Corso di Laurea in Fisica - UNITS

ISTITUZIONI DI FISICA

PER IL SISTEMA TERRA

Born of the Wave Equation

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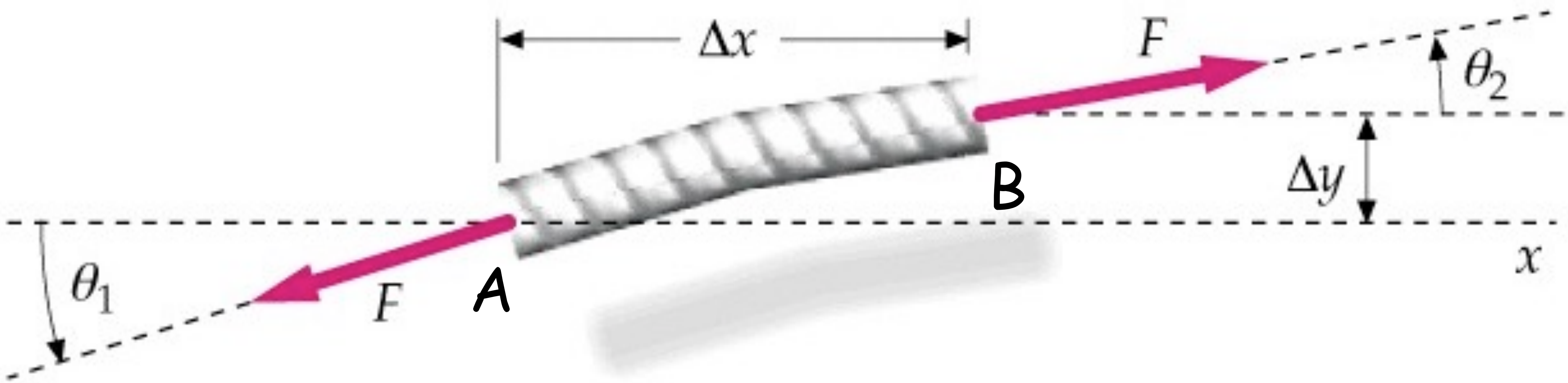
What is a **wave?** – 2

Small perturbations of a **stable** equilibrium point → **Linear restoring force** → **Harmonic Oscillation**

Coupling of harmonic oscillators → the disturbances can propagate, superpose and **stand**

Normal modes of the system

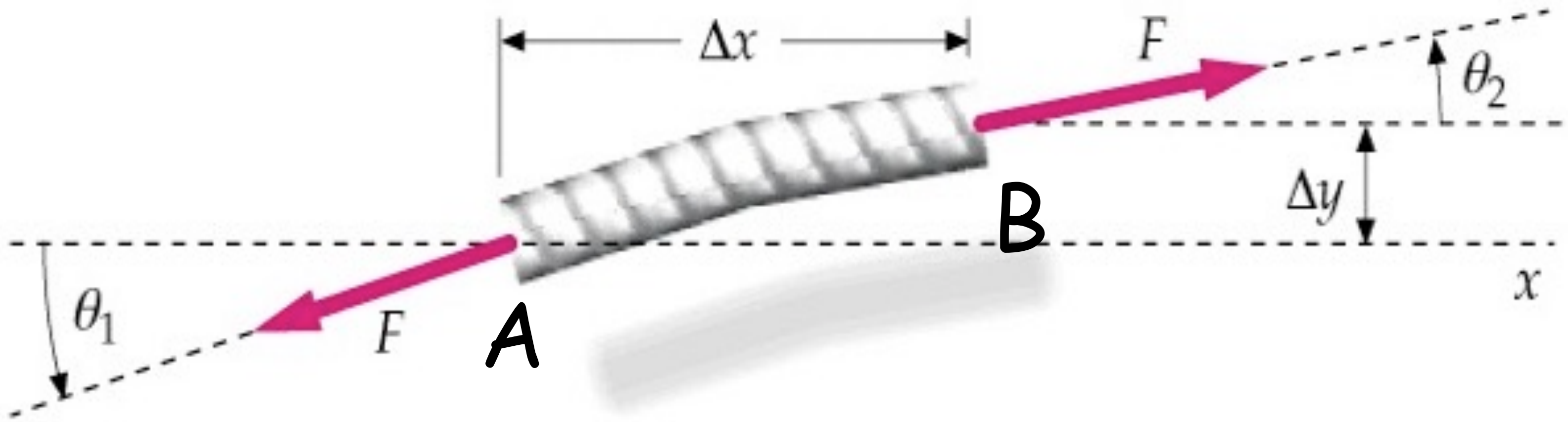
Derivation of the wave equation



Consider a small segment of string of length Δx and tension F

The ends of the string make **small** angles θ_1 and θ_2 with the x-axis.

