The Abdus Salam International Centre for Theoretical Physics

Postgraduate Diploma Programme

Earth System Physics

Wave physics

Sound and Fourier analysis

Fabio ROMANELLI

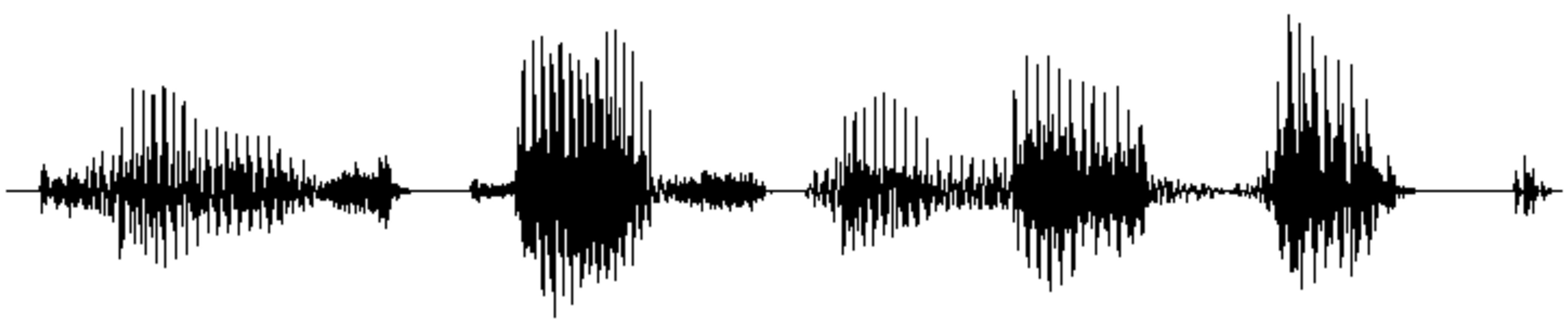
Dept. Mathematics & Geosciences

Università degli studi di Trieste

romanel@units.it

Complex waves

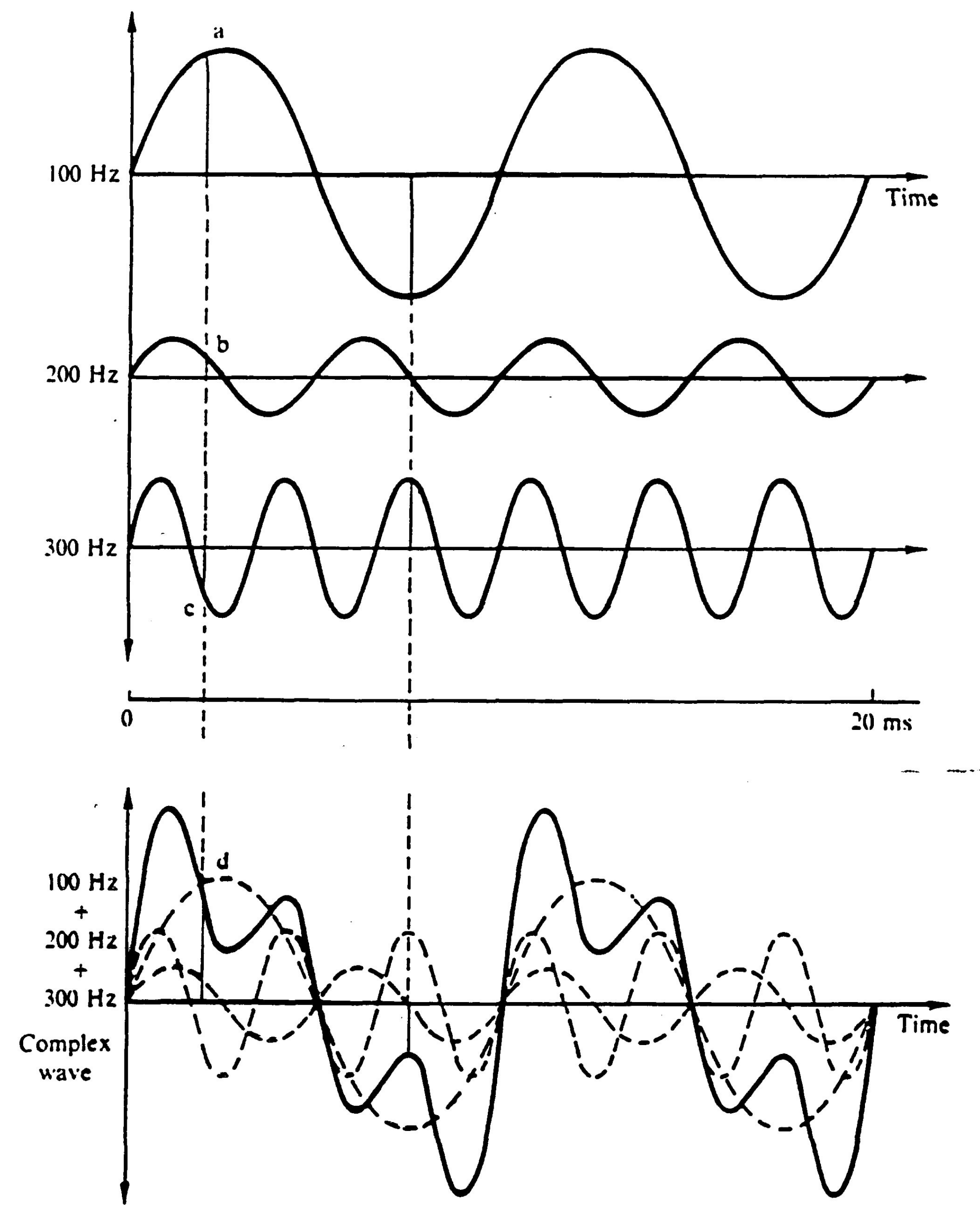
- Sound waves are not simple sinusoidal waves, but are complex. We can see this visually with a waveform:



Complex waves

Complex waves can be broken down mathematically (by Fourier analysis) into a series of simple waves:

Fourier analysis (simplified): at a given point in time, add the increases in air pressure and subtract the decreases in air pressure



Complex waves

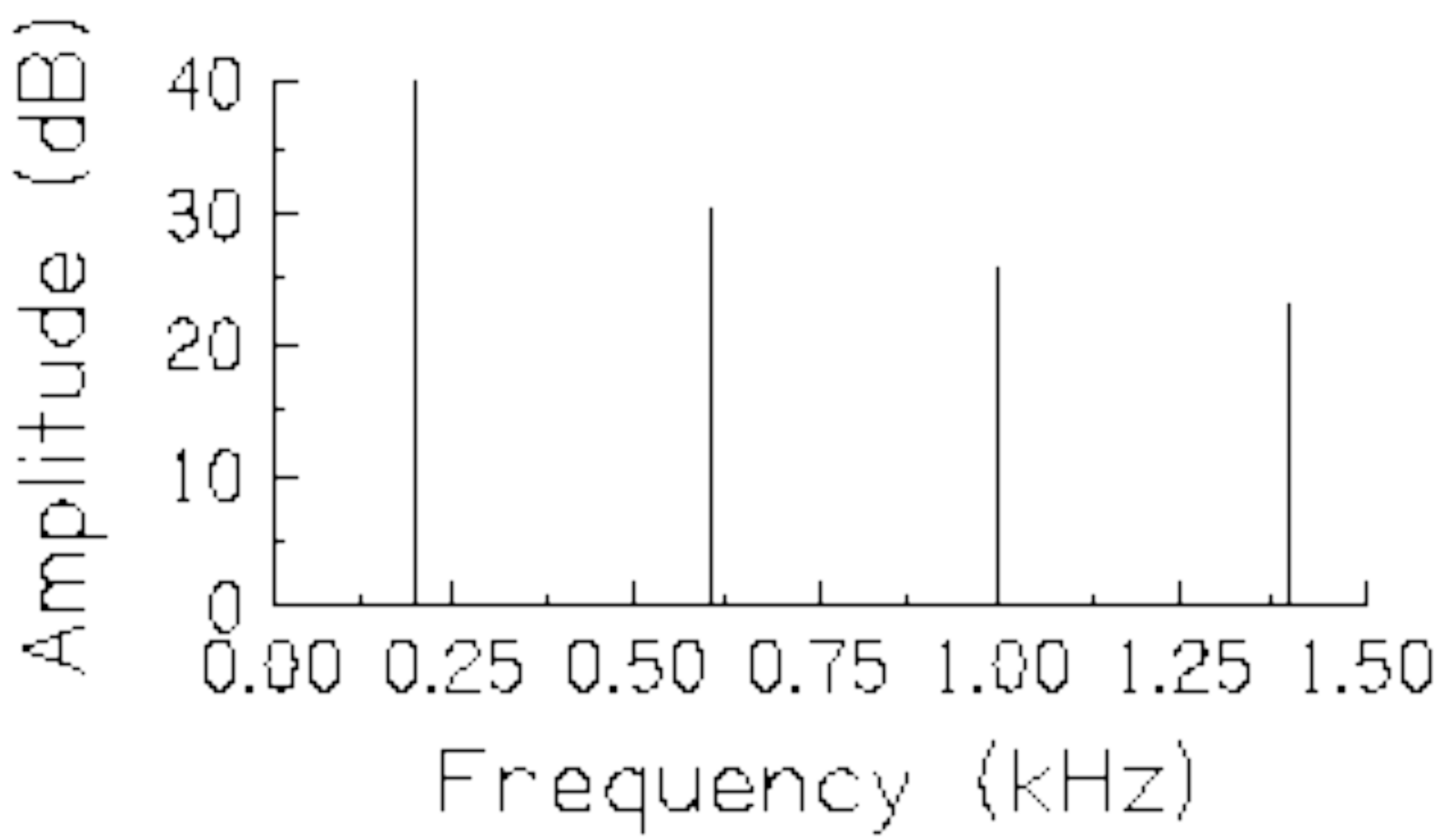
- The frequency of the complex wave is the fundamental period and is the same as the frequency of the first component of a complex wave. This is the fundamental frequency or f_1, that corresponds to the "**pitch**" of the sound

