Corso di Laurea in Fisica - UNITS

ISTITUZIONI DI FISICA PER IL SISTEMA TERRA

Modes

in strings, air columns & membranes

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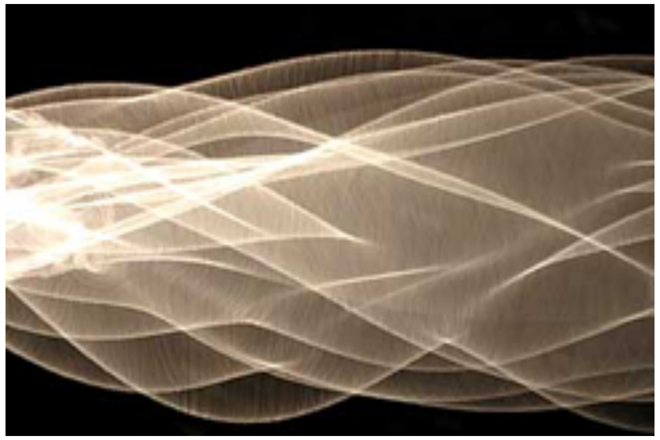
Separation of variables

- A starting point to study differential equations is to guess solutions of a certain form (ansatz). Dealing with linear PDEs, the superposition principle principle guarantees that linear combinations of separated solutions will also satisfy both the equation and the homogeneous boundary conditions.
    - Separation of variables: a PDE of n variables $\Rightarrow$ n ODEs
    - Solving the ODEs by BCs to get normal modes (solutions satisfying PDE and BCs).
- The proper choice of linear combination will allow for the initial conditions to be satisfied
    - Determining exact solution (expansion coefficients of modes) by ICs

Separation of variables: string



$$\frac{\partial^2 y(x,t)}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 y(x,t)}{\partial t^2} = 0$$

and if it has separable solutions:

$$y(x,t) = X(x)T(t)$$

$$\frac{d^2X(x)}{dx^2} + k^2X(x) = 0$$

$$X(x) = A\cos(kx) + B\sin(kx)$$

$$T''(t) + c^2k^2T(t) = 0$$

$$T(t) = C\cos(\omega t) + D\sin(\omega t)$$

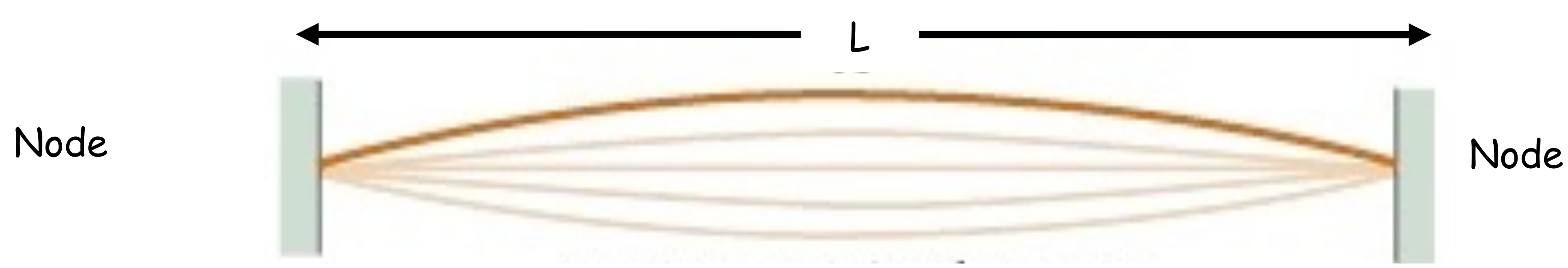
$\omega = ck$

To be determined by initial and boundary conditions

Consider a string of length L and fixed at both ends

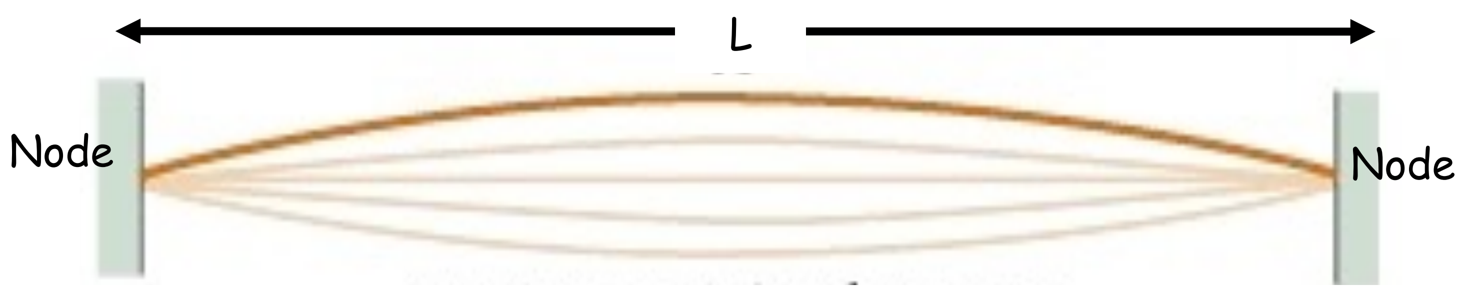
The string has a number of natural patterns of vibration called **NORMAL MODES**

Each normal mode has a characteristic frequency which we can easily calculate



When the string is displaced at its mid point the centre of the string becomes an antinode.

Standing waves in a string fixed at both ends



String is fixed at both ends $\therefore$ $y(x,t) = 0$ at $x = 0$ and L

$y(0,t)=0$ when $x = 0$ as $\sin(kx) = 0$ at $x = 0$

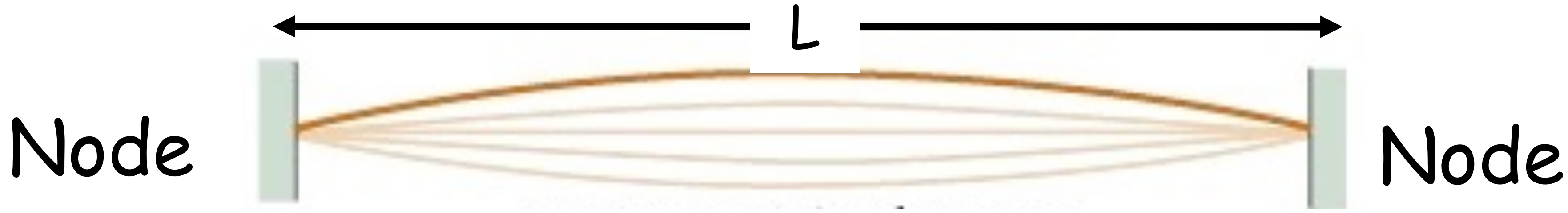
$$y(x,t) = 2A_0 \sin(kx)\cos(\omega t)$$

$y(L,t) = 0$ when $\sin(kL) = 0$ ie $k_n L = n\pi$ $n=1,2,3....$

but $k_n = 2\pi / \lambda$ $\therefore$ $(2\pi / \lambda_n)L = n\pi$ or $\lambda_n = 2L/n$

Standing waves in a string fixed at both ends

For first normal mode $L = \lambda_1 / 2$



The next normal mode occurs when the length of the string L = one wavelength, i.e. $L = \lambda_2$