The vertical displacement Δy is very **small** compared to the length of the string



Resolving forces vertically

$$\sum F_y = F\sin\theta_2 - F\sin\theta_1$$
$$= F(\sin\theta_2 - \sin\theta_1)$$

From small angle approximation
$\sin\theta \sim \tan\theta$

$$\sum F_y \approx F(\tan\theta_2 - \tan\theta_1)$$

The tangent of angle A (B) = pendence of the curve in A (B) given by $\frac{\partial y}{\partial x}$

$$\therefore \quad \sum F_y \approx F\left(\left(\frac{\partial y}{\partial x}\right)_B - \left(\frac{\partial y}{\partial x}\right)_A\right)$$

We now apply N2 to segment

$$\sum F_y = ma = \mu\,\Delta x\left(\frac{\partial^2 y}{\partial t^2}\right)$$

$$\mu\,\Delta x\left(\frac{\partial^2 y}{\partial t^2}\right) = F\left(\left(\frac{\partial y}{\partial x}\right)_B - \left(\frac{\partial y}{\partial x}\right)_A\right)$$

$$\frac{\mu}{F}\left(\frac{\partial^2 y}{\partial t^2}\right) = \frac{\left[(\partial y/\partial x)_B - (\partial y/\partial x)_A\right]}{\Delta x}$$

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The derivative of a function is defined as

$$\left(\frac{\partial f}{\partial x}\right) = \lim_{\Delta x \to 0} \frac{\left[f(x+\Delta x) - f(x)\right]}{\Delta x}$$

If we associate f(x+Δx) with $(\partial y/\partial x)_B$ and f(x) with $(\partial y/\partial x)_A$

as Δx -> 0

$$\frac{\mu}{F}\left(\frac{\partial^2 y}{\partial t^2}\right) = \frac{\partial^2 y}{\partial x^2}$$

This is the linear wave equation for waves on a string

Solution of the wave equation

Consider a solution of the form $y(x,t) = A\sin(kx-\omega t)$

$$\frac{\partial^2 y}{\partial t^2} = -\omega^2 A\sin(kx-\omega t) \qquad \frac{\partial^2 y}{\partial x^2} = -k^2 A\sin(kx-\omega t)$$

If we substitute these into the linear wave equation

$$\frac{\mu}{F}(-\omega^2 A\sin(kx-\omega t)) = -k^2 A\sin(kx-\omega t)$$

$$\frac{\mu}{F}\omega^2 = k^2$$

and, using $\omega^2/k^2 = F/\mu = v^2$, i.e. $v = \omega/k$

$$v = \sqrt{F/\mu}$$

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$$

General form of LWE

Speed of waves

A general property of waves is that the speed of a wave depends on the properties of the medium, but is independent of the motion of the source of the waves.

Consider a wave moving along a rope experimentally we find

(i) the greater the tension in a rope the faster the waves propagate

(ii) waves propagate faster in a light rope than a heavy rope

ie $v \propto$ tension (F) and $v \propto 1/\text{mass}$

known as **Mersenne's law**

Mersenne's law

L'Harmonie Universelle (1637)

