -\frac{dH}{dy} = -1$

(iv) $\lambda(T)=0$

From (i) we get $u(t) = \frac{\lambda}{2}$

Replacing in (ii) → $y' = \frac{\lambda}{2}$ [1]

Integrating (iii) we get $\lambda(t) = c_1 - t$

Using (iv) $\lambda(1) = c_1 - 1 = 0$ → $c_1 = 1$ → $\lambda^* (t) = 1 - t$

Then replacing in [1] → $y' = \frac{1-t}{2}$ [2]

and integrating

$$y(t) = \frac{1}{2}t - \frac{1}{4}t^2 + c_2$$

Using the initial condition $y(0) = 5$

$y(0) = \frac{1}{2}0 - \frac{1}{4}0^2 + c_2 = 5 \rightarrow c_2 = 5$

$$y^*(t) = \frac{1}{2}t - \frac{1}{4}t^2 + 5$$

Replacing [2] in (ii)

$$u^*(t) = \frac{1}{2}(1 - t)$$

Example 3

$$\max \int_0^2 (2y - 3\,u)\ dt$$

$$subject\ to$$

$$y' = y + u$$

$$y(0) = 4 \quad y(2)\ free \ \text{ and } u(t) \in [0, 2]$$

The Hamiltonian function is

$$H = 2y - 3u + \lambda(y + u)$$

$$H = 2y - 3u + \lambda(y + u)$$

(i) $H(t, y, u^*, \lambda) \geq H(t, y, u, \lambda)\ \forall\ t \in [0, T]$

(ii) $y' = \frac{dH}{d\lambda}$ state equation

(iii) $\lambda' = -\frac{dH}{dy}$ costate equation

(iv) $\lambda(T)=0$ transversality condition

(i) $\frac{dH}{du} = -3 + \lambda = 0$

(ii) $y' = y + u$

(iii) $\lambda' = -\frac{dH}{dy} = -2 - \lambda$

(iv) $\lambda(T)=0$

(i) $\frac{dH}{du} = -3 + \lambda = 0$

(ii) $y' = y + u$

(iii) $\lambda' = -\frac{dH}{dy} = -2 - \lambda$

(iv) $\lambda(T) = 0$

Consider condition (i) $\frac{dH}{du} = \lambda - 3 = 0$

If $\lambda > 3$ function H is maximized for $u = 2$

If $\lambda < 3$ function H is maximized for $u = 0$

Consider (iii) (costate equation) solving this differential equation

$$\lambda(t) = Ae^{-t} - 2$$

Using (iv) $\lambda(2) = Ae^{-2} - 2 = 0 \;\rightarrow\; A = 2e^{2}$

$$\lambda^{*}(t) = 2e^{2-t} - 2$$

Note that $\lambda^*(t) = 2e^{2-t} - 2$ is a decreasing function of t

$$\lambda^*(0) = 2e^2 - 2 = 12.8$$

$$\lambda^*(2) = 2e^0 - 2 = 0$$

Then there is a critical time τ such that

$\lambda^*(t) > 3 \; if \; t < \tau \rightarrow u^* = 2$

$\lambda^*(t) < 3 \; if \; t > \tau \rightarrow u^* = 0$

To find a critical time τ we solve by τ the condition

$$\lambda^*(\tau) = 2e^{2-\tau} - 2 = 3$$

Solution is $\tau = 1.08$

Then

$$u^*(t) = 2 \;\; if \; t < 1.08$$

$$u^*(t) = 0 \; if \; t > 1.08$$

Alternative terminal conditions

1) Fixed terminal point → $Y(T) = y_T$ with T and y_T given.

   no transversality condition is necessary

2) Horizontal terminal line

   Transversality condition is $H_{t=T} = 0$

3) Truncated vertical line

   Transversality condition is

$$\lambda(T) \geq 0, y_T \geq y_{min}\,,\ (y_T - y_{min})\lambda(T) = 0$$

4) Truncated horizontal vertical line

   Transversality condition is

$$H_{t=T_{max}} \geq 0, \qquad T \leq T_{max}, \qquad (T - T_{max})H_{t=T_{max}} = 0$$