The third normal mode occurs when $L = 3\lambda_3 /2$

Generally normal modes occur when $L = n\lambda_n /2$

ie $\lambda_n = \frac{2L}{n}$ where $n = 1,2,3........$

Standing waves in a string fixed at both ends

The natural frequencies associated with these modes can be derived from $f = v/\lambda$

$$f = \frac{v}{\lambda} = \frac{n}{2L}v \quad \text{with } n = 1,2,3.....$$

For a string of mass/unit length μ, under tension F we can replace v by $(F/\mu)^{\frac{1}{2}}$

$$f = \frac{n}{2L}\sqrt{\frac{F}{\mu}} \quad \text{with } n = 1,2,3.....$$

The lowest frequency (**fundamental**) corresponds to n = 1

$$\text{ie} \quad f = \frac{1}{2L}v \qquad \text{or} \quad f = \frac{1}{2L}\sqrt{\frac{F}{\mu}}$$

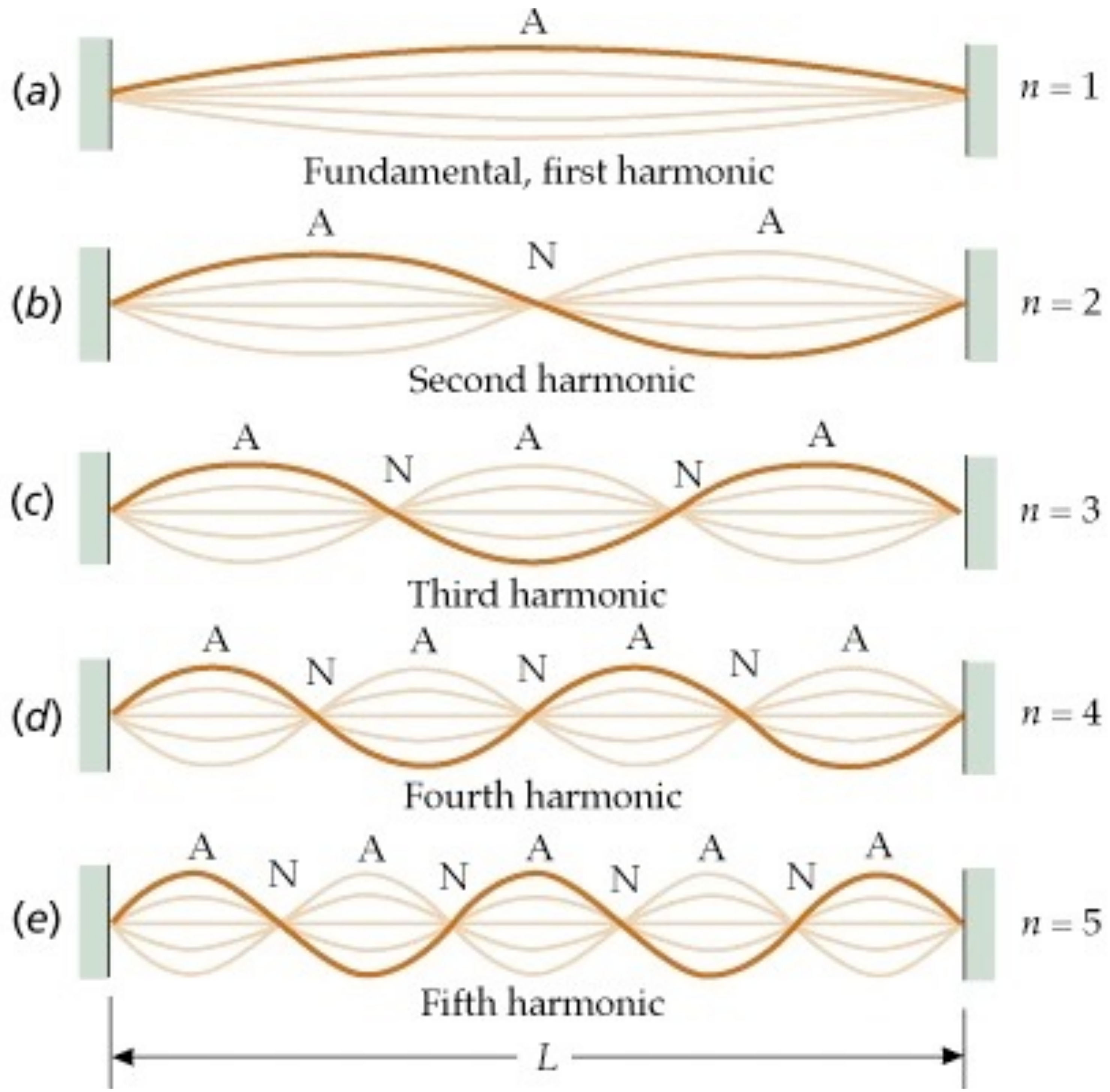
Musical Interpretation

The frequencies of modes with n = 2, 3, … (**harmonics**) are integral multiples of the fundamental frequency, $2f_1$, $3f_1$……

These higher natural frequencies together with the fundamental form a **harmonic series**.

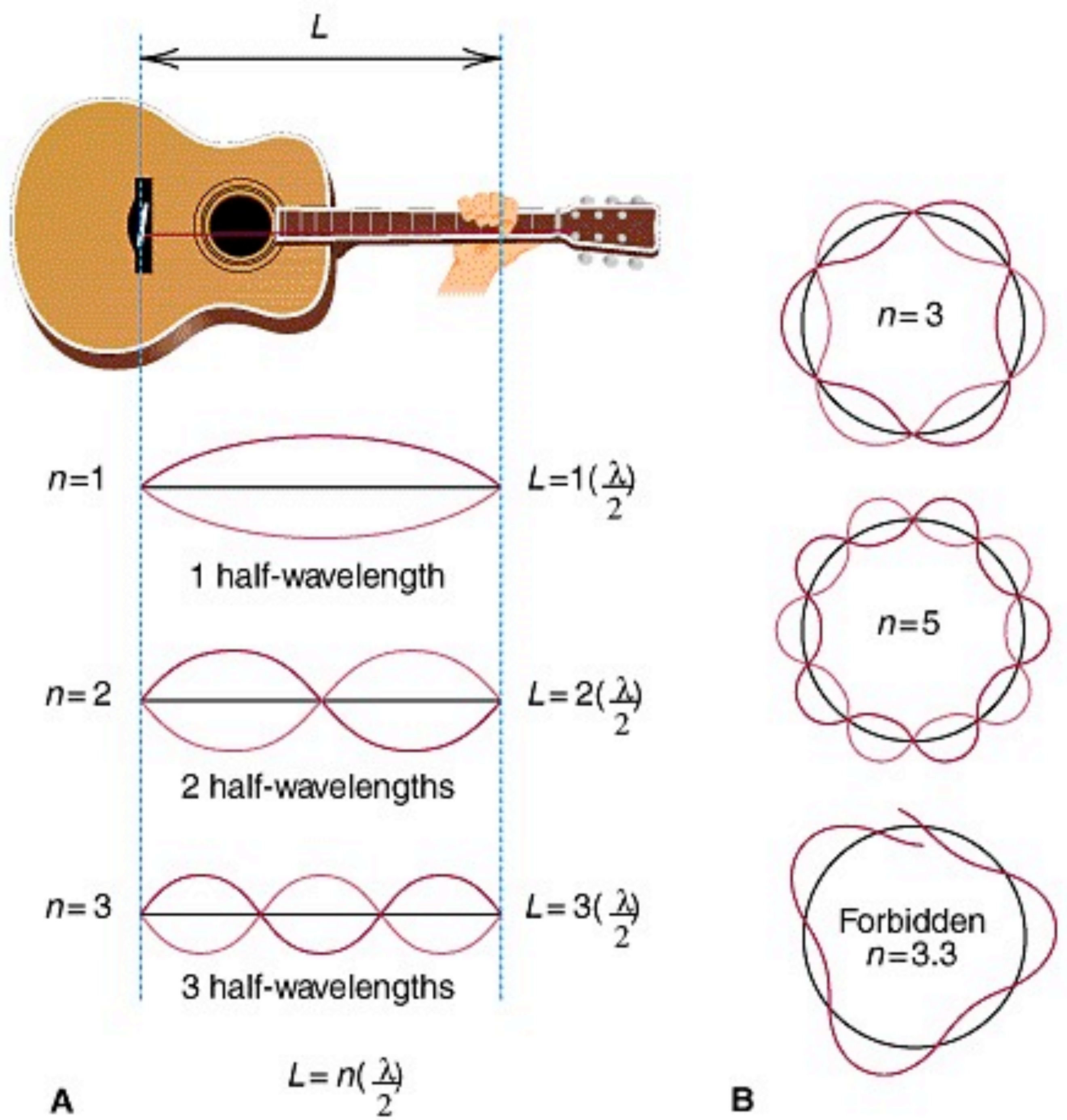
The fundamental f_1 is the first harmonic, $f_2 = 2f_1$ is the second harmonic, $f_n = nf_1$ is the nth harmonic

