The Abdus Salam
International Centre
for Theoretical Physics

Postgraduate Diploma Programme

Earth System Physics

Wave physics
PDE

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Mathematical reference: PDE

Second order **PDE** of two variables can written in the form:

$$a\frac{\partial^2 f(x,y)}{\partial x^2}+b\frac{\partial^2 f(x,y)}{\partial x\,\partial y}+c\frac{\partial^2 f(x,y)}{\partial y^2}+d\frac{\partial f(x,y)}{\partial x}+e\frac{\partial f(x,y)}{\partial y}=F(x,y)$$

that can be classified according to:

- **$b^2-4ac<0$** **ELLIPTIC** equations produce stationary or energy minimizing solutions (e.g. Laplace eqn.)
- **$b^2-4ac=0$** **PARABOLIC** equations produce smooth spreading flow of an initial disturbance (e.g. Diffusion eqn.)
- **$b^2-4ac>0$** **HYPERBOLIC** equations produce propagating disturbance (e.g. Wave eqn.)

Mathematical reference: PDE

Second order Linear PDE of two variables
can be written in the form:

$$a\frac{\partial^2 f(x,y)}{\partial x^2} + b\frac{\partial^2 f(x,y)}{\partial x\,\partial y} + c\frac{\partial^2 f(x,y)}{\partial y^2} + d\frac{\partial f(x,y)}{\partial x} + e\frac{\partial f(x,y)}{\partial y} = F(x,y)$$

that can be classified according to:

Discriminant	TYPE	Example eqn.	Soln. Behaviour
$b^2-4ac<0$	ELLIPTIC	Laplace	stationary
$b^2-4ac<0$	PARABOLIC	Diffusion	spreading flow
$b^2-4ac<0$	HYPERBOLIC	Wave	propagation

Classification of Partial Differential Equations (PDE)

Second-order PDEs of two variables are of the form:

$$a\frac{\partial^2 f(x,y)}{\partial x^2} + b\frac{\partial^2 f(x,y)}{\partial x \partial y} + c\frac{\partial^2 f(x,y)}{\partial y^2} + d\frac{\partial f(x,y)}{\partial x} + e\frac{\partial f(x,y)}{\partial y} = F(x,y)$$

$b^2 - 4ac < 0$	elliptic	LAPLACE equation
$b^2 - 4ac = 0$	parabolic	DIFFUSION equation
$b^2 - 4ac > 0$	hyperbolic	WAVE equation

Elliptic equations produce **stationary and energy-minimizing** solutions

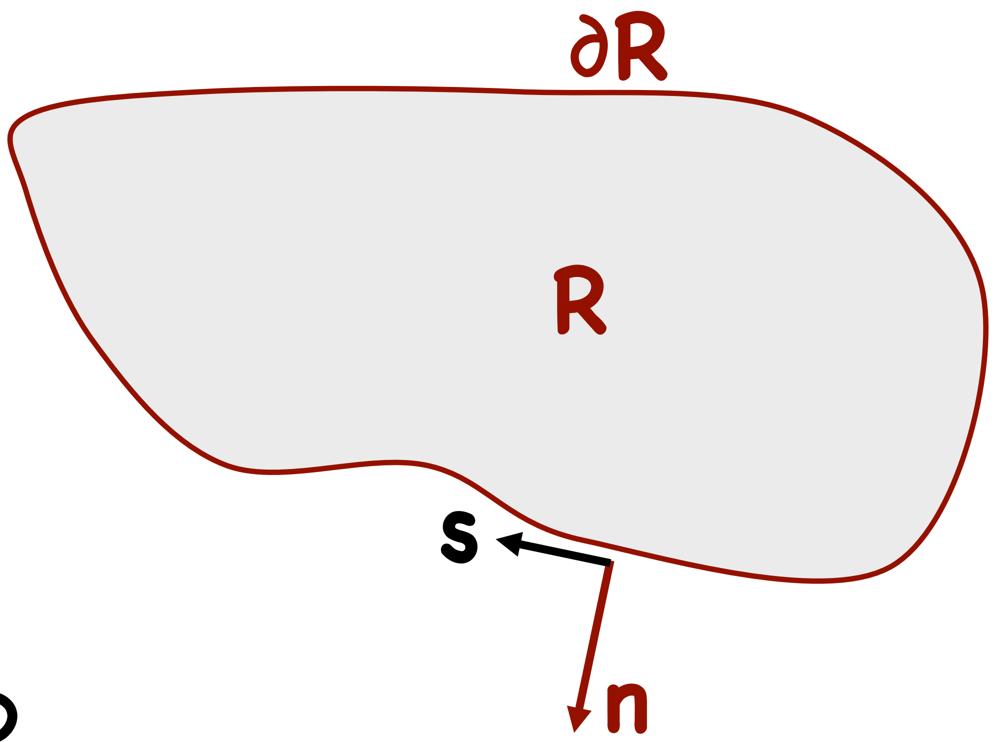
Parabolic equations a **smooth-spreading flow** of an initial disturbance