Example: Lifetime utility maximization

$$\max \int_0^T U(C(t))e^{-\delta t}\ dt$$

$$subject\ to$$

$$K' = rK(t) - C(t)$$

$$K(0) = K_0 \quad K(T) \geq 0$$

We assume that $U(C)$ is increasing and concave

The Hamiltonian function is

$$H = U(C(t))e^{-\delta t} + \lambda(rK(t) - C(t))$$

$$H = U(C(t))e^{-\delta t} + \lambda(rK(t) - C(t))$$

(i) $\frac{dH}{dC} = U'\big(C(t)\big)e^{-\delta t} - \lambda = 0$

(ii) $K' = rK(t) - C(t)$

(iii) $\lambda' = -r\lambda$

(iv) $\lambda(T) \geq 0, K(T) \geq 0$, $(K(T))\lambda(T) = 0$

(Truncated vertical line)

Differentiating (i) we get

$U''(C)C'e^{-\delta t} - \delta U'(C)e^{-\delta t} = \lambda'$ [1]

Replacing (i) in (iii) we get

$$\lambda' = -rU'(C)e^{-\delta t}$$

Replacing in [1] we get

Replacing in [1] we

$$U''(C)C'e^{-\delta t} - \delta U'(C)e^{-\delta t} = -rU'(C)e^{-\delta t}$$

$$U''(C)C' - \delta U'(C) \ = -rU'(C)$$

$$U''(C)C' + (r-\delta)U'(C) = 0$$

$$(r-\delta) = -\frac{U''(C)C'}{U'(C)}$$

Note C will be increasing over time only if $r > \delta$

Solving (iii) we get

$$\lambda(t) = \lambda_0 e^{-rt}$$

Replacing in (i) $U'(C) = \lambda_0 e^{(\delta - r)t}$

Marginal utility will be increasing over time if $\delta > r$

Note $\lambda(T) > 0$ then the terminal condition has to be $K(T) = 0$

Dynamic maximization in discrete time using Euler equations

Problem

An individual has to choose the optimal levels of consumption in future periods.

Individual starts with a given endowment that evolves by a given (exogenous) rate of interest.

We assume that this individual is characterized by an exponential discounting.

Two class of problems:

1. finite number of periods
2. infinite number of periods.

Finite Time

At the begin of each period t an individual is endowed by x_t.

The endowment produces interests, so the budget for period t is given by Rx_t , where $R > 1$.

This individual has to decide the share of the budget (Rx_t) to consume (c_t).

The share of the budget that is not consumed in period t will be the endowment in period $t + 1$, x_{t+1}.

Then $\rightarrow$ $R\, x_t = x_{t+1} + c_t$

The problem is:

$$\begin{cases} \max\limits_{\forall c_t} \sum_{t=0}^{n} U(c_t)\, \delta^t \quad \textit{subject to} \\ \qquad R\, x_t = x_{t+1} + c_t \\ c_t \geq 0, \quad x_t \geq 0, \quad x_0 \text{ given} \end{cases}$$

δ^t is the discount function where $0 < \delta < 1$.

x_0 is the endowment at time 0

Maximization is with respect to variables c_t for each $t \in \{0, 1, 2, \dots, n\}$

c_t is the control variable

x_t is the state variable

Consider the constraint $R\,x_t = x_{t+1} + c_t$

Then $c_t = R\,x_t - x_{t+1}$

Replacing in the objective function the problem can be written as:

$$\begin{cases} \max\limits_{\forall \mathrm{x_t}} \sum_{t=0}^{n} U(R\,x_t - x_{t+1})\,\delta^t \\ \text{such that:} \\ 0 \le x_{t+1} \le Rx_t \\ x_0 \ \text{given} \end{cases}$$

Note that in the utility function appears only state variables x_t

So maximization is with respect to state variables x_t .