In music the allowed frequencies are called **overtones** where the second harmonic is the first overtone, the third harmonic the second overtone etc.



Tipler Fig 16-11

Guitars and Quantum mechanics



Musical Instruments

When a stretched string is distorted so that the initial shape corresponds to a harmonic it will vibrate at the frequency of that harmonic.

If the string is struck (piano) or bowed (violin) the resulting vibration will include many frequencies. Waves of the "wrong" frequency will destroy themselves when travelling between the fixed ends of the string and the string effectively "selects" the normal mode frequencies.

The frequency and pitch of a stringed instrument can be changed either by varying the tension F or the length L (guitar, violin)

Modal summation on a string

Recall modes on a string:

$$u(x, t) = \sum_{n=0}^{\infty} A_n \, U_n(x, \omega_n) \cos(\omega_n t)$$

This is the sum of standing waves or *eigenfunctions,* $U_n(x, \omega_n)$, each of which is weighted by the amplitude A_n and vibrates at its *eigenfrequency* ω_n.

The eigenfunctions and eigenfrequencies are constants due to the physical properties of the string.

The amplitudes depend on the position and nature of the source that excited the motion.

The eigenfunctions were constrained by the boundary conditions, so that

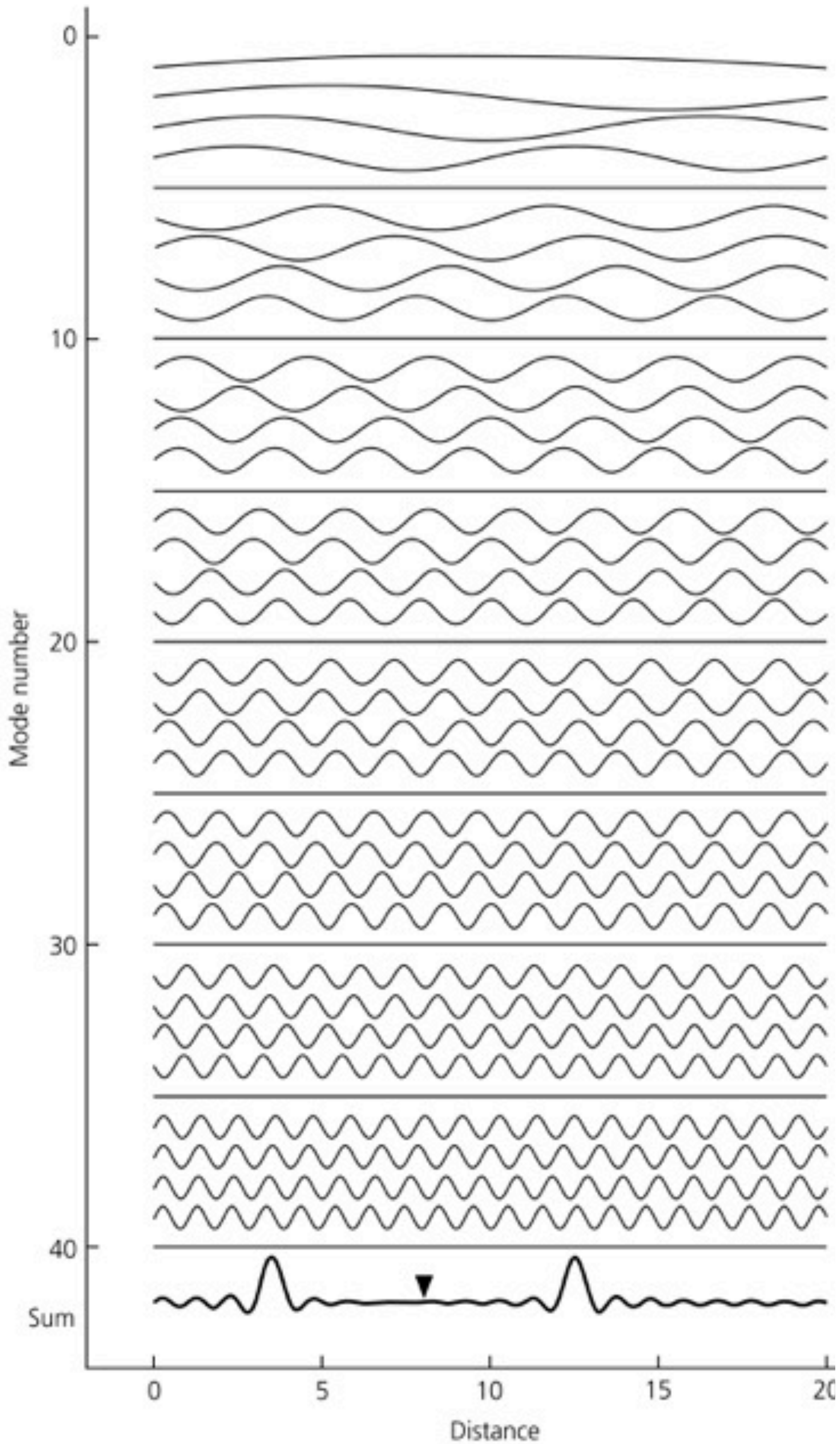
$$U_n(x, \omega_n) = \sin(n\pi x/L) = \sin(\omega_n x/\mathrm{v}) \qquad \omega_n = n\pi \mathrm{v}/L = 2\pi \mathrm{v}/\lambda$$

$$u(x, t) = \sum_{n=0}^{\infty} \sin(n\pi x_s/L)\, F(\omega_n)\, \sin(n\pi x/L)\, \cos(\omega_n t)$$

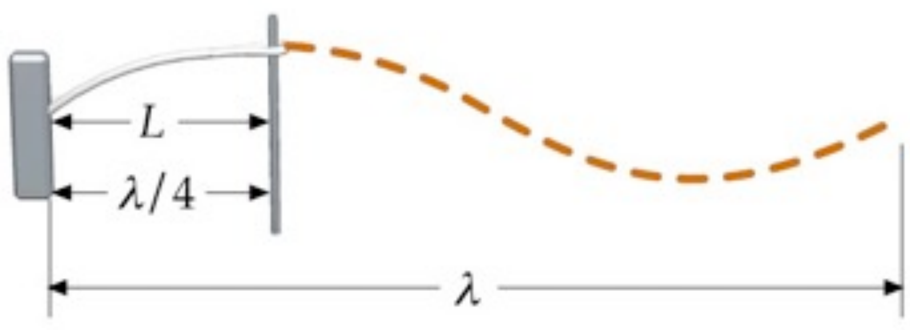
The source, at $x_s = 8$, is described by