Hyperbolic equations a **propagating disturbance**

Boundary and Initial conditions

Initial conditions: starting point for propagation problems

Boundary conditions: specified on domain boundaries to provide the interior solution in computational domain



Dirichlet: $u=f$ on ∂R

Neumann: $\dfrac{\partial u}{\partial n}=f$ or $\dfrac{\partial u}{\partial s}=g$ on ∂R

Robin: $\dfrac{\partial u}{\partial n}+ku=f$ on ∂R

Elliptic PDEs

Steady-state two-dimensional **heat conduction** equation is prototypical **elliptic PDE**

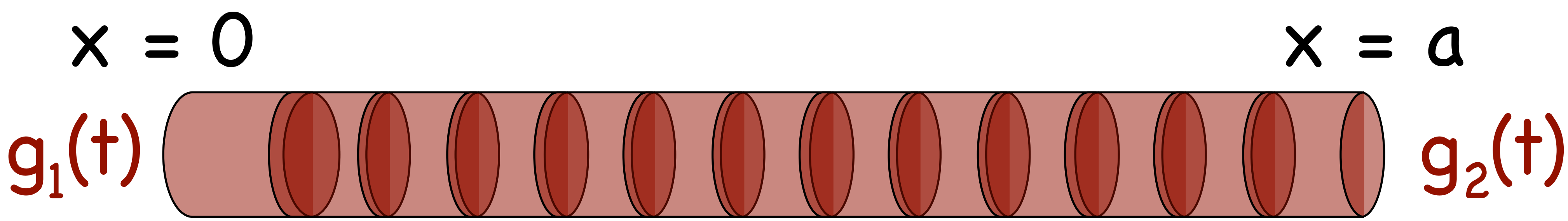
Laplace equation - no heat generation

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

Poisson equation - with heat source

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = f(x, y)$$

Heat transfer in a one-dimensional rod



$$\frac{\partial u}{\partial t} = d\,\frac{\partial^2 u}{\partial x^2} \qquad 0 \le x \le a, \quad 0 \le t \le T$$

I.C.s $\quad u(x,0) = f(x) \qquad 0 \le x \le a$

B.C.s $\quad \begin{cases} u(0,t) = g_1(t) \\ u(a,t) = g_2(t) \end{cases} \qquad 0 \le t \le T$