- Each of the higher-frequency simple waves is called a **harmonic**. In naturally occurring vibrations, there is a harmonic at each multiple of the fundamental frequency, so if f_1 = 100Hz, 2nd harmonic = 200Hz, 3rd harmonic = 300Hz , etc...

- Different patterns of harmonics are part of what contributes to the distinction between the sound of different musical instruments playing the same note

Spectrum

- Because each complex wave has many components or harmonics, it is difficult to see in a waveform in which the X-axis is time. We can also plot amplitude with respect to frequency in a **spectrum**, a useful way of interpreting complex waves:

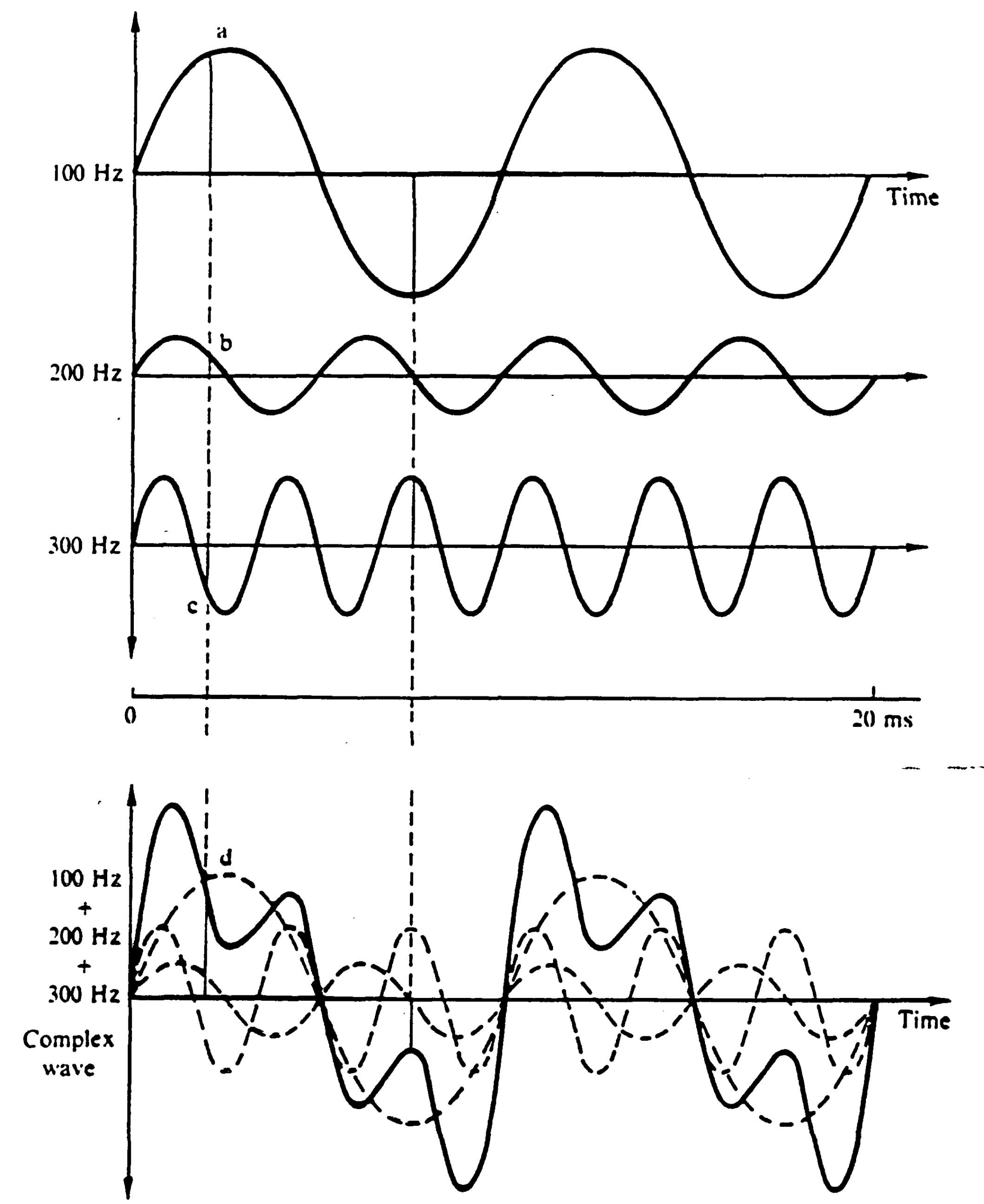


Each line in a spectrum represents a simple wave and indicates its amplitude and frequency

Spectrum

Consider the complex wave again.

