The Abdus Salam International Centre for Theoretical Physics

Postgraduate Diploma Programme

Earth System Physics

Wave physics

2&N DOF systems: Coupled oscillators

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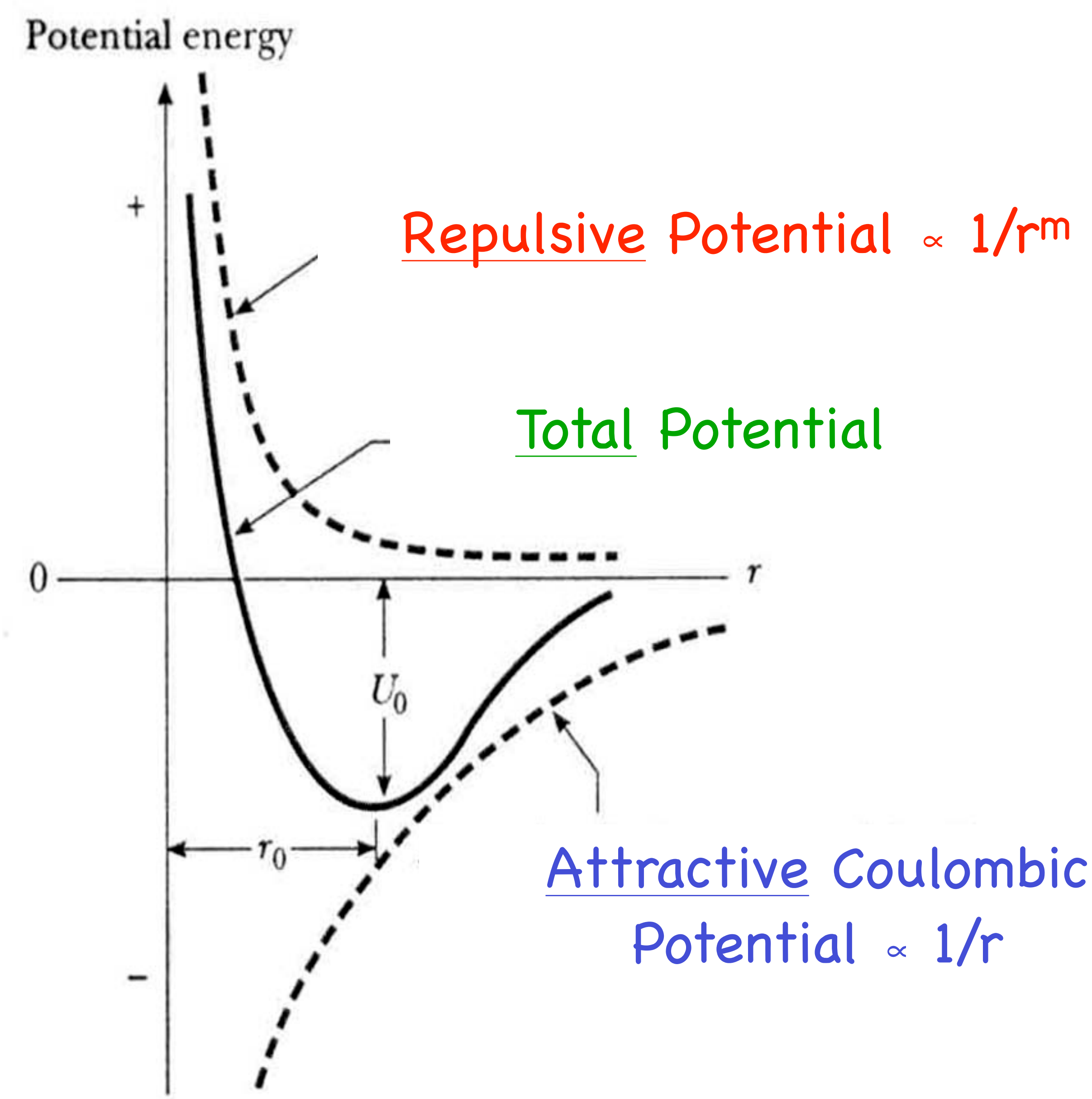
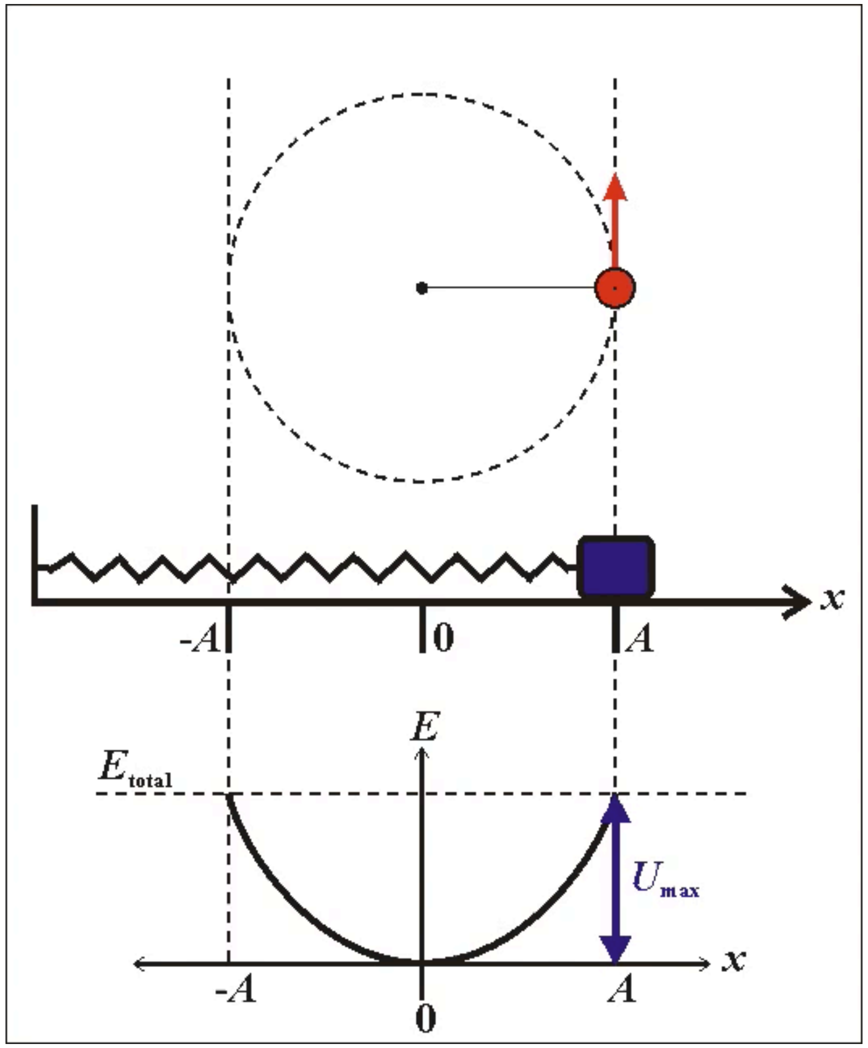
What is a wave?

Small perturbations of a **stable** equilibrium point → **Linear restoring force** → **Harmonic Oscillation**

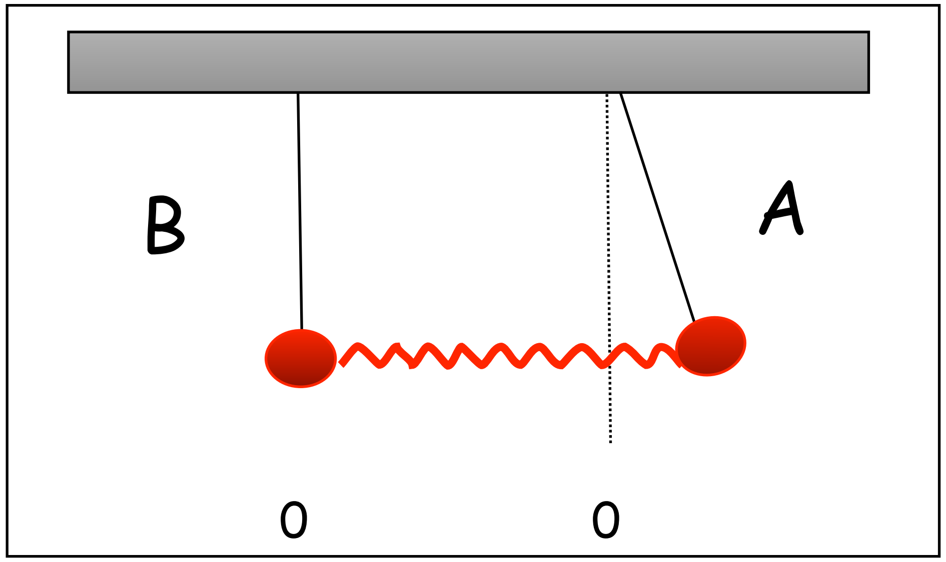




Coupled systems

Small perturbations of a **stable** equilibrium point ⟶ **Linear restoring force** ⟶ **Harmonic Oscillation**