Hyperbolic PDE: wave equation

$b^2 - 4ac = 0 - 4(1)(-c^2) > 0$: Hyperbolic

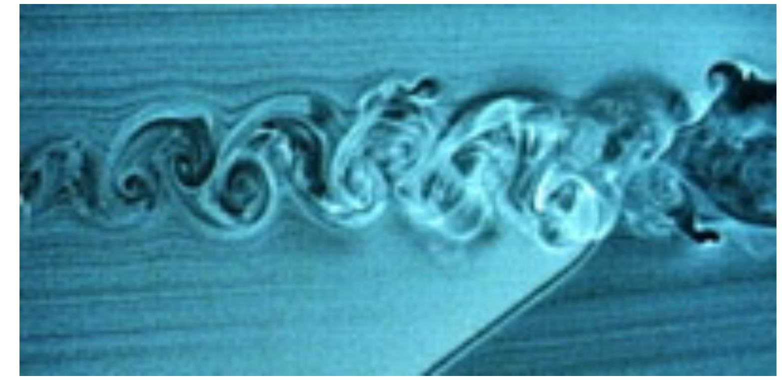
$$\frac{\partial^2 u}{\partial t^2} = v^2 \frac{\partial^2 u}{\partial x^2} \qquad 0 \le x \le a, \quad 0 \le t$$

I.C.s
$$\begin{cases} u(x,0) = f_1(x) \\ u_t(x,0) = f_2(x) \end{cases} \qquad 0 \le x \le a$$

B.C.s
$$\begin{cases} u(0,t) = g_1(t) \\ u(a,t) = g_2(t) \end{cases} \qquad t > 0$$