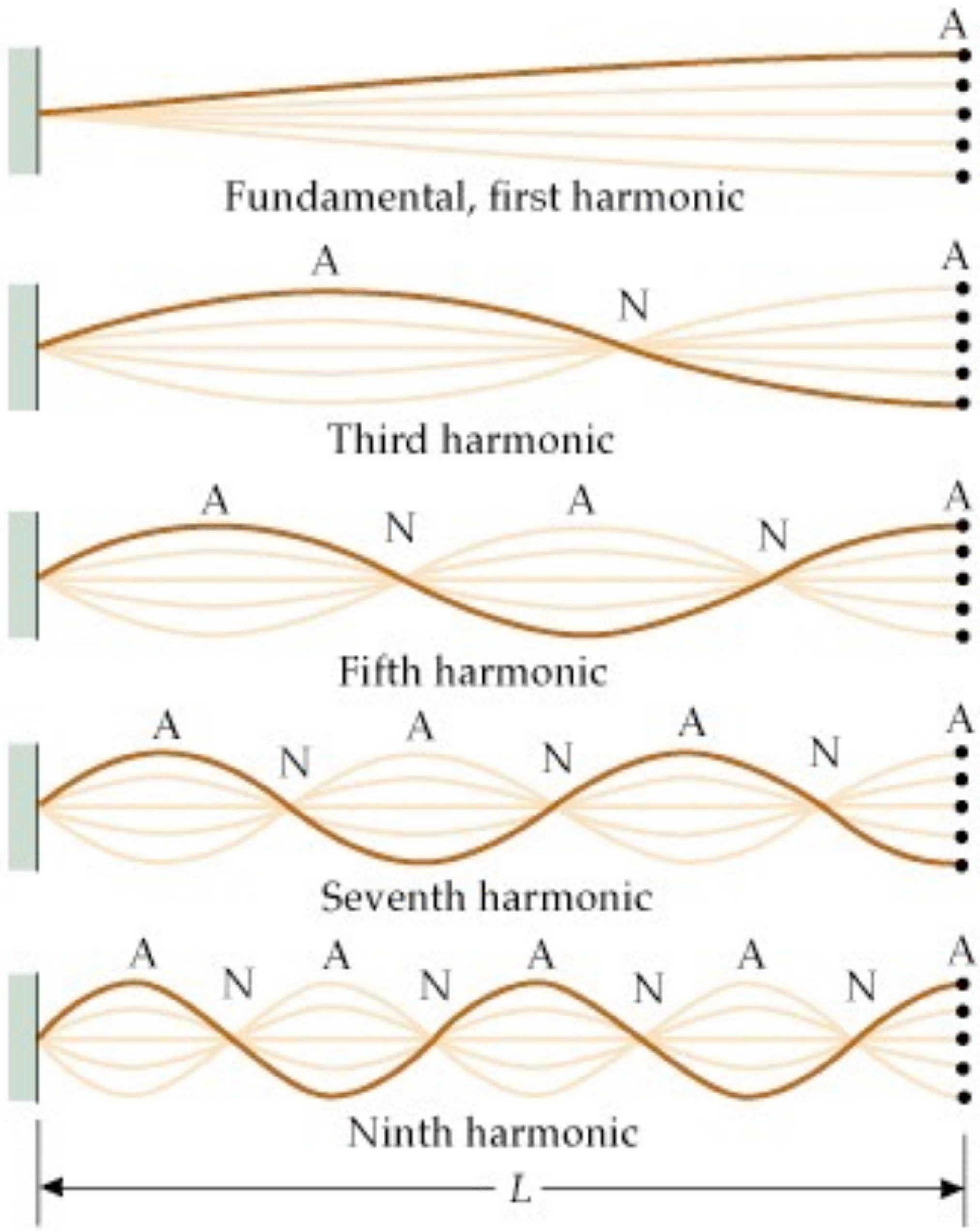
$$F(\omega_n) = \exp[-(\omega_n \tau)^2/4]$$

with $\tau = 0.2$.



Tipler Fig 16-17





Tipler fig 16-15

One end of string is node, the other an antinode.

For fundamental mode of vibration
$L = \lambda_1/4$

For next highest mode of vibration
$L = 3\lambda_3/4$

Generally $\boxed{L = n\lambda_n/4}$ with n=1,3,5...

Standing waves with string free at one end

if $L = n\lambda_n/4$ then $\lambda_n = 4L/n$

Resonant frequencies are given by $f_n = v/\lambda_n$

$$\therefore \quad f_n = n\frac{v}{4L}$$

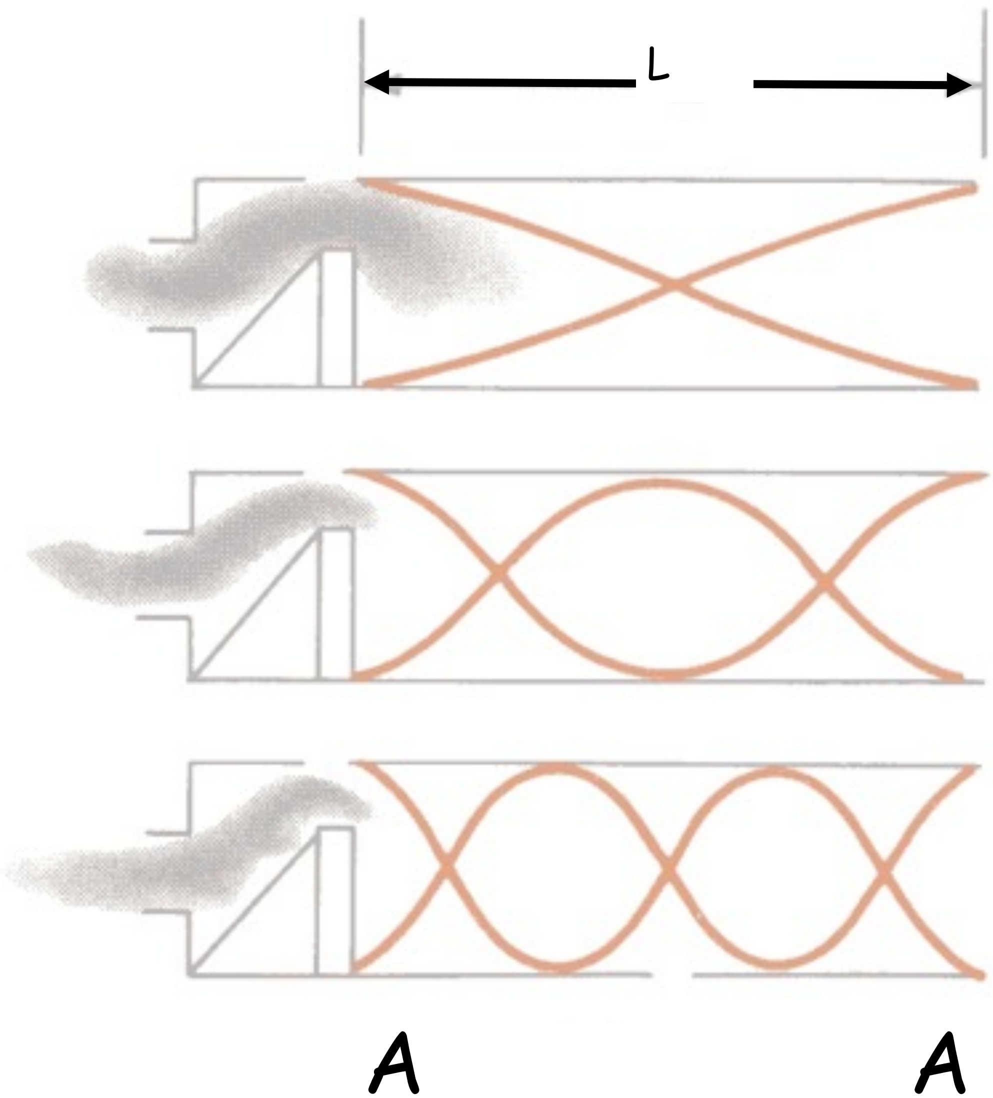
$$= nf_1 \qquad \text{with } n = 1,3,5,....$$

where f_1 is the fundamental frequency.