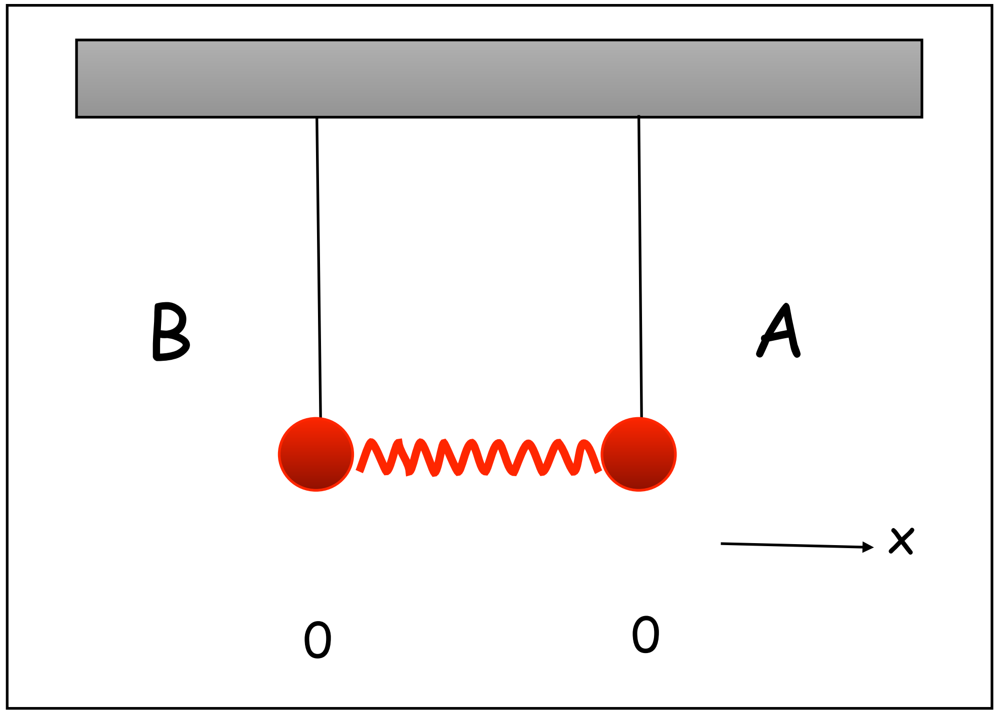
Coupling of harmonic oscillators ⟶ the disturbances can **propagate**



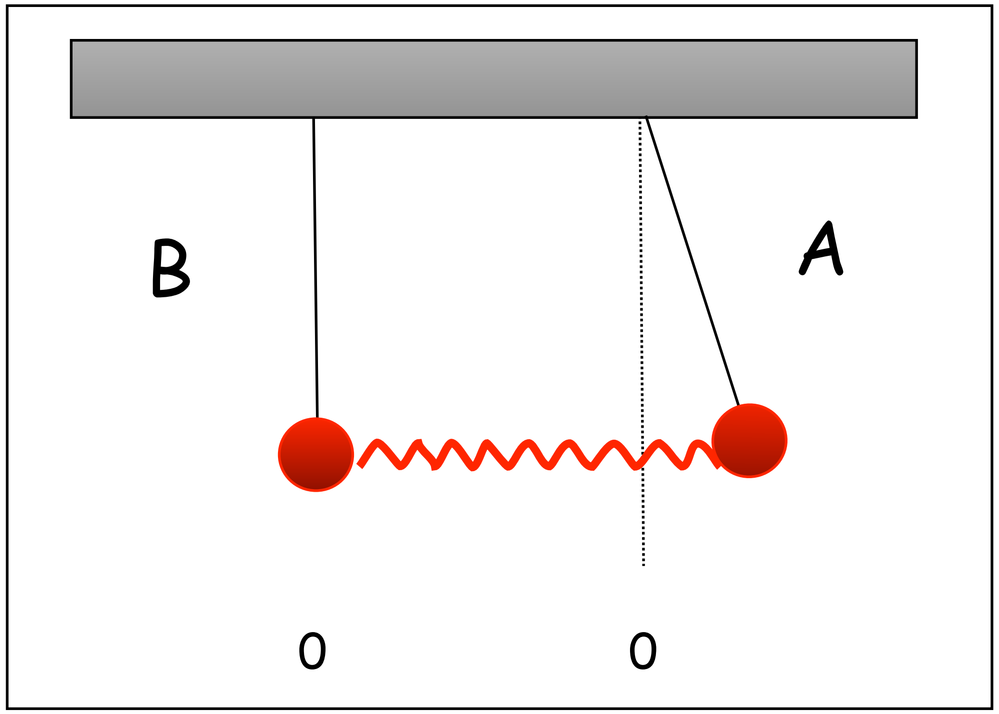
Coupled systems

Simplest example: Two identical pendula A and B connected by a light unstretched spring.

(a) Move A to one side, while holding B fixed, then release.



A will oscillate with gradually decreasing amplitude.



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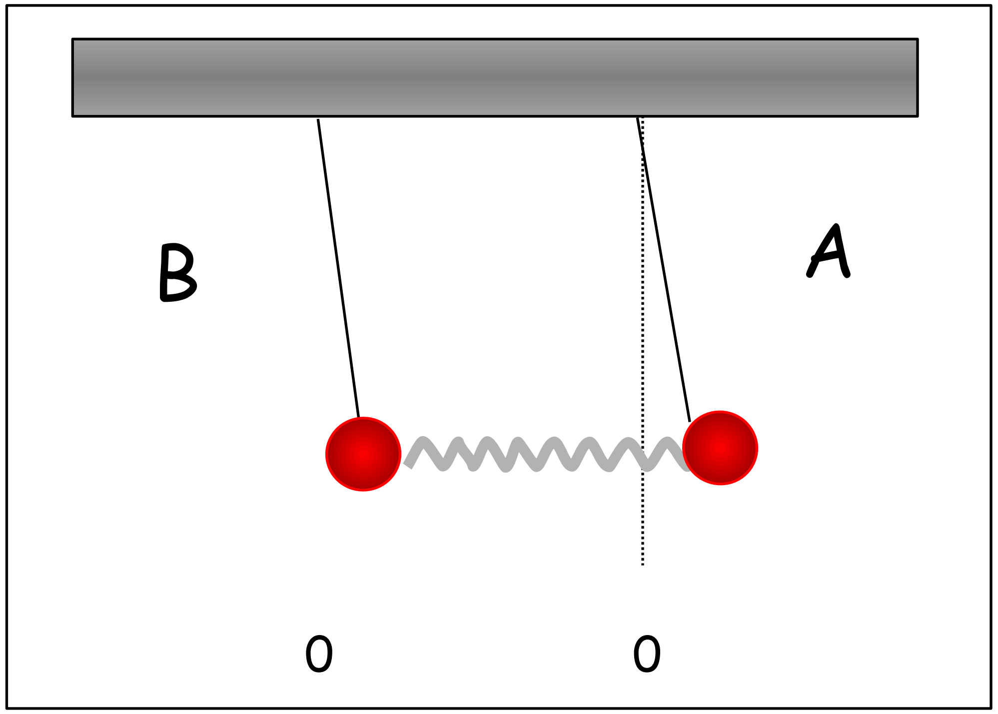
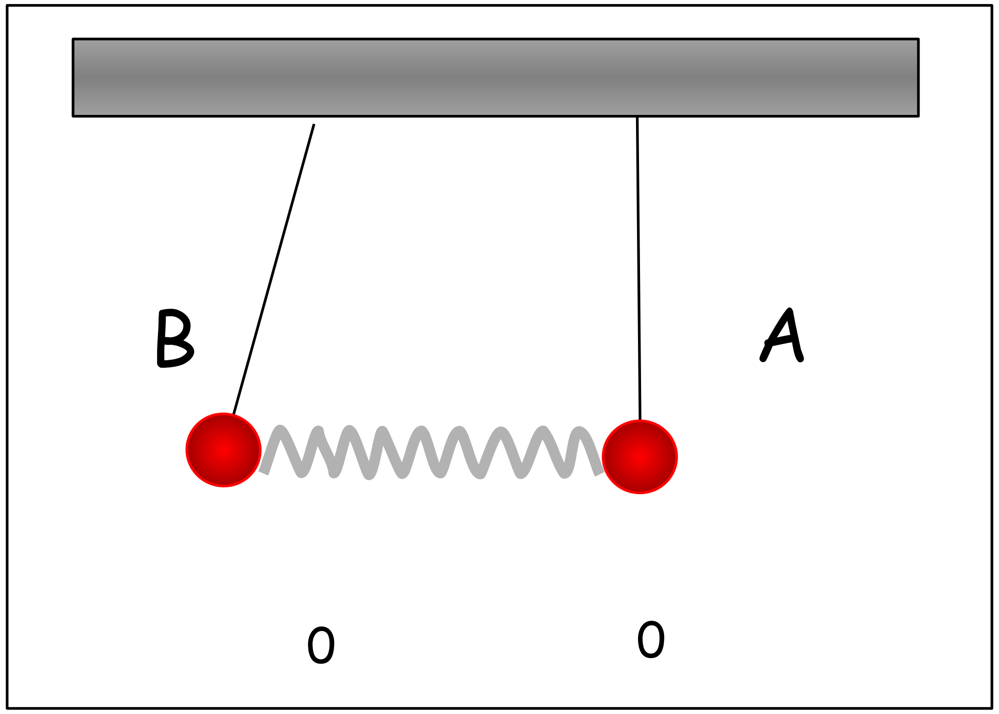


A will oscillate with gradually decreasing amplitude.