Coupled PDE

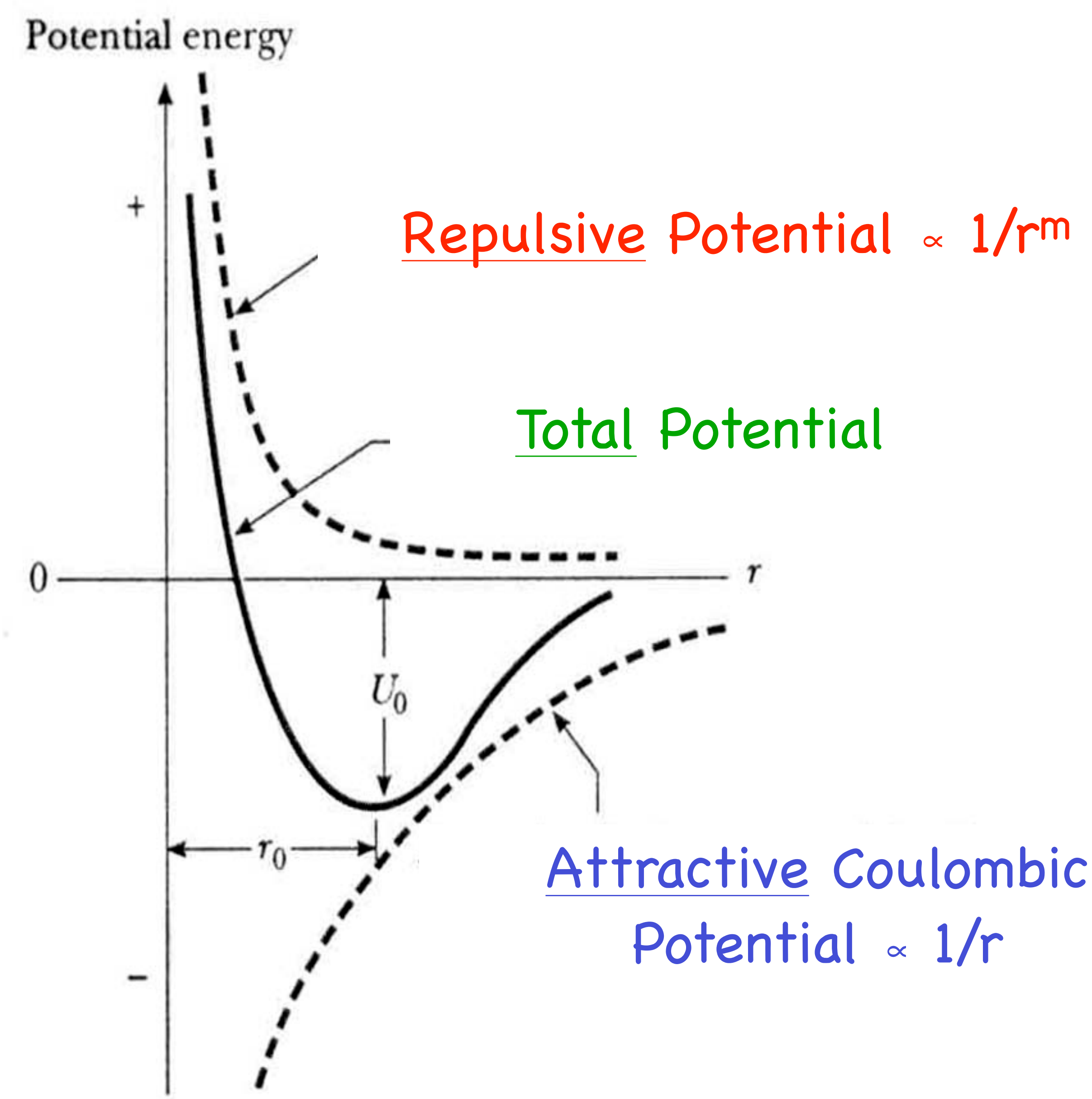
Navier-Stokes Equations

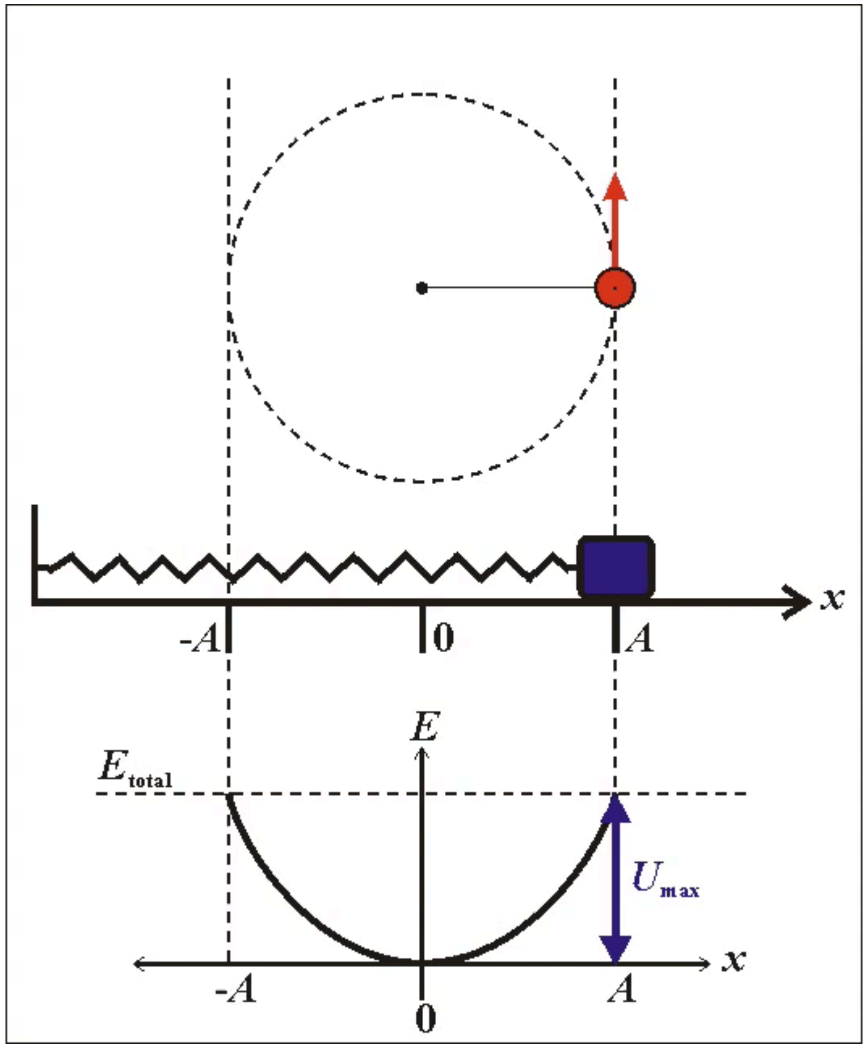


$$
\begin{cases}
\dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} = 0 \\[2ex]
\dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial x} + v\dfrac{\partial u}{\partial y} = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial x} + \nu\left(\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}\right) \\[2ex]
\dfrac{\partial v}{\partial t} + u\dfrac{\partial v}{\partial x} + v\dfrac{\partial v}{\partial y} = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial y} + \nu\left(\dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2}\right)
\end{cases}
$$

What is a wave?

Small perturbations of a **stable** equilibrium point → **Linear restoring force** → **Harmonic Oscillation**

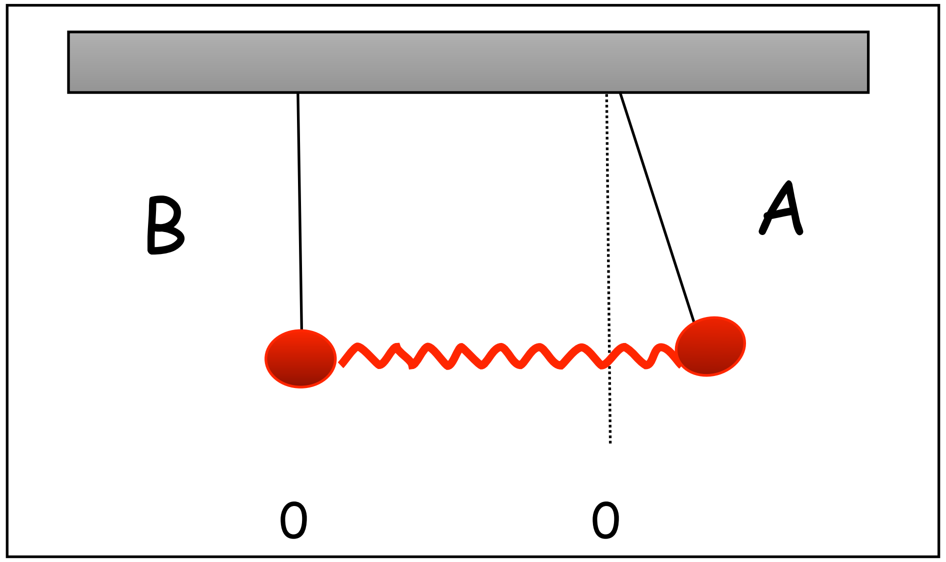




What is a **wave**? - 2

Small perturbations of a **stable** equilibrium point → **Linear restoring force** → **Harmonic Oscillation**

Coupling of **harmonic oscillators** → the disturbances can **propagate**



What is a **wave**? - 2

Small perturbations of a **stable** equilibrium point → **Linear restoring force** → **Harmonic Oscillation**

Coupling of **harmonic oscillators** → the disturbances can **propagate**, **superpose** and **stand**

Normal modes of the system

What is a **wave**? - 3

Small perturbations of a **stable** equilibrium point → **Linear restoring force** → **Harmonic Oscillation**

Coupling of **harmonic oscillators** → the disturbances can **propagate**, **superpose** and **stand**

General form of LWE

$$\frac{\partial^2 \psi(x,t)}{\partial t^2} = v^2 \frac{\partial^2 \psi(x,t)}{\partial x^2}$$

WAVE: organized **propagating imbalance**, satisfying differential equations of motion

What is a **wave**? - 4

Small perturbations of a **stable** equilibrium point → **Linear restoring force** → **Harmonic Oscillation**

Coupling of **harmonic oscillators** → the disturbances can **propagate**, superpose, stand and be **dispersed**

WAVE: organized propagating imbalance, satisfying differential equations of motion

Organization can be destroyed, when interference is destructive

non linearity → **Turbulence**

strong scattering → **Diffusion**

Exceptions

Solitons

Phonons