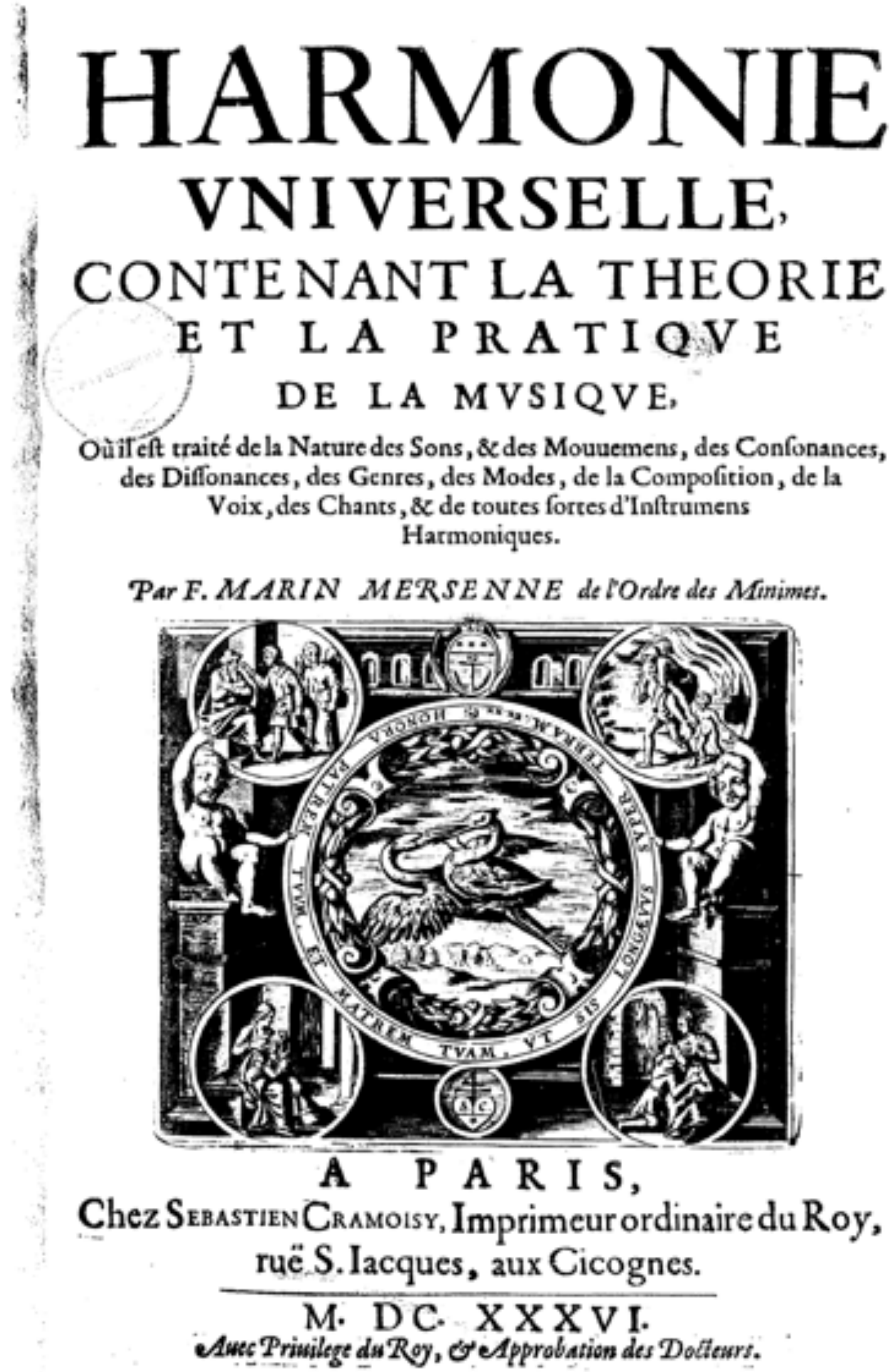
This book contains (Marine) Mersenne's laws which describe the frequency of oscillation of a stretched string.

This frequency is:

a) Inverse proportional to the length of the string (this was actually known to the ancients, and is usually credited to Pythagoras himself).

b) Proportional to the square root of the stretching force, and

c) Inverse proportional to the square root of the mass per unit length.



HARMONIE
VNIVERSELLE,
CONTENANT LA THEORIE
ET LA PRATIQVE
DE LA MVSIQVE,

Où il est traité de la Nature des Sons, & des Mouuemens, des Consonances, des Dissonances, des Genres, des Modes, de la Composition, de la Voix, des Chants, & de toutes sortes d'Instrumens Harmoniques.

Par F. MARIN MERSENNE de l'Ordre des Minimes.

A PARIS,
Chez SEBASTIEN CRAMOISY, Imprimeur ordinaire du Roy, ruë S. Iacques, aux Cicognes.