B (initially undisplaced) begins to oscillate with gradually increasing amplitude



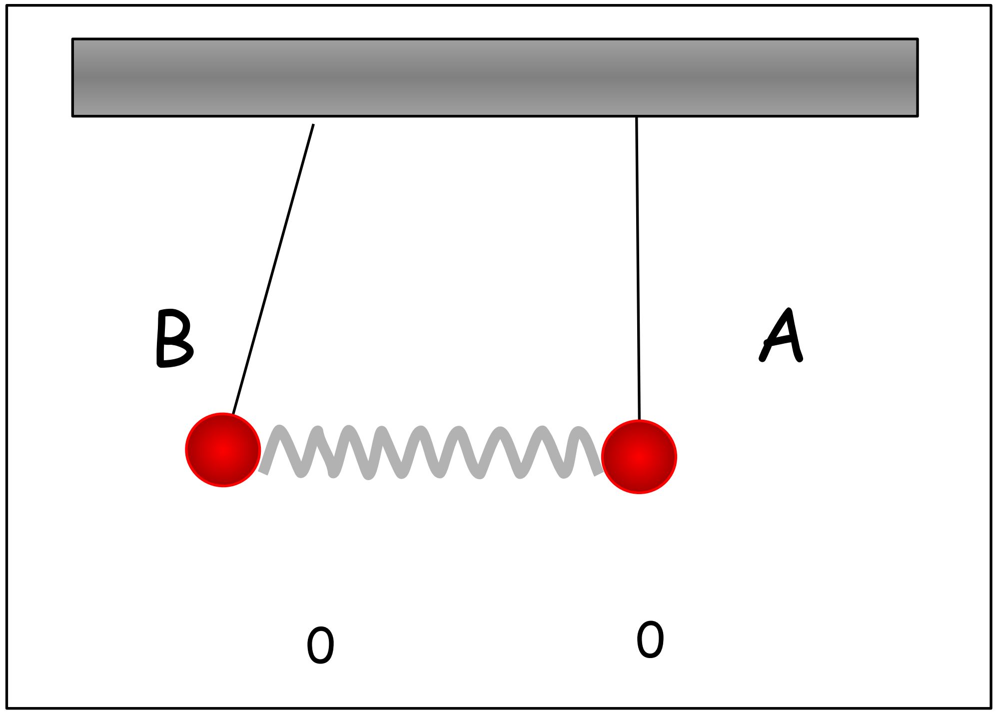
Case a)

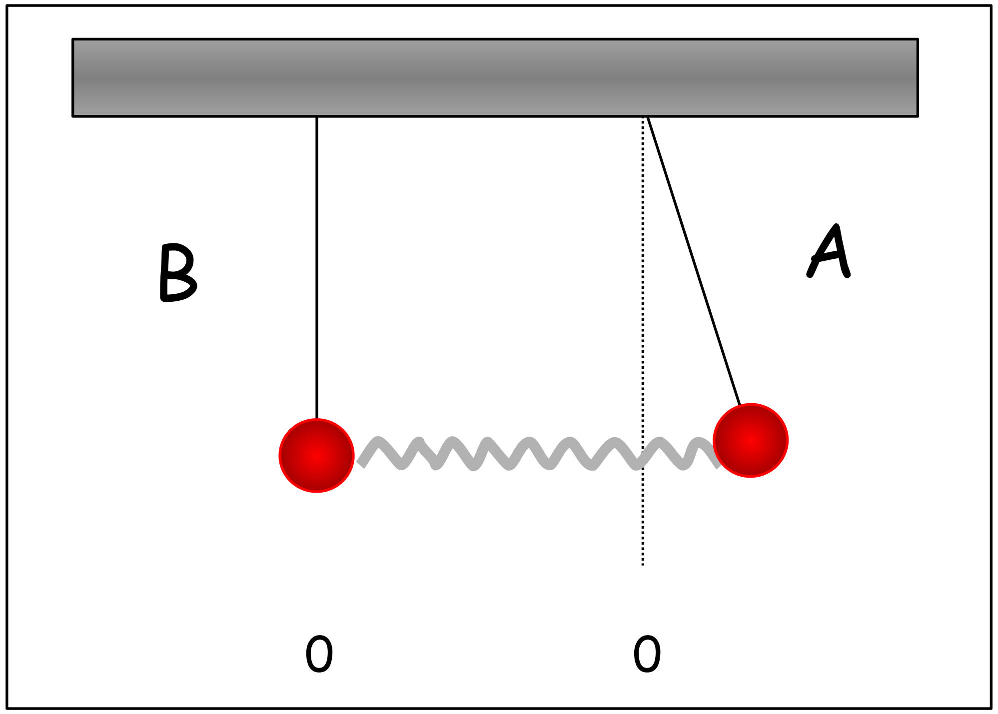
At some point A and B will be oscillating with equal amplitudes.

Is this equilibrium ? NO !

The process continues - the amplitude of oscillation of A continues to decrease and that of B increase until B is oscillating with approximately the initial amplitude of A

Energy shifts backwards and forwards from A to B





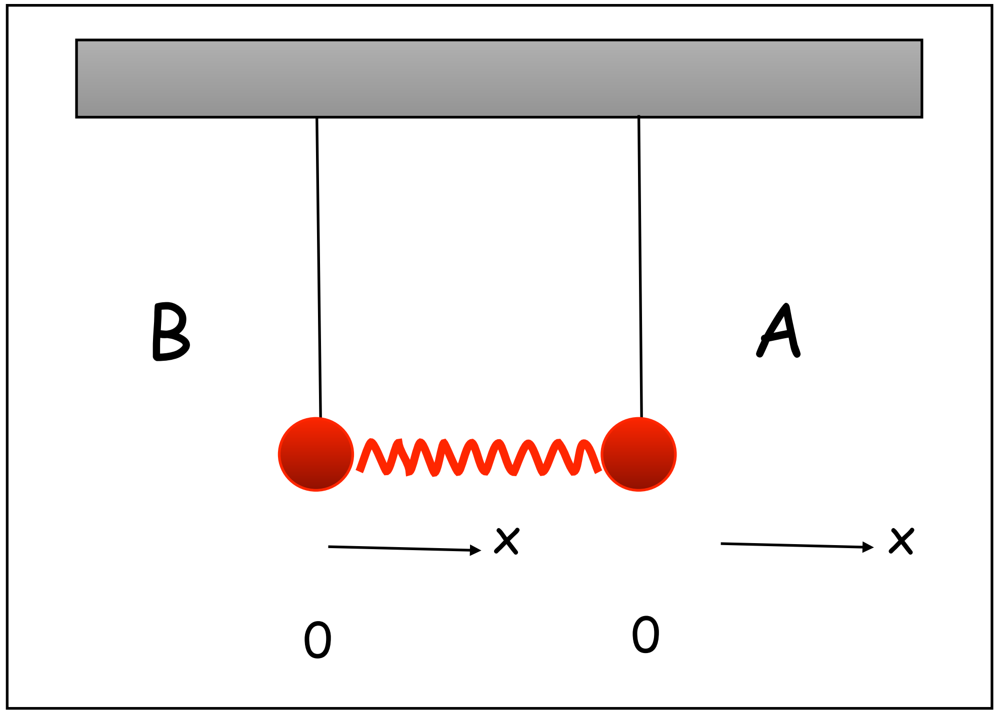
Case b)