To solve this problem we have to compute the first order conditions for all *n* state variables.

$$\ldots + \delta^{t-1}U(Rx_{t-1} - x_t) + \delta^t U(Rx_t - \boldsymbol{x_{t+1}})$$
$$+ \delta^{t+1}U(R\,\boldsymbol{x_{t+1}} - x_{t+2}) + \delta^{t+2}U(R\,x_{t+2} - x_{t+3}) + \cdots.$$

Note $\boldsymbol{x_{t+1}}$ appears only two times

Then the partial derivative respect to x_t is:

$$-U'(R\,x_{t-1} - x_t)\delta^{t-1} + U'(R\,x_t - x_{t+1})R\,\delta^t = 0\ \forall t \in \{1,2,..n\}$$

These conditions are called Eulero Lagrange conditions.

Note that in this problem, characterized by an exponential discount, the *n* equations are linear transformations each other. Therefore we can consider only one, that is:

$$-U'(R\,x_{t-1} - x_t) + U'(R\,x_t - x_{t+1})R\delta = 0$$

and using the relation $R\,x_t = x_{t+1} + c_t$ we have:

$-U'(c_{t-1}) + U'(c_t)R\,\delta = 0 \rightarrow U'(c_{t-1}) = U'(c_t)R\,\delta$

Note: If $U(.)$ is increasing and concave, the first order conditions are necessary and sufficient for a maximum.

If $R\,\delta \geq 1$ (≤ 1) the marginal utility is decreasing (increasing) on the time; it follows that consumption is increasing (decreasing) on the time.

A Note on first other conditions

$$\ldots + \delta^{t-1}U(Rx_{t-1} - x_t) + \delta^t U(Rx_t - \boldsymbol{x_{t+1}})$$
$$+ \delta^{t+1}U(R\,\boldsymbol{x_{t+1}} - x_{t+2}) + \delta^{t+2}U(R\,x_{t+2} - x_{t+3}) + \cdots.$$

Note $\boldsymbol{x_{t+1}}$ appears only two times

Condition of Eulero-Lagrange:

Assume $x^* = (x_1^*, x_2^*, \ldots \ldots, x_n^*)$ is the solution

Then

$$U(Rx_t^* - x_{t+1}) + \delta U(R\,x_{t+1} - x_{t+2}^*)$$
$$\leq$$
$$U(Rx_t^* - x_{t+1}^*) + \delta U(Rx_{t+1}^* - x_{t+2}^*)$$

Using first order conditions

$$U'(c_{t-1}) = U'(c_t)R\ \delta$$

we find a relation that links consumption at time t with consumption at time $t-1$, which is:

$$c_t = f\ (c_{t-1})$$

We note that in an optimal consumption plan, if the utility function is characterized by Non Satiation the individual has to consume the endowment as possible.

This means that $x_{n+1} = 0$

Sometime $x_{n+1} \neq 0$ (bequest)

An optimum plan has to satisfy the following condition:

$$\sum_{t=0}^{n} \frac{c_t}{R^{t+1}} = x_0$$

(If $x_{n+1} \neq 0$ then $\sum_{t=0}^{n} \frac{c_t}{R^{t+1}} = x_0 - x_{n+1}$ has to be satisfied)

Using the relation $c_t = f\,(c_{t-1})$ we can solve it by c_0.

Indeed every c_t can be written as a function of c_0

$$c_1 = f\,(c_0)$$

$$c_2 = f\,(c_1) = f\left(f\,(c_0)\right)$$

$$c_3 = f\,(c_2) = f\left(f\,(c_1)\right) = f\left(f\left(f\,(c_0)\right)\right)$$

.....

Example 1 (logarithmic utility in finite time):

Assume $U(c_t) = \ln c_t$.