Standing waves in air columns

Standing longitudinal waves can be set up in a tube of air (eg organ pipe).

Consider a pipe open at both ends:



1st harmonic

$L = \lambda_1/2 \qquad f_1 = v/2L$

2nd harmonic

$L = \lambda_2 \qquad f_2 = 2v/2L$

3rd harmonic

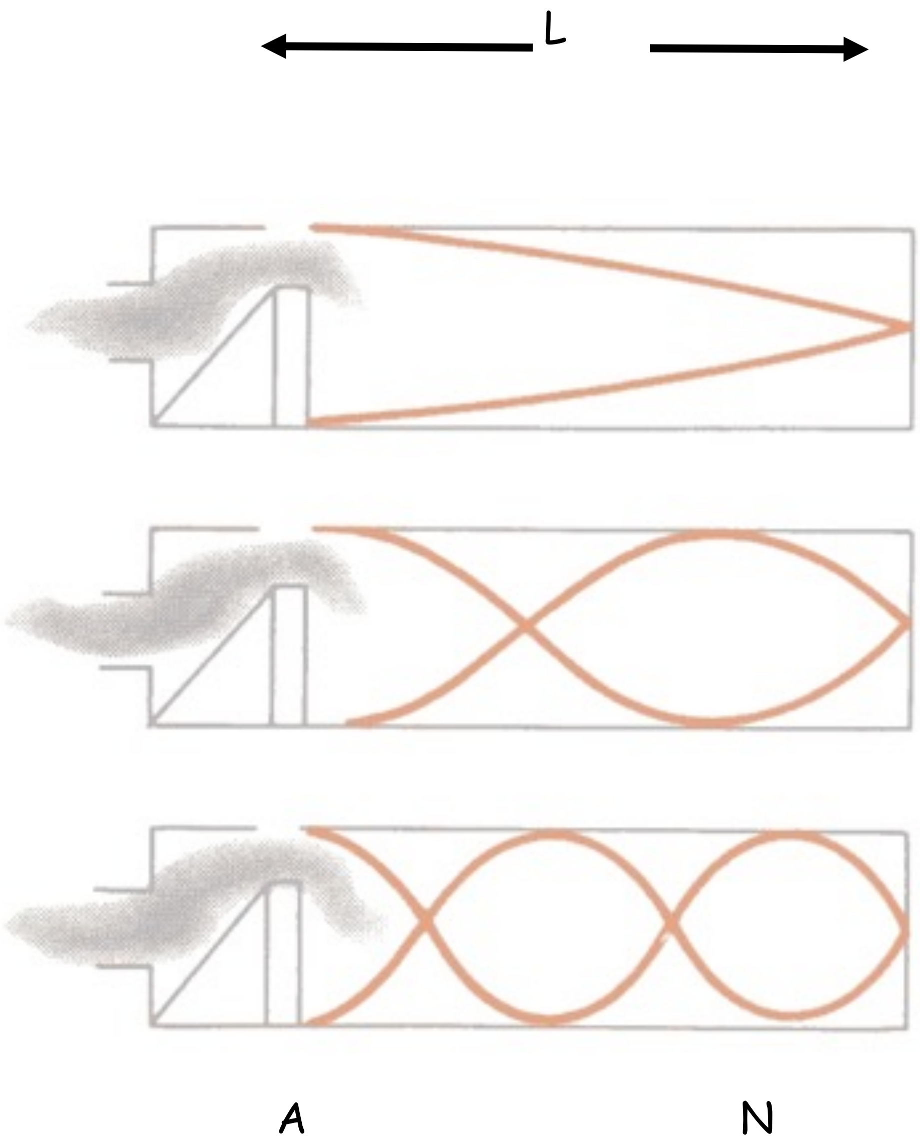
$L = 3\lambda_3/2 \qquad f_3 = 3v/2L$

Generally $$f_n = n\frac{v}{2L}$$ with $n = 1,2,3,....$

In a pipe open at both ends, the natural frequencies of vibration form a harmonic series, ie the overtones are integral multiples of the fundamental frequency.