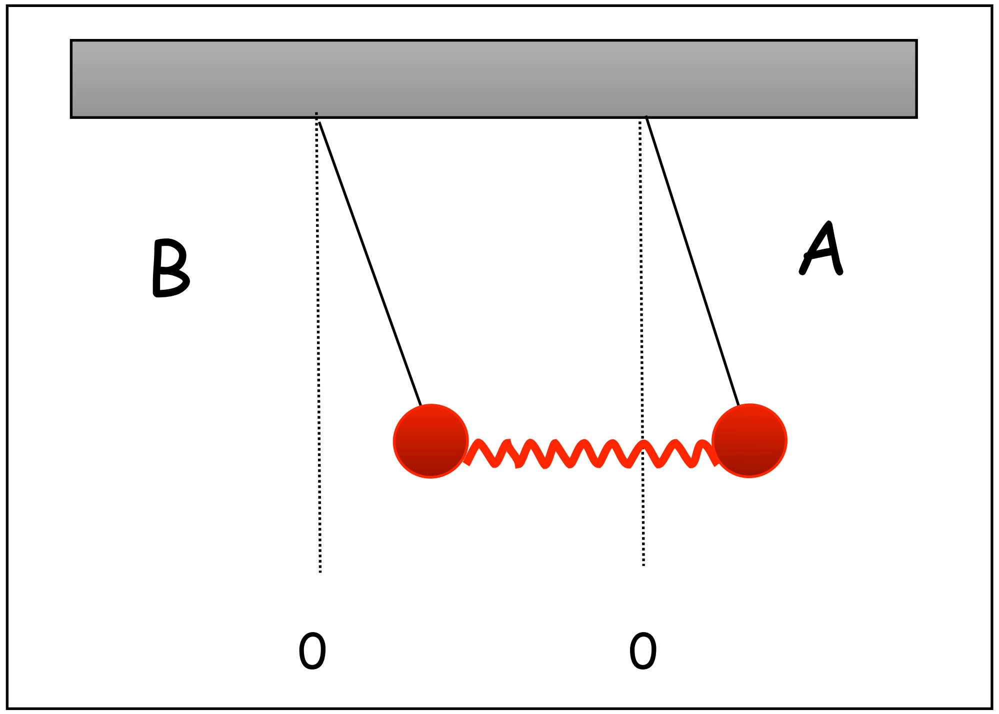
(b) Move A and B to same side by equal amounts then release.

The distance between A and B is constant and equal to the relaxed length of the spring, and the spring exerts no force on either mass.

Each pendulum is essentially free and oscillating with its natural frequency

$$\omega_0 = \sqrt{\frac{g}{L}}$$





Case b)

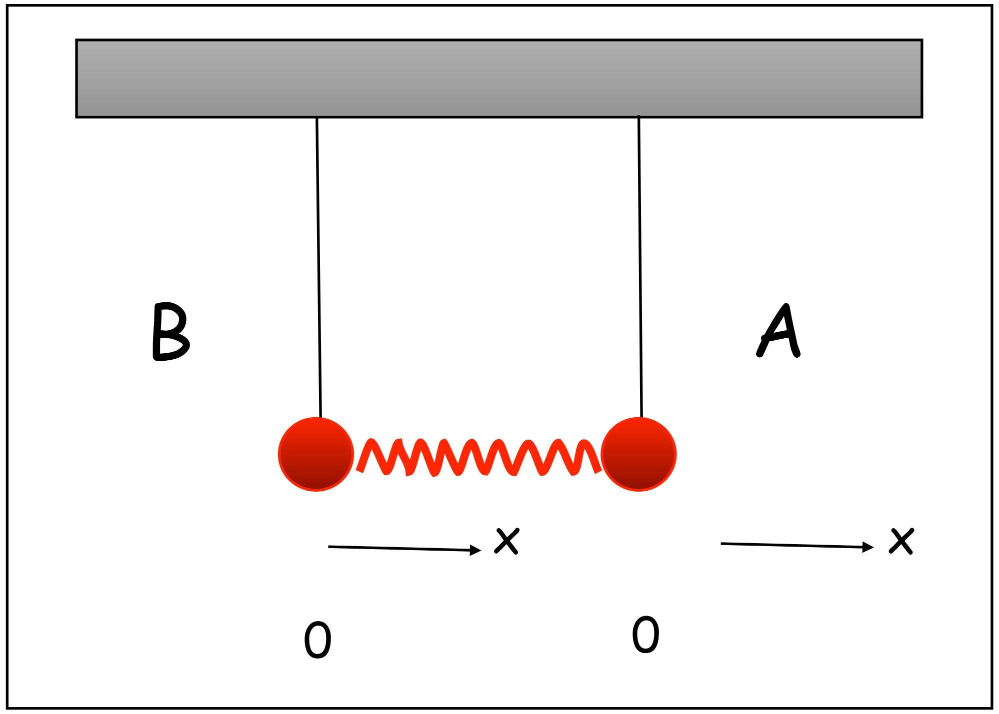
If x_A and x_B are the displacements of A and B resp.

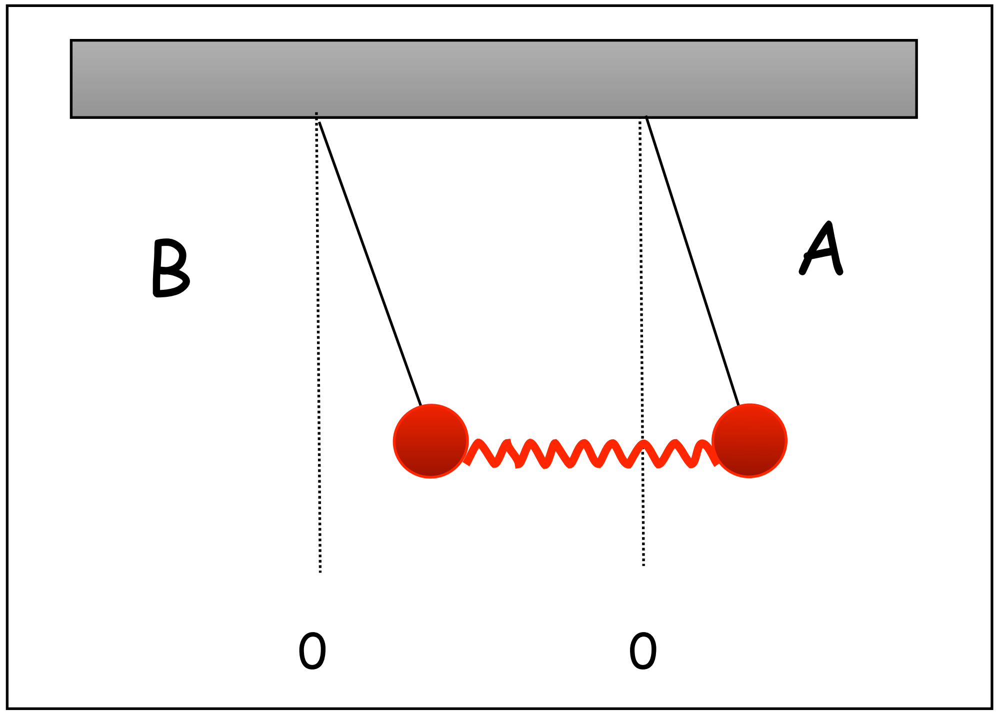
The equations of motion are

$$x_A = c\cos(\omega_0 t)$$
$$x_B = c\cos(\omega_0 t)$$

ie both masses vibrate at the **same** frequency and with the **same** amplitude.