M. DC. XXXVI.

Auec Priuilege du Roy, & Approbation des Docteurs.

The linear wave equation

Earlier we introduced the concept of a wavefunction to represent waves travelling on a string.

All wavefunctions $y(x,t)$ represent solutions of the

LINEAR WAVE EQUATION

The wave equation provides a complete description of the wave motion and from it we can derive the wave velocity

The most general solution is, for 1D homogeneous medium,

$$y(x,t)=g(x+vt)+f(x-vt)$$

D'Alembert's solution

D'Alembert (1747) "Recherches sur la courbe que forme une corde tendue mise en vibration" (Researches on the curve that a tense cord forms [when] set into vibration), Histoire de l'académie royale des sciences et belles lettres de Berlin, vol. 3, pages 214-219.

D'Alembert (1750) "Addition au mémoire sur la courbe que forme une corde tenduë mise en vibration," Histoire de l'académie royale des sciences et belles lettres de Berlin, vol. 6, pages 355-360.

$$y(x,t) \rightarrow y(\xi,\eta) \text{ with } \xi=x-vt,\ \eta=x+vt$$

$$y_x = \frac{\partial y}{\partial x} = y_\xi \xi_x + y_\eta \eta_x = y_\xi + y_\eta;\ y_{xx} = \frac{\partial}{\partial x}\left(y_x\right) = y_{\xi\xi} + 2y_{\xi\eta} + y_{\eta\eta},\ y_{tt} = v^2(y_{\xi\xi} - 2y_{\xi\eta} + y_{\eta\eta})$$

$$\Rightarrow y_{\xi\eta} = \frac{\partial^2 y}{\partial\xi\,\partial\eta} = \frac{\partial}{\partial\xi}\left(\frac{\partial y}{\partial\eta}\right) = 0$$

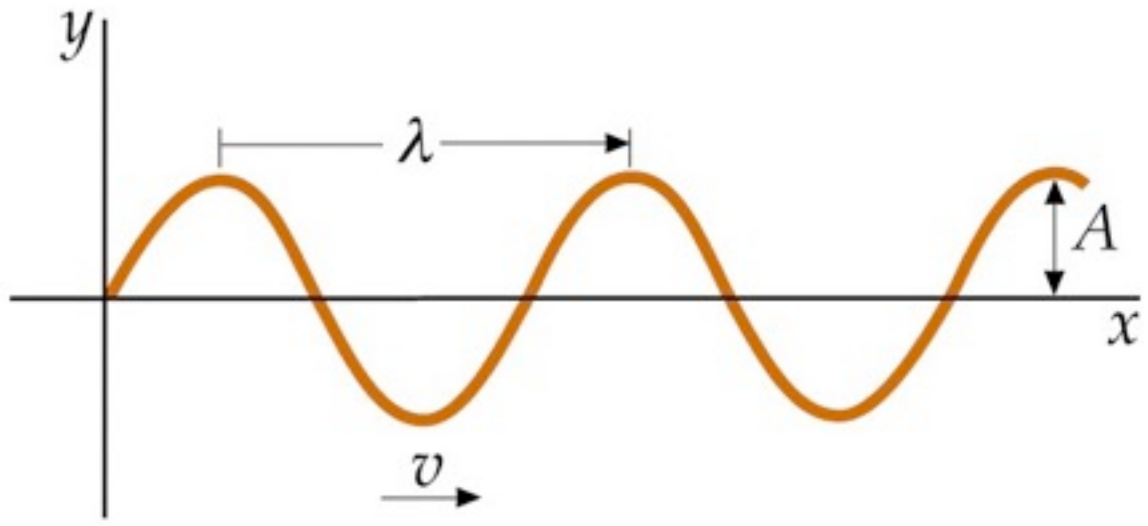
$$y = h(\xi) + g(\eta) \ \Rightarrow\ y(x,t) = h(x-vt) + g(x+vt)$$

and if the initial conditions are y(x,0)=f(x) and initial velocity=0

$$y(x,t) = \frac{1}{2}\left[f(x-vt) + f(x+vt)\right]$$

Harmonic Waves

A harmonic wave is sinusoidal in shape, and has a displacement y at time t=0



$$y = A\sin\left(\frac{2\pi}{\lambda}x\right)$$

A is the **amplitude** of the wave and λ is the **wavelength** (the distance between two crests);

if the wave is moving to the right with speed v, the wavefunction at some t is given by:

$$y = A\sin\left[\frac{2\pi}{\lambda}(x - vt)\right]$$

Time taken to travel one wavelength is the **period** T

Velocity, wavelength and period are related by

$$v = \frac{\lambda}{T} \quad \text{or} \quad \lambda = vT$$

$$\therefore \quad y = A\sin\left[2\pi\left(\frac{x}{\lambda} - \frac{t}{T}\right)\right]$$