$f_1 = v/2L$ $f_2 = 2v/2L = 2f_1$ $f_3 = 3v/2L = 3f_1$

Consider a pipe open at one end and closed at the other

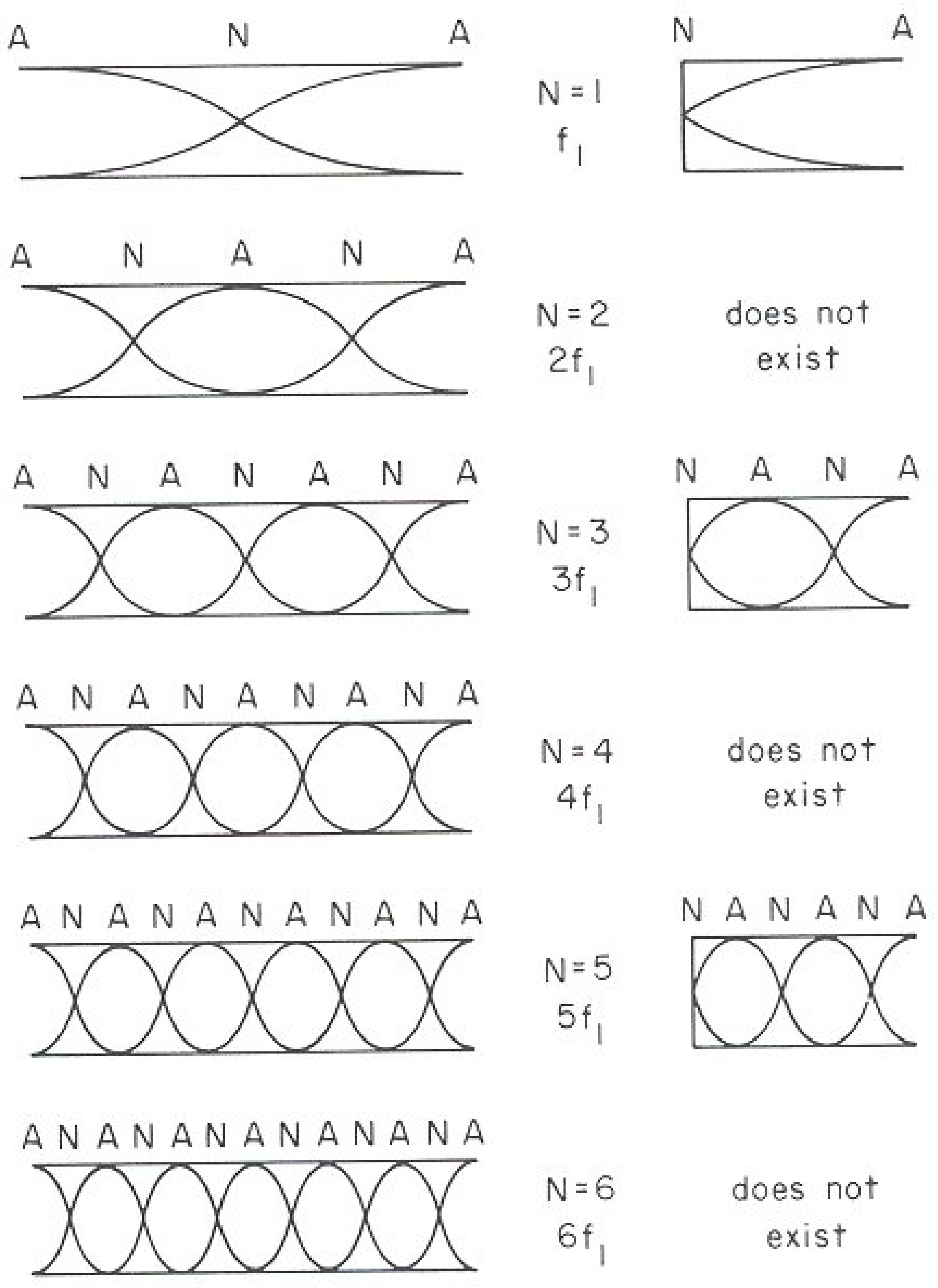


1st harmonic $L = \lambda_1/4$ $f_1 = v/4L$

2nd harmonic $L = 3\lambda_2/4$ $f_2 = 3v/4L$

3rd harmonic $L = 5\lambda_3/4$ $f_3 = 5v/4L$

Open-closed tube comparison



The boundary:
A closed end allows large pressures but no motions.
An open ends allows motions but no pressure changes.

At a closed boundary:

the wave reflects if it has a high pressure at the wall; the air compresses at the wall and then bounces back.

Wave equation & Laplacian

Wave equation

$$v^2\nabla^2\mathbf{u} = v^2\Delta\mathbf{u} = \mathbf{u}_{tt}$$

Laplacian in Cylindrical and Spherical systems

$$\Delta f = \frac{\partial^2 f}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial f}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial \varphi^2} + \frac{\partial^2 f}{\partial z^2}$$

$$\Delta f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial \varphi^2}$$

Special Coordinate systems

In these cases, the variable separation approach also facilitates the solution. In the Euclidian case the eigenfunctions were Fourier series. Here, after the substitution:

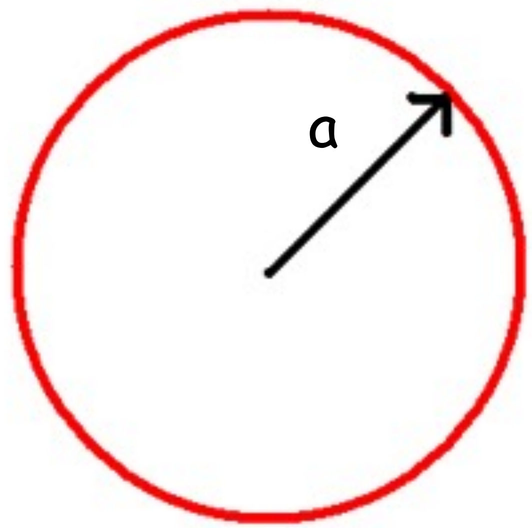
$$f = P(\rho) \cdot \Phi(\varphi) \cdot Z(z)$$

$$f = R(r) \cdot \Phi(\varphi) \cdot \Theta(\theta)$$

The differential equations arise, which solutions are special functions like Legendre polinomials or Bessel functions.

Circular Membrane Problem

A thin circular elastic membrane has a radius a:



and the wave equation, with a circular boundary condition, is:

$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}-\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}=0$$

and if it has separable solutions:

$$u(r,\theta,t)=R(r)\Theta(\theta)T(t)$$

Variable separation

$$\Theta''(\theta) + m^2\Theta(\theta) = 0$$

$$\Theta(\theta) = C\cos(m\theta) + D\sin(m\theta)$$

m is a positive integer

$$T''(t) + c^2k^2T(t) = 0$$

$$T(t) = A\cos(\omega t) + B\sin(\omega t)$$

$\omega = ck$

$$\frac{1}{R}\frac{\partial^2 R}{\partial r^2} + \frac{1}{rR}\frac{\partial R}{\partial r} - \frac{m^2}{r^2} = \frac{1}{c^2T}\frac{\partial^2 T}{\partial t^2} = -k^2$$

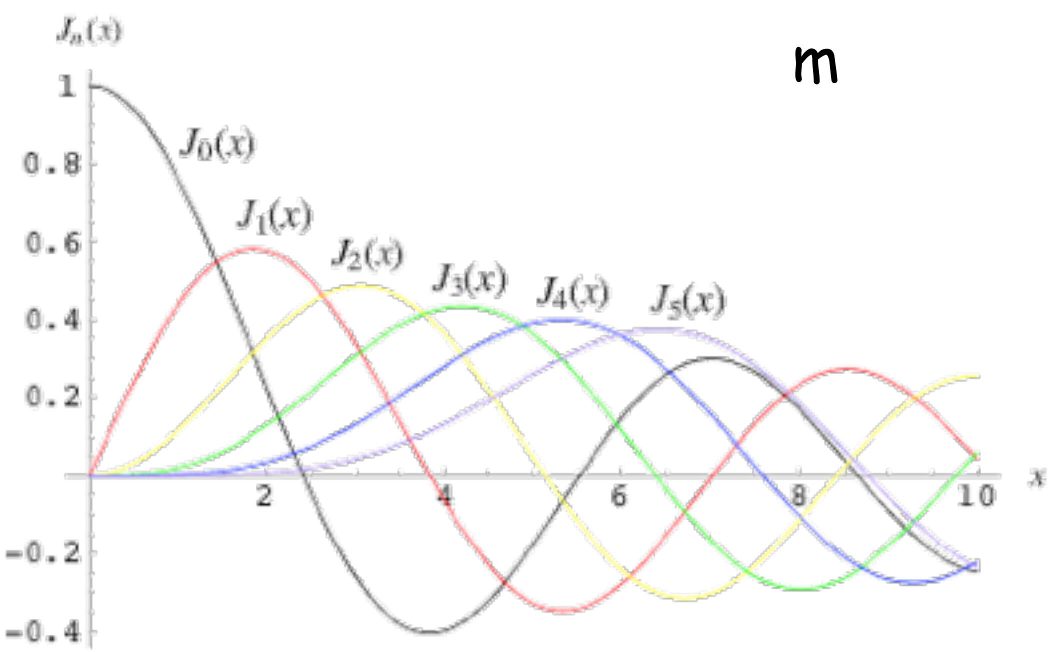
$$s^2\frac{d^2R}{ds^2} + s\frac{dR}{ds} + \left(s^2 - m^2\right)R = 0; \quad s = kr$$