The first order condition is

$$-\frac{1}{R\,x_{t-1} - x_t} + \frac{R\delta}{R\,x_t - x_{t+1}} = 0$$

$$-\frac{1}{c_{t-1}} + \frac{R\delta}{c_t} = 0$$

that gives

$$c_t = R\,\delta c_{t-1}$$

We can explicit each c_t as

$$c_t = (R\,\delta)^t c_0.$$

Then

$$\sum_{t=0}^{n} \frac{c_t}{R^{t+1}} = x_0$$

$$\rightarrow$$

$$\sum_{t=0}^{n} \frac{(R\,\delta)^t c_0}{R^{t+1}} = x_0$$

$$\rightarrow$$

$$\frac{c_0}{R} \sum_{t=0}^{n} \delta^t = x_0$$

$$\rightarrow$$

$$\frac{c_0}{R} \frac{1-\delta^{n+1}}{1-\delta} = x_0$$

$$\rightarrow$$

$$c_0 = R \frac{1-\delta}{1-\delta^{n+1}} x_0$$

Infinite time

In infinite time the problem (1) is written as:

$$\begin{cases} \max\limits_{\forall c_t} \sum_{t=0}^{\infty} U(c_t)\, \delta^t \\ \quad \text{subject to} \\ R\, x_t = x_{t+1} + c_t \\ \quad c_t \geq 0 \\ \quad x_t \geq 0 \\ \quad x_0 \ \text{given} \end{cases}$$

The solution is similar to the finite time problem and the levels of consumption are linked by the relation of Eulero - Lagrange

$$-U'(R\, x_{t-1} - x_t) + U'(R\, x_t - x_{t+1}) R\delta = 0$$

and using the relation $R\, x_t = x_{t+1} + c_t$ we have:

$$-U'(c_{t-1}) + U'(c_t) R\, \delta = 0 \rightarrow U'(c_{t-1}) = U'(c_t) R\, \delta$$

A candidate solution is given by:

$$\sum_{t=0}^{\infty} \frac{c_t}{R^{t+1}} = x_0 \quad [1]$$

Using the relation

$$U'(c_{t-1}) = U'(c_t)R\ \delta$$

we get a relation

$$c_t = f\ (c_{t-1})$$

Given that c_t can be written as a function of c_0, we can solve the condition [1] by c_0

The transversality condition $x_{n+1} = 0$ now is replaced by:

$$\lim_{t\to\infty} \delta^t P_t x_t = 0, \text{ where } P_t = \frac{\partial U(R\ x_t - x_{t+1})}{\partial x_t}.$$

Then we have to check if this condition is satisfied by the candidate solution.

Example 2 (logarithmic utility in infinite time):

The first order condition is the same than the finite time problem.

So, from the previous example, we have that:

$c_t = R\,\delta c_{t-1}$

and

$c_t = (R\,\delta)^t c_0.$

Replacing c_t in $\sum_{t=0}^{\infty} \frac{c_t}{R^{t+1}} = x_0$

and solving by c_o we get

$c_0 = R(1-\delta)x_0.$

Using these relations we find that

$$x_t = R\,\delta x_{t-1} = (R\,\delta)^t x_0.$$

Then

$$P_t = \frac{\partial ln(R\,x_t - x_{t+1})}{\partial x_t} = \frac{R}{Rx_t - x_{t+1}}.$$

Therefore we can check the end condition is satisfied, that is:

$$\lim_{t\to\infty} \frac{\delta^t R}{Rx_t - x_{t+1}} (R\,\delta)^t x_0 = \lim_{t\to\infty} \frac{\delta^t R}{(R\,\delta)^t c_0} (R\,\delta)^t x_0$$

$$= \lim_{t\to\infty} \delta^t R\,\frac{x_0}{c_0} = 0$$

Model of growth with one sector

Suppose a simple economy where individuals produce only one good (y) that is used for consumption and investment.

i. in each period t, the good y is produced in quantity $y_t = f(k_t)$ where k_t is the stock of capital.

ii. In each period t the output can be divided by consumption c_t and investment i_t.

iii. $k_{t+1} = i_t$ (the capital depreciates completed in each period)

iv. $k_{t+1} + c_t = f(k_t)$

v. The representative individual has to decide the quantity to consume in each period c_t.