The wavefunction shows the periodic nature of y:

at any time t y has the same value at x, x+λ, x+2λ.........

and at any x y has the same value at times t, t+T, t+2T......

It is convenient to express the harmonic wavefunction by defining the **wavenumber k**, and the **angular frequency** ω

$$\text{where} \quad k = \frac{2\pi}{\lambda} \quad \text{and} \quad \omega = \frac{2\pi}{T}$$

$$\therefore \quad y = A\sin(kx - \omega t)$$

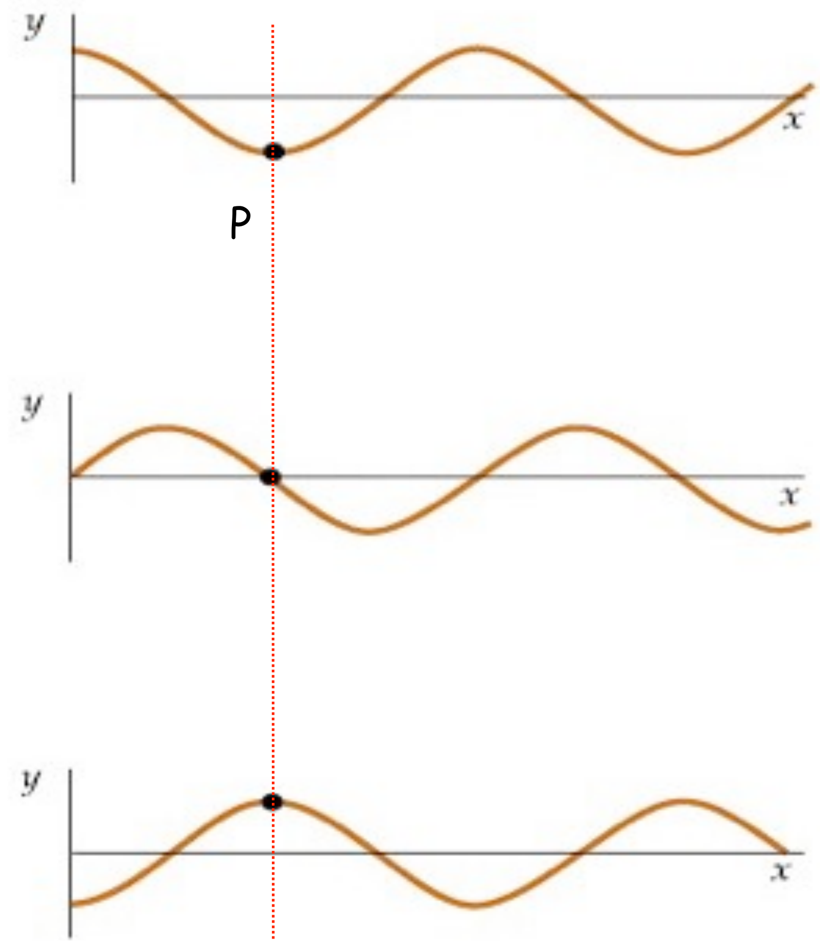
This assumes that the displacement is zero at x=0 and t=0. If this is not the case we can use a more general form

$$y = A\sin(kx - \omega t - \phi)$$

where ϕ is the **phase constant** and is determined from initial conditions

The wavefunction can be used to describe the motion of any point P.