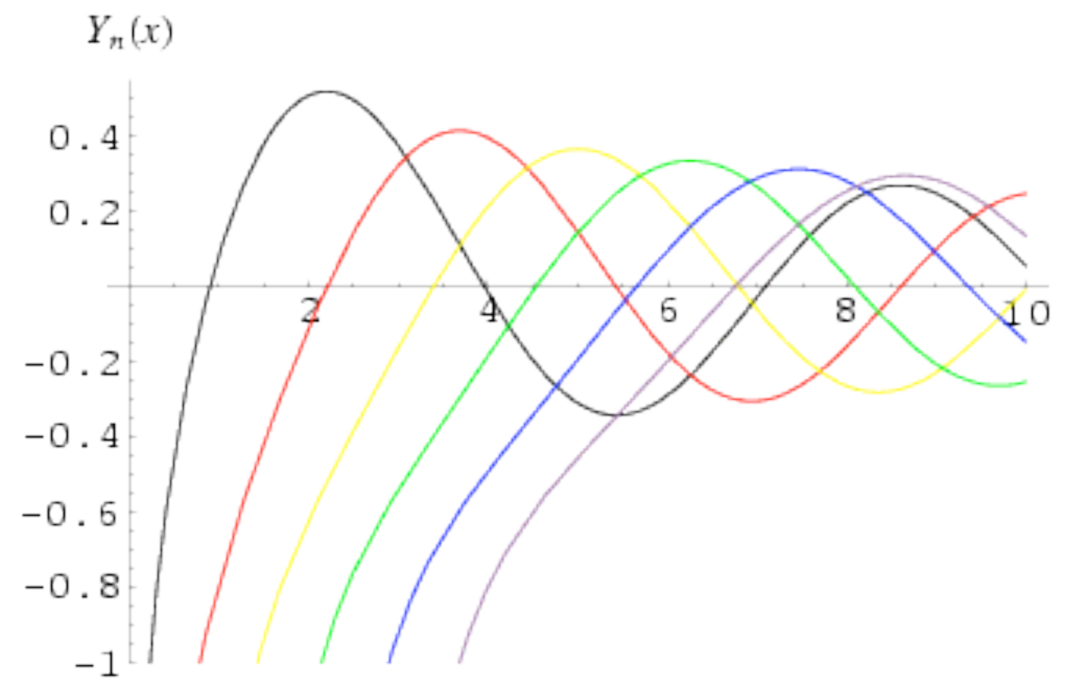
that is a Bessel equation of order m

$$x^2y'' + xy' + \left(x^2 - m^2\right)y = 0$$

and the general solution is:

$$y = \sum_m A_{1m} J_m(x) + A_{2m} Y_m(x)$$





that are to cylindrical waves what cosines/sines are to waves on a straight line.

The BC at the (regular singular) origin point is: R(0) is finite

$$R(s) = R(kr) = \sum_m A_m J_m(kr)$$

The radial factor of the solution is a **Bessel function of the first kind**: NOT **periodic and the distance between zeros is NOT constant.**

The other **BOUNDARY CONDITION** of the circular membrane problem is:

u=0 at r=a

this implies that $$J_m(ka)=0$$

Therefore $$\frac{\omega a}{c} = \gamma_{mn}$$ nth positive zero of J_m

$$\omega_{mn} = \frac{c}{a}\gamma_{mn}$$

$$R(kr) \propto J_m\left(\frac{\gamma_{mn}}{a} r\right)$$

General solution

and the general solution is:

$$u = R(r)\Theta(\theta)T(t) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\left[C_{mn}\cos(m\theta) + D_{mn}\sin(m\theta)\right]\left[A_{mn}\cos(ck_{mn}t) + B_{mn}\sin(ck_{mn}t)\right]J_m(k_{mn}r)$$

but if we assume that the initial conditions are rotationally **symmetric**, i.e. goes like f(r), we have that we need only m=0

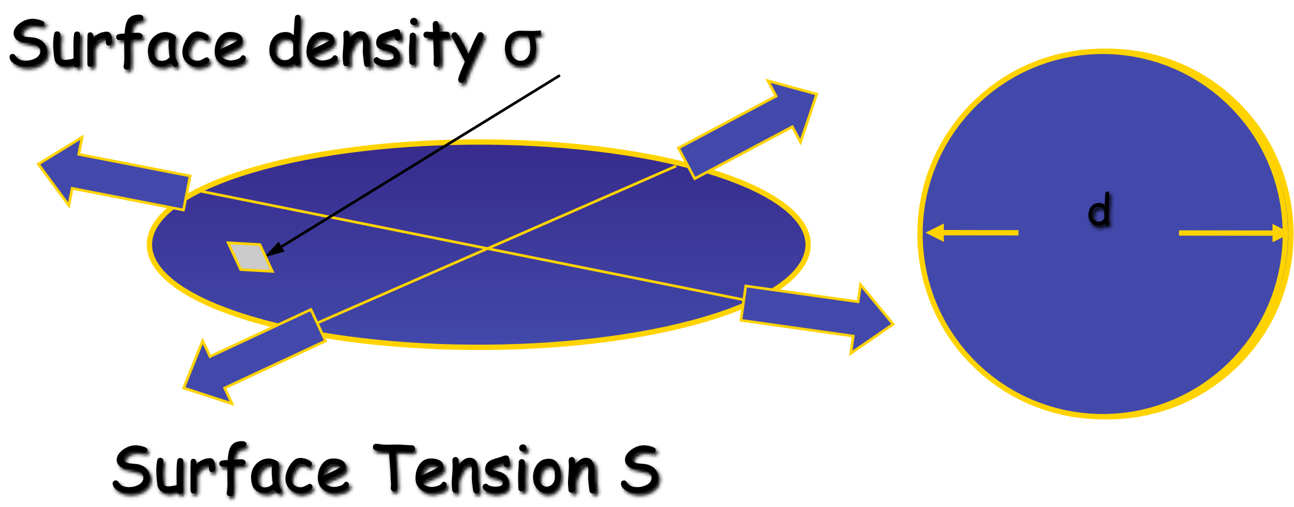
$$u = R(r)\Theta(\theta)T(t) = \sum_{n=1}^{\infty}\left[A_n\cos(ck_nt) + B_n\sin(ck_nt)\right]J_0(k_nr)$$

with

$$k_n = \frac{\gamma_n}{a} = \frac{\gamma_{0n}}{a} = \frac{\omega_{0n}}{c}$$

to be determined with the proper **initial conditions!**

Oscillations of a Clamped Membrane



Mode: (0,1)

$$f_{0\,1} = v/\lambda; \quad v = \sqrt{(S/\sigma)}$$

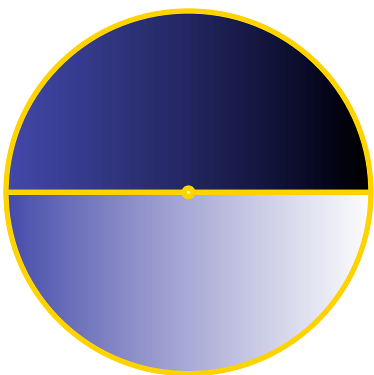
$$f_{0\,1} = x_{0\,1}/(\pi d) \cdot \sqrt{(S/\sigma)}$$

$$x_{0\,1} = 2.405$$

Surface density σ= mass/area σ= density · thickness

Surface Tension S= force/length

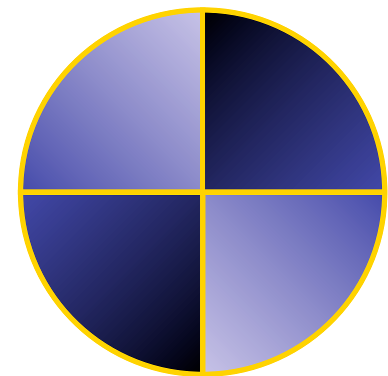
Membrane vs string



Mode: (1,1)