This is one NORMAL MODE of a coupled system



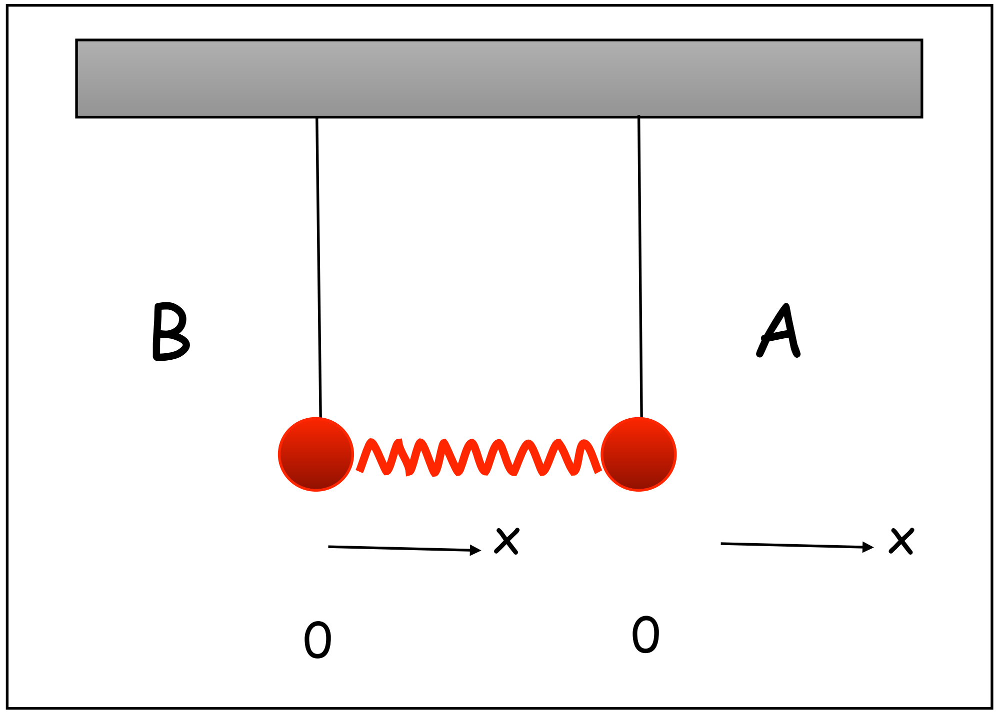


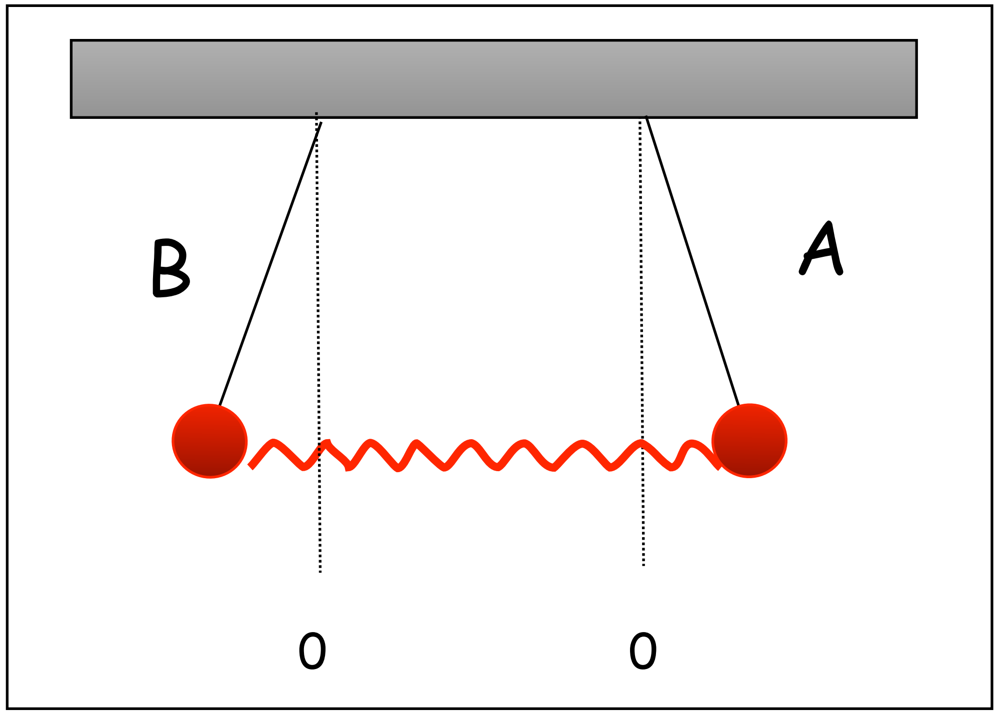
Move A and B to opposite sides by equal amounts then release.

The spring is first stretched then a half cycle later it will be compressed.

The motions of A and B will be mirror images.

This is the second NORMAL MODE of the system

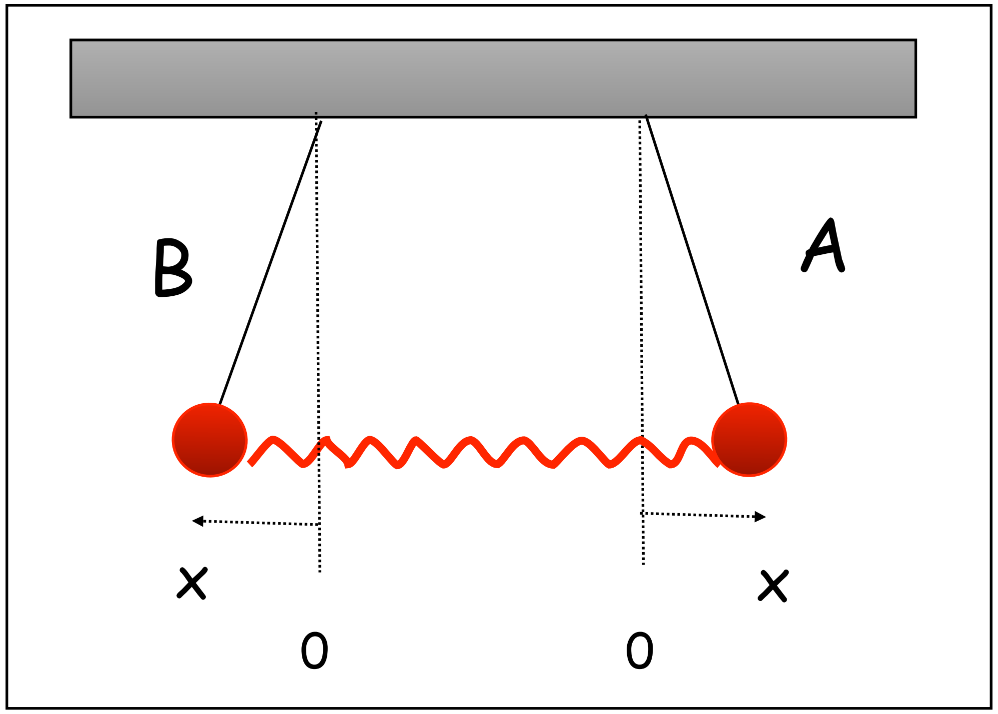




Motion equations

Consider the situation where both masses are free to move and both are displaced a small distance x

The spring is stretched by 2x and exerts a restoring force of 2kx on the masses



Equation of motion for mass A

$$m\frac{d^2x_A}{dt^2} + m\omega_0^2 x_A + 2kx_A = 0$$

$$\frac{d^2x_A}{dt^2} + (\omega_0^2 + 2\omega_c^2)x_A = 0 \qquad \text{where} \qquad \omega_c^2 = k/m$$

if $\sqrt{(\omega_0^2 + 2\omega_c^2)} = \omega'$ then $\omega' = \sqrt{\left(\frac{g}{l} + \frac{2k}{m}\right)}$

This has the solution $x_A = D \cos \omega' t$

Motion of B is the mirror image of A $x_B = -D \cos \omega' t$

Each pendulum oscillates with SHM.

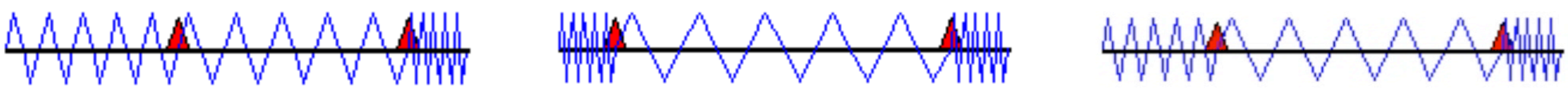
The coupling spring has increased the restoring force and therefore increased the frequency of oscillation.

A and B are always 180^o (π) out of phase.

What is a **wave**? - 2

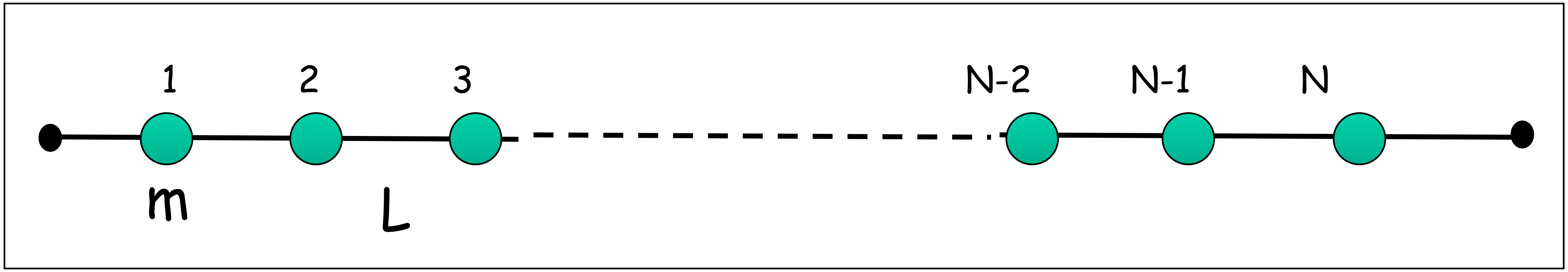
Small perturbations of a **stable** equilibrium point → **Linear restoring force** → **Harmonic Oscillation**

Coupling of harmonic oscillators → the disturbances can propagate, superpose and **stand**