The problem is:

$$\begin{cases} \max\limits_{\forall \mathrm{c_t}} \sum_{t=0}^{\infty} U(c_t)\,\delta^t \quad \textit{subject to} \\ f(k_t) = k_{t+1} + c_t \\ c_t \geq 0, \quad k_t \geq 0, \quad k_0 \text{ given} \end{cases}$$

δ^t is the discount function where $0 < \delta < 1$.

k_0 is the capital at time 0

Maximization is with respect to variables c_t for each $t \in \{0, 1, 2, \ldots, n\}$

Eulero Lagrange condition is

$$U'(f(k_{t-1}) - k_t) = U'(f(k_t) - k_{t+1})f'(k_t)\delta$$

This condition is sufficient for an optimum if the function $f(.)$ is concave and is satisfied the following transversality condition

$\lim_{t\to\infty} \delta^t P_t k_t = 0$, where $P_t = \frac{\partial U(f(k_t)-k_{t+1})}{\partial k_t}$.

Assume

$$U(c_t) = \ln c_t$$
$$f(k_t) = k_t^\alpha$$

Eulero Lagrange condition is:

$$\frac{1}{k_{t-1}^\alpha - k_t} = \frac{\alpha\delta k_t^{\alpha-1}}{k_t^\alpha - k_{t+1}}$$

Solving by k_{t+1} we get:

$$k_{t+1} = k_t^\alpha(1+\alpha\delta) - \alpha\delta k_t^{\alpha-1} k_{t-1}^\alpha$$

There are many paths that satisfy the Eulero Lagrange condition.

One can be the following. Divide the above condition by ${k_t}^\alpha$:

$$\frac{k_{t+1}}{k_t^\alpha} = (1+\alpha\delta) - \alpha\delta\frac{k_{t-1}^\alpha}{k_t}$$

Let be $Z_h = \frac{k_h}{k_{h-1}^\alpha}$ then:

$$Z_{t+1} = (1+\alpha\delta) - \frac{\alpha\delta}{Z_t}$$

$$Z_{t+1} = (1 + \alpha\delta) - \frac{\alpha\delta}{Z_t}$$

Assuming $Z_{t+1} = Z_t = Z$

$$Z = (1 + \alpha\delta) - \frac{\alpha\delta}{Z}$$

Solving by Z we get $Z = \alpha\delta$ and $Z = 1$.

a) $k_t = \alpha\delta k_{t-1}^{\alpha}$

b) $k_t = k_{t-1}^{\alpha}$

In case b) consumption is 0 in all periods, then utility is not defined

$(\lim_{x \to 0} \ln x = -\infty)$

We check solution a) $k_t = \alpha\delta k_{t-1}^{\alpha}$

$\lim_{t\to\infty} \delta^t P_t k_t = 0$, where $P_t = \frac{\partial ln(k_t^{\alpha} - k_{t+1})}{\partial k_t} = \frac{\alpha k_t^{\alpha-1}}{k_t{}^{\alpha} - k_{t+1}}$.

$$\delta^t P_t k_t = \delta^t \frac{\alpha k_t^{\alpha-1}}{k_t^{\alpha} - k_{t+1}} k_t = \delta^t \frac{\alpha k_t^{\alpha}}{k_t^{\alpha} - k_{t+1}} \quad [1]$$

Using $k_{t+1} = \alpha\delta k_t^{\alpha}$ we get $k_{t+1} - k_t^{\alpha} = \alpha\delta k_t^{\alpha} - k_t^{\alpha} = k_t^{\alpha}(\alpha\delta - 1)$

$k_t^{\alpha} - k_{t+1} = k_t^{\alpha}(1 - \alpha\delta)$ replacing in [1]

$$\delta^t P_t k_t = \frac{\delta^t \alpha}{(1 - \alpha\delta)}$$

then

$$\lim_{t\to\infty} \delta^t P_t k_t = \lim_{t\to\infty} \frac{\delta^t \alpha}{(1 - \alpha\delta)} = 0$$