$$f_{11} = (x_{11} / x_{01})\, f_{01}$$

$$x_{11} / x_{01} = 1.594$$

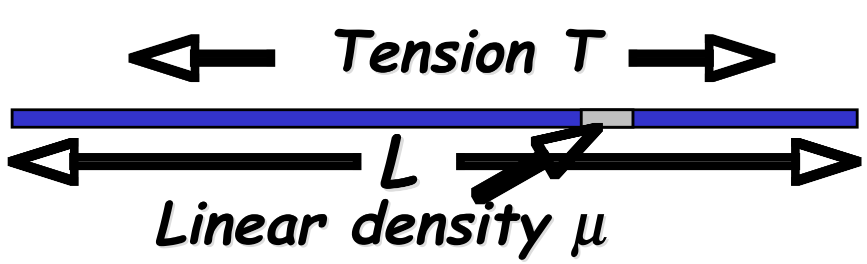


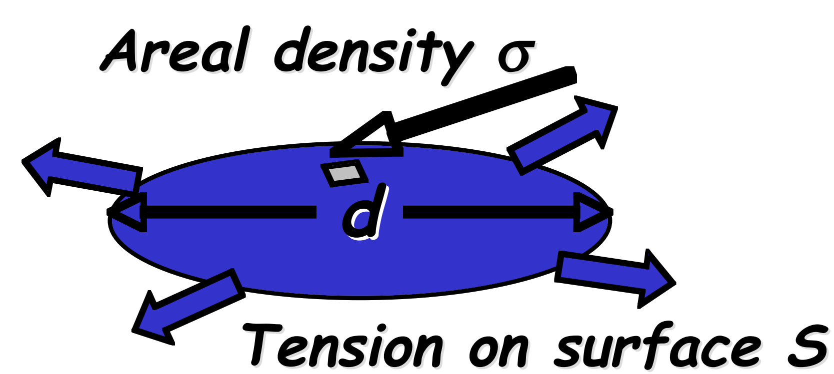
Mode: (2,1)

$$f_{21} = (x_{21} / x_{01})\, f_{01}$$

$$x_{21} / x_{01} = 2.136$$

http://www.kettering.edu/~drussell/Demos/MembraneCircle/Circle.html

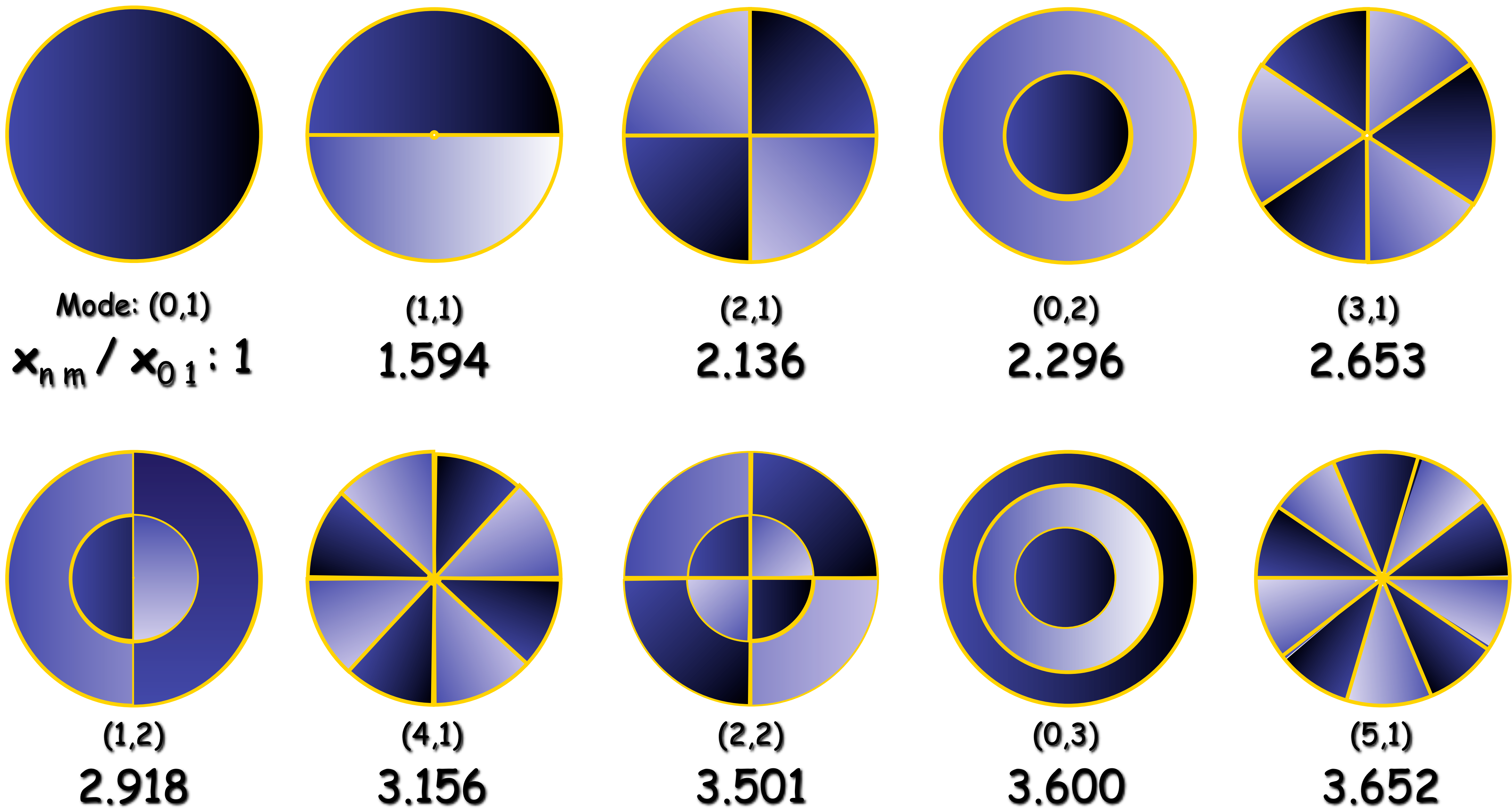




$$f_n = n/(2L)\,(T/\mu)^{1/2}$$
$$n = 1, 2, 3, 4, 5, 6, \ldots$$

$$f_{nm} = x_{nm}/(\pi d)(S/\sigma)^{1/2}$$
$$x_{01} = 2.405$$

Membrane modes



Sturm-Liouville Problem

The special functions, which arise in these homogeneous Boundary Value Problems (BVPs) with homogeneous boundary conditions (BCs) are mostly special cases of Sturm-Liouville Problem, given by:

$$\frac{d}{dx}\left[p(x)\frac{d}{dx}y(x)\right]+\left[q(x)+\lambda r(x)\right]y(x)=0$$

On the interval $a \le x \le b$, with the homogeneous boundary conditions

$$c_1 y(a)+c_2 y'(a)=0$$
$$k_1 y(b)+k_2 y'(b)=0$$

The values λ_n, that yield the nontrivial solutions are called **eigenvalues**, and the corresponding solutions $y_n(x)$ are **eigenfunctions**.

The set of eigenfunctions, $\{y_n(x)\}$, form an **orthogonal** system with respect to the weight function, $r(x)$, over the interval.

If $p(x)$, $q(x)$, and $r(x)$ are real, the eigenvalues are also real