Normal modes of the system

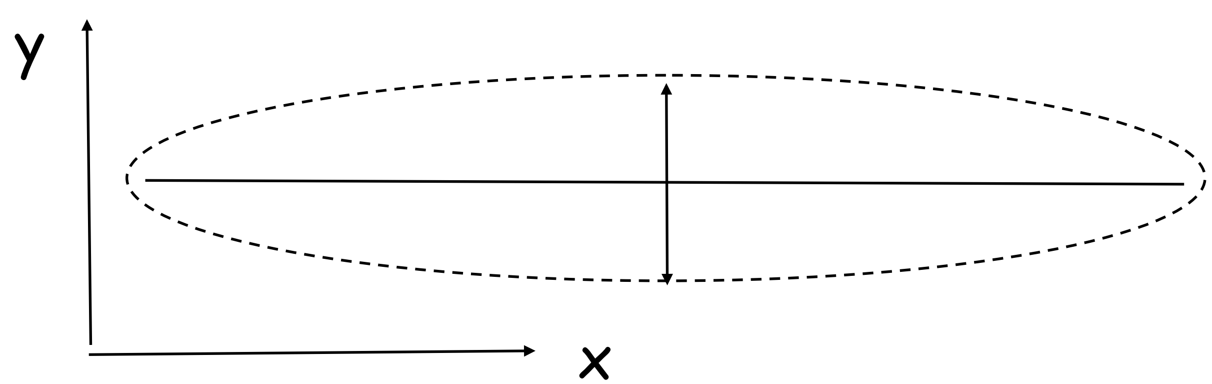
N coupled oscillators



Consider a flexible elastic string to which are attached N identical particles, each mass m, equally spaced a distance L apart.

The ends of the string are fixed a distance L from mass 1 and mass N. The initial tension in the string is T.

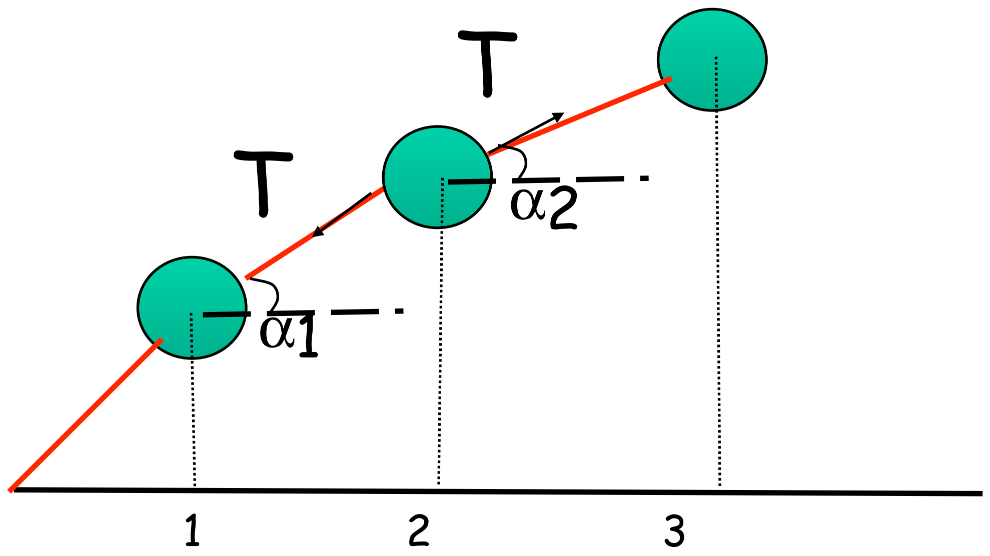
Consider small transverse displacements of the masses



Restoring forces

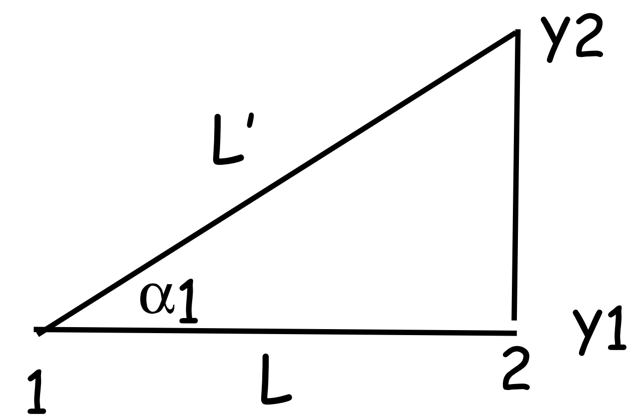
Suppose particle 1 is displaced to y_1,

particle 2 to y_2 etc



Length of string between

particles 1 and 2



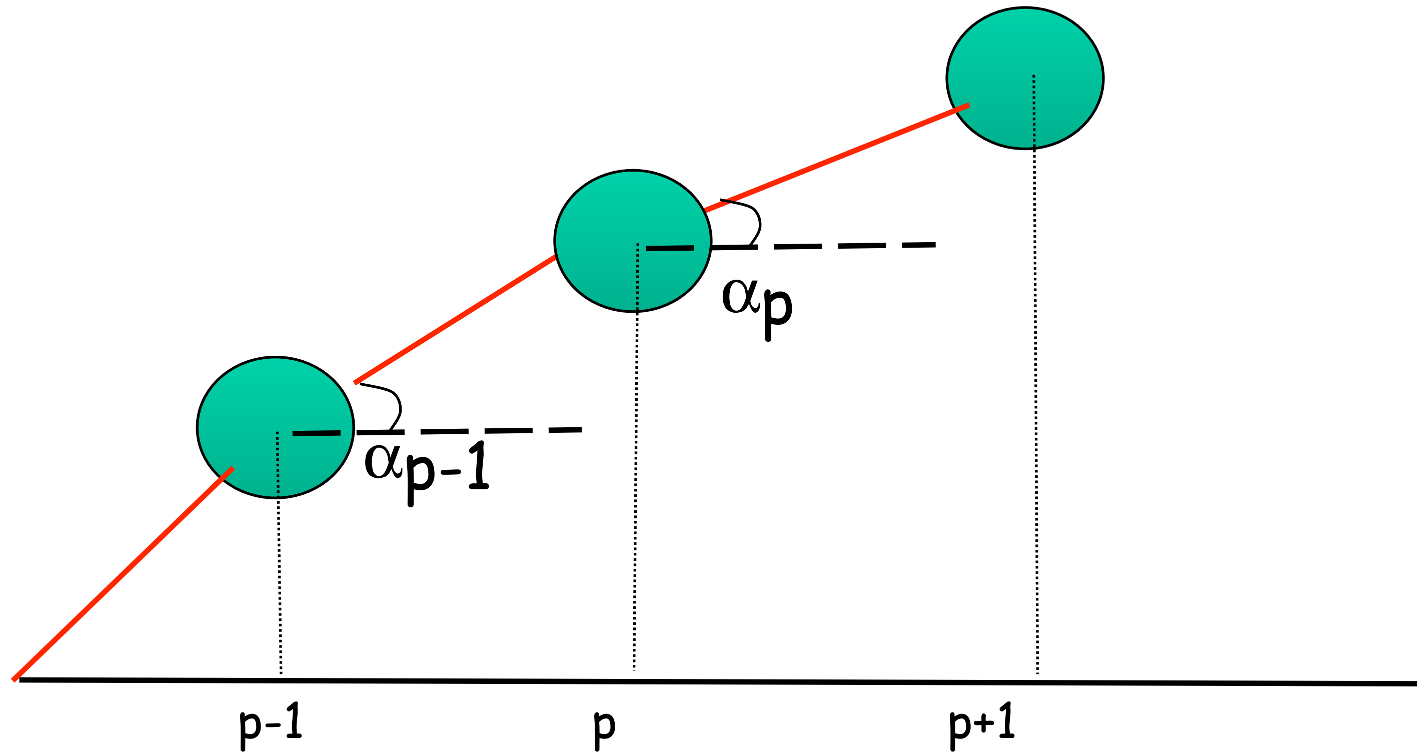
$= \cos \alpha_1 = L / L'$

ie $\quad L' = L / \cos \alpha_1$

for small α $\cos \alpha \sim (1 - \alpha_1^2 / 2)$

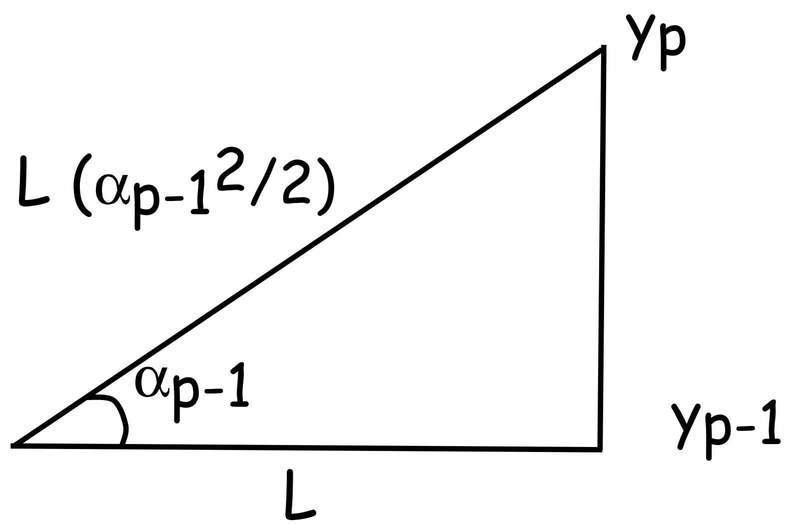
$L' \sim L (1 + \alpha_1^2 / 2)$ ie the increase in length $= L (\alpha_1^2 / 2)$