If $y = A\sin(kx - \omega t)$



Transverse velocity v_y

$$v_y = \left.\frac{dy}{dt}\right|_{x=\mathrm{constant}}$$

$$= \frac{\partial y}{\partial t}$$

$$= -\omega A\cos(kx - \omega t)$$

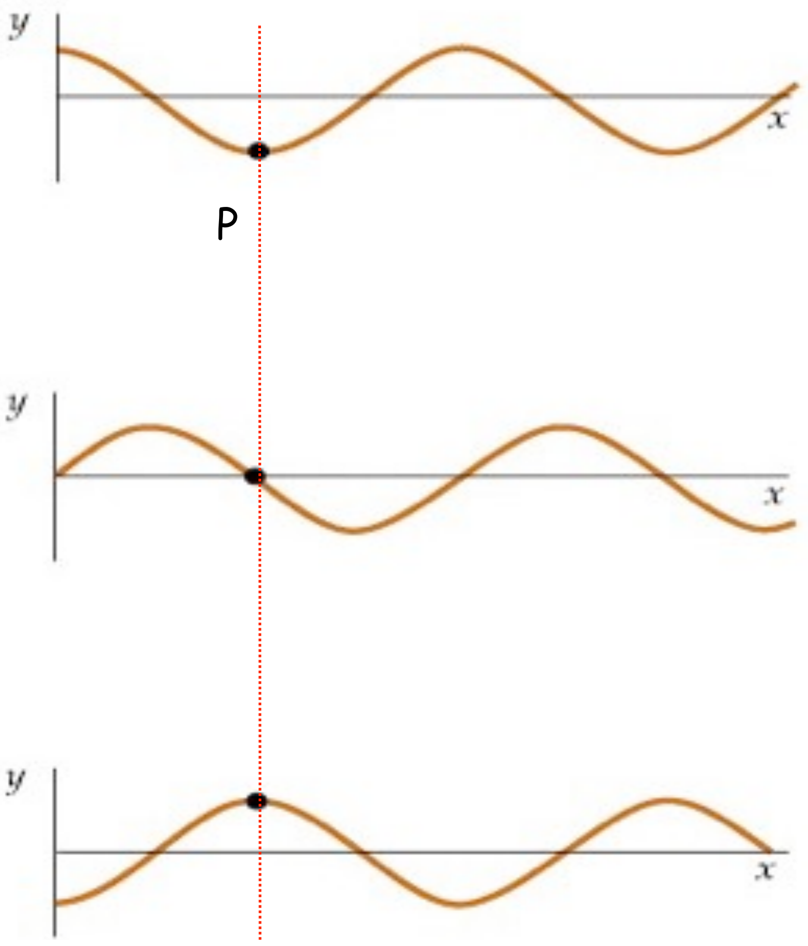
which has a maximum value, $(v_y)_{max} = \omega A$, when $y = 0$

Transverse acceleration a_y

$$a_y = \left.\frac{dv_y}{dt}\right|_{x=\text{constant}}$$

$$= \frac{\partial v_y}{\partial t}$$

$$= -\omega^2 A \sin(kx - \omega t)$$



which has a maximum absolute value, $(a_y)_{max} = \omega^2 A$, when t=0

NB: x-coordinates of P are constant

Example

A harmonic wave on a rope is given by the expression

$$y(x,t) = 10\sin(2x - 5t)$$

where the amplitude is in mm, k in rad m^{-1}, and ω in rad s^{-1}

(a) Determine the velocity and acceleration for each element of the rope.

(b) What are the maximum values of the acceleration and velocity ?

(c) Is the displacement +ve or -ve at x=1m and t=0.2s ?

(a) Determine the velocity and acceleration for each element of the rope.

Generally $y(x,t) = A\sin(kx - \omega t) \quad \therefore \quad v_y = -\omega A\cos(kx - \omega t)$

$$y(x,t) = 10\sin(2x - 5t)$$

$$\therefore \quad v_y = -5 \times 10\cos(2x - 5t)$$

$$v_y = -50\cos(2x - 5t)$$

Generally $y(x,t) = A\sin(kx - \omega t) \quad \therefore \quad a_y = -\omega^2 A\sin(kx - \omega t)$

$$\therefore \quad a_y = -5^2 \times 10\sin(2x - 5t)$$

$$a_y = -250\sin(2x - 5t)$$

(b) What are the maximum values of the acceleration and velocity ?

$$(a_y)_{max} = \omega^2 A \qquad (v_y)_{max} = \omega A$$

$$(a_y)_{max} = 5^2 \times 10 \qquad (v_y)_{max} = 5 \times 10$$

$$(a_y)_{max} = 250 mms^{-2} \qquad (v_y)_{max} = 50 mms^{-1}$$

(c) Is the displacement +ve or -ve at x=1m and t=0.2s ?