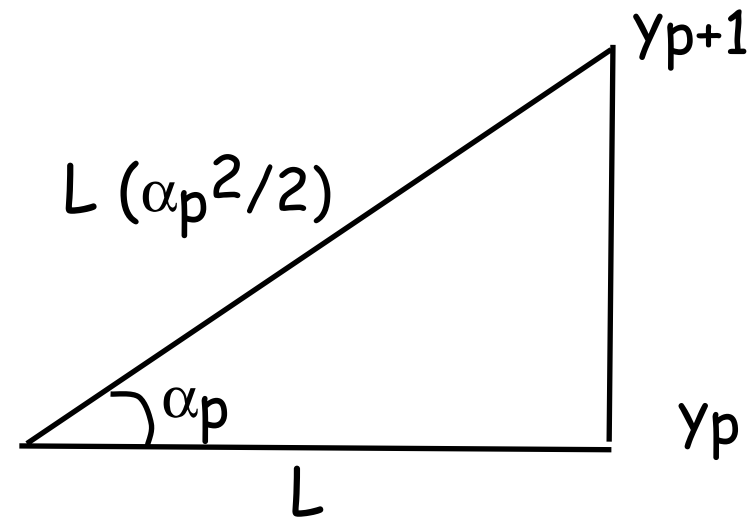
for small angles this increase is small and can be ignored consider masses p-1, p and p+1 at some point along string



for small displacements y (compared to L)

Resultant Force on p = $-T \sin \alpha_{p-1} + T \sin \alpha_p$





$$\sin\alpha_{p-1} = \frac{y_p - y_{p-1}}{L\,(\alpha_{p-1}^2/2)} \sim \frac{y_p - y_{p-1}}{L} \quad \text{for small } \alpha$$

$$\sin\alpha_{p} = \frac{y_{p+1} - y_{p}}{L\,(\alpha_{p}^2/2)} \sim \frac{y_{p+1} - y_{p}}{L} \quad \text{for small } \alpha$$

Force on particle: $$F_p = -T\left(\frac{y_p - y_{p-1}}{L}\right) + \left(\frac{y_{p+1} - y_p}{L}\right)$$

Equations of motion

$$F_p = -T\left(\frac{y_p - y_{p-1}}{L}\right) + \left(\frac{y_{p+1} - y_p}{L}\right)$$

but $F_p = m_p a_p$

$$\therefore \quad m\frac{d^2y_p}{dt^2} = -T\left(\frac{y_p - y_{p-1}}{L}\right) + \left(\frac{y_{p+1} - y_p}{L}\right)$$

Substitute $T/mL = \omega_0{}^2$

$$\therefore \quad \frac{d^2y_p}{dt^2} = -\omega_0^2\,(y_p - y_{p-1}) + \omega_0^2\,(y_{p+1} - y_p)$$

N-coupled system

or $$\frac{d^2 y_p}{dt^2} + 2\omega_0^2 y_p - \omega_0^2 (y_{p+1} - y_{p-1}) = 0$$

We can write a similar expression for all N particles

Therefore we have a **set** of **N (coupled)** differential equations one for each value of p from p=1 to p=N.

N.B. at fixed ends: $y_0 = 0$ and $y_{N+1} = 0$

Generally $$\frac{d^2y_p}{dt^2} + 2\omega_0^2\, y_p - \omega_0^2\,(y_{p+1} - y_{p-1}) = 0$$

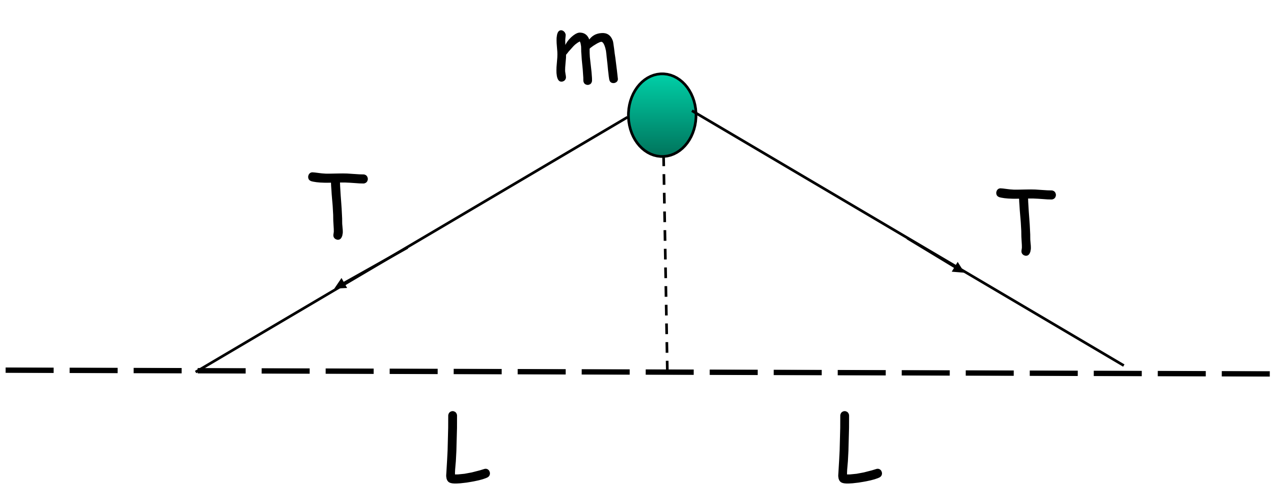
$$\frac{d^2y_1}{dt^2} + 2\omega_0^2\, y_1 - \omega_0^2\,(y_2 - y_0) = 0 \qquad \text{for } N = 1$$

no particle (y_2) =0 (y_0)

$$\frac{d^2y_1}{dt^2} + 2\omega_0^2\, y_1 = 0$$

This is transverse harmonic motion with angular frequency

$$2\omega_0^2 = 2\frac{T}{mL}$$



Special case N=2

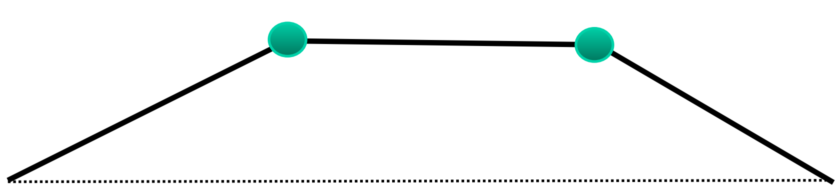
$$\frac{d^2y_1}{dt^2} + 2\omega_0^2 y_1 - \omega_0^2 y_2 = 0$$

$$\frac{d^2y_2}{dt^2} + 2\omega_0^2 y_2 - \omega_0^2 y_1 = 0$$

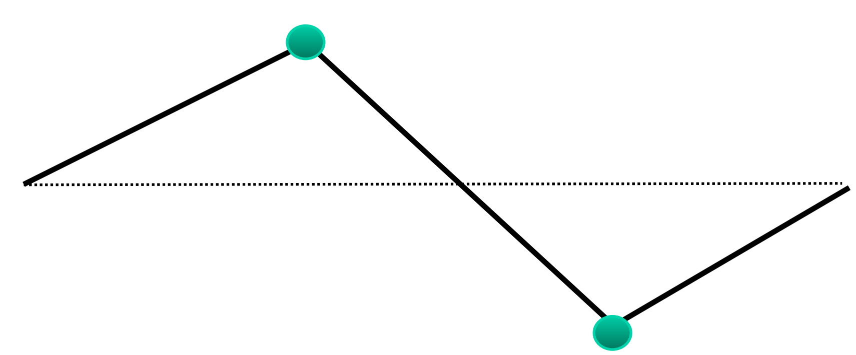
These are similar to the equations for coupled pendula but here we have the simplification that $\omega_0 = \omega_C$