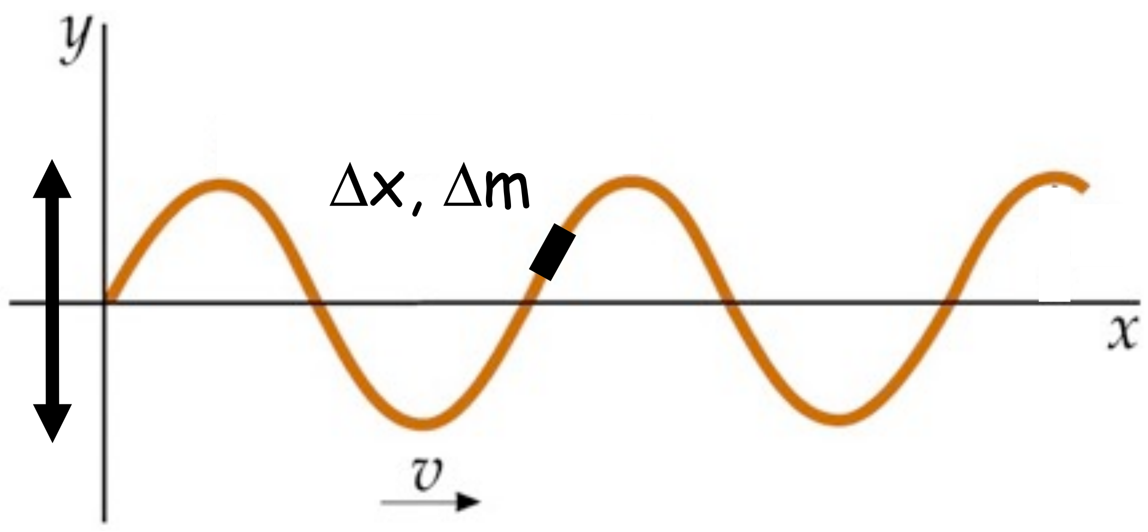
$$y(1,0.2) = 10\sin((2x1) - (5x0.2))$$

$$y(1,0.2) = 8.415$$

Displacement is +ve

Energy of waves on a string

Consider a harmonic wave travelling on a string.



Source of energy is an external agent on the left of the wave which does work in producing oscillations.

Consider a small segment, length Δx and mass Δm.

The segment moves vertically with SHM, frequency ω and amplitude A.

Generally $$E = \frac{1}{2} m\omega^2 A^2$$

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If we apply this to our small segment, the total energy of the element is

$$\Delta E = \frac{1}{2} (\Delta m)\, \omega^2 A^2$$

If μ is the mass per unit length, then the element Δx has mass $\Delta m = \mu\, \Delta x$

$$\Delta E = \frac{1}{2} (\mu\, \Delta x)\, \omega^2 A^2$$

If the wave is travelling from left to right, the energy ΔE arises from the work done on element Δm_i by the element Δm_{i-1} (to the left).

Similarly Δm_i does work on element Δm_{i+1} (to the right) ie. energy is transmitted to the right.

The rate at which energy is transmitted along the string is the power and is given by dE/dt.

If $\Delta x \to 0$ then

$$\text{Power} = \frac{dE}{dt} = \frac{1}{2}\left(\mu \frac{dx}{dt}\right)\omega^2 A^2$$

but dx/dt = speed

$$\therefore \quad \text{Power} = \frac{1}{2}\mu\, \omega^2 A^2 v$$

$$\text{Power} = \frac{1}{2}\mu\, \omega^2 A^2 v$$

Power transmitted on a harmonic wave is proportional to

(a) the wave speed v

(b) the square of the angular frequency ω

(c) the square of the amplitude A

All harmonic waves have the following general properties:

The power transmitted by any harmonic wave is proportional to the square of the frequency and to the square of the amplitude.

What is a *wave*? - 3

Small perturbations of a **stable** equilibrium point → **Linear restoring force** → **Harmonic Oscillation**

Coupling of harmonic oscillators → the disturbances can **propagate**, superpose and stand

WAVE: organized **propagating imbalance**, satisfying differential equations of motion

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$$

General form of LWE