We get two normal modes of oscillation

Lower mode $\omega = \omega_0$



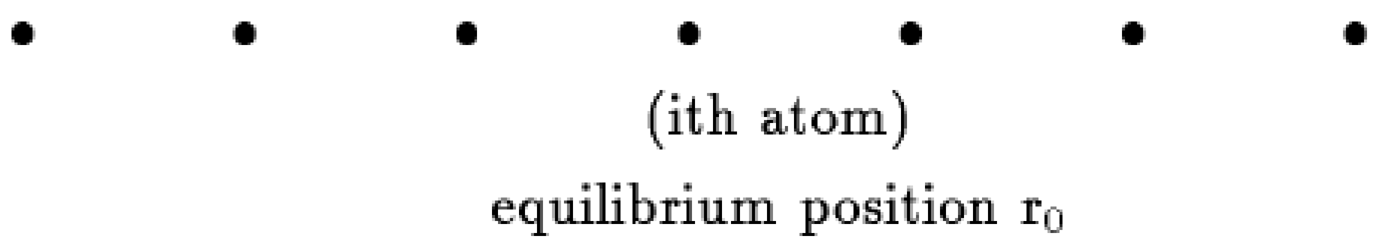
Upper mode $\omega = \sqrt{3}\omega_0$

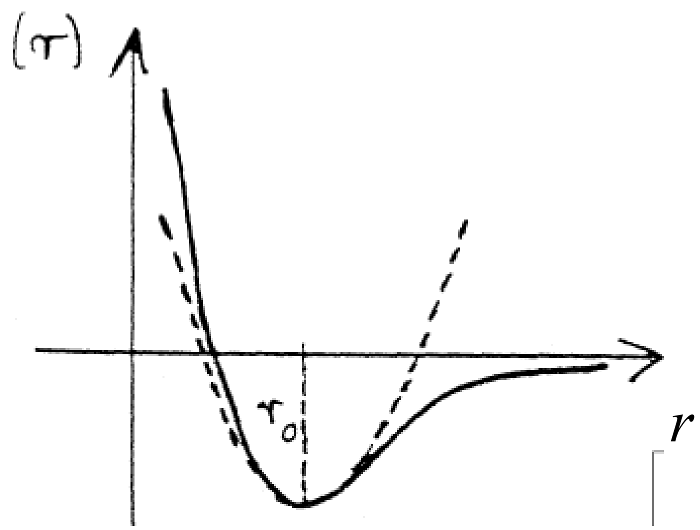


Monoatomic 1D lattice

Interatomic potential

Now we consider a monatomic 1-D lattice in the x-direction. The lattice atoms are very close to eqilibrium. Let us examine a single i-th atom and find the r_i potential as a function of displacement from equilibrium, $U(r_i)$.





We expand this potential into a Taylor's series:

$$U(r_i) = U(r_0) + (r_i - r_0)\left(\frac{dU}{dr_i}\right)_{r_0} + \frac{1}{2}(r_i - r_0)^2\left(\frac{d^2U}{dr_i^2}\right)_{r_0} + \frac{1}{6}(r_i - r_0)^3\left(\frac{d^3U}{dr_i^3}\right)_{r_0} + \dots$$

The first term of this expansion is just the equilibrium binding energy (≡ const). The second term is the slope of the potential at its minimum (= 0). The fourth and higher terms become increasingly smaller. We are therefore left with the third term as the only significant change in the potential energy for a small displacement $u = r_i - r_o$. This has the form

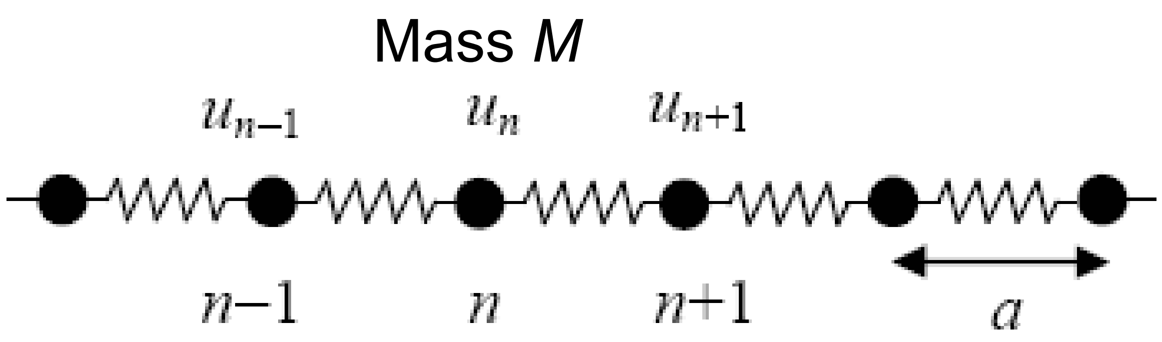
$$\Delta U = \tfrac{1}{2} C u^2 \quad (C = d^2U/dr_i^2 \text{ at } r_i = r_0)$$

representing the *harmonic approximation*, since it is the same as the energy stored in a spring, or the potential energy of a harmonic oscillator. Our simple model of the dynamic crystal structure should therefore be a "ball and spring" model, with the lengths of the springs equivalent to the equilibrium separations of the ion cores.

Monatomic 1D lattice

Let us examine the simplest periodic system within the context of harmonic approximation ($F = dU/du = Cu$) - a one-dimensional crystal lattice, which is a sequence of masses m connected with springs of force constant C and separation a.

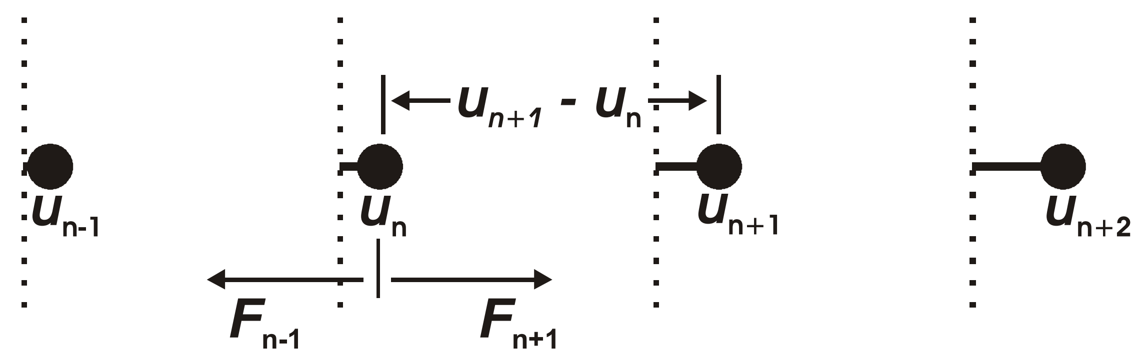
The collective motion of these springs will correspond to solutions of a wave equation. Note: by construction we can see that 3 types of wave motion are possible, 2 transverse, 1 longitudinal (or compressional)



How does the system appear with a longitudinal wave?:

The force exerted on the n-th atom in the lattice is given by

$F_n = F_{n+1,n} - F_{n-1,n} = C[(u_{n+1} - u_n) - (u_n - u_{n-1})]$.



Applying Newton's second law to the motion of the n-th atom we obtain

$$M\frac{d^2u_n}{dt^2} = F_n = -C(2u_n - u_{n+1} - u_{n-1})$$

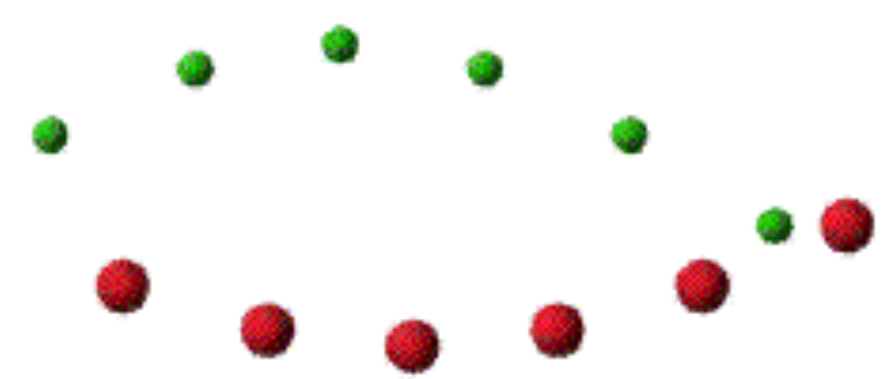
Note that we neglected hereby the interaction of the n-th atom with all but its nearest neighbors. A similar equation should be written for each atom in the lattice, resulting in N coupled differential equations, which should be solved simultaneously (N - total number of atoms in the lattice). In addition the boundary conditions applied to end atoms in the lattice should be taken into account.

Acoustic and optical modes

Monoatomic chain
acoustic longitudinal mode

Monoatomic chain
acoustic transverse mode

Diatomic chain
acoustic transverse mode



Diatomic chain
optical transverse mode