It has three components with different frequencies and amplitudes



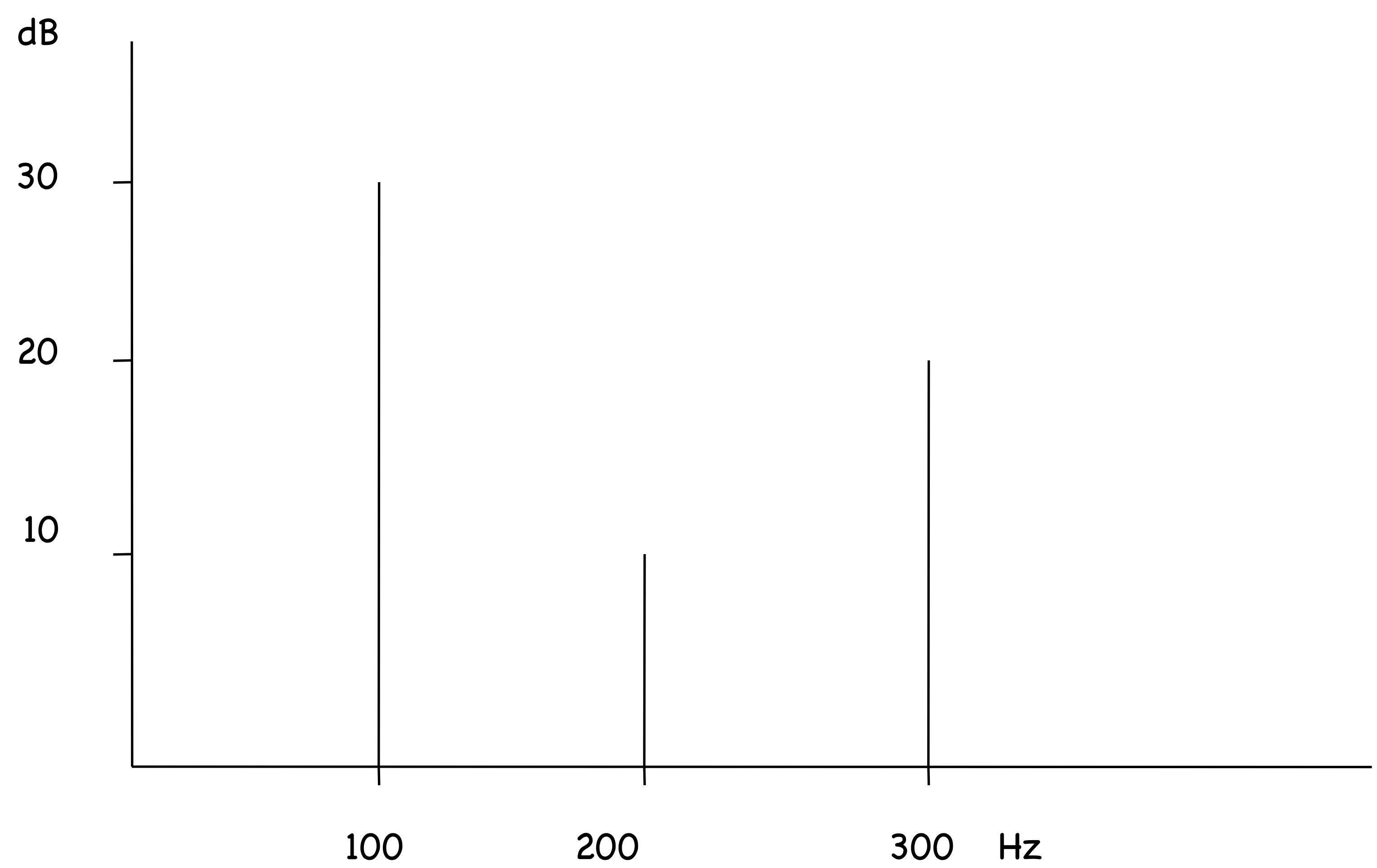
	Frequency	Amplitude
1	100Hz	30dB
2	200Hz	10dB
3	300Hz	20dB

Spectrum

The three component waves have the following properties:

	Frequency	Amplitude
1	100Hz	30dB
2	200Hz	10dB
3	300Hz	20dB

The spectrum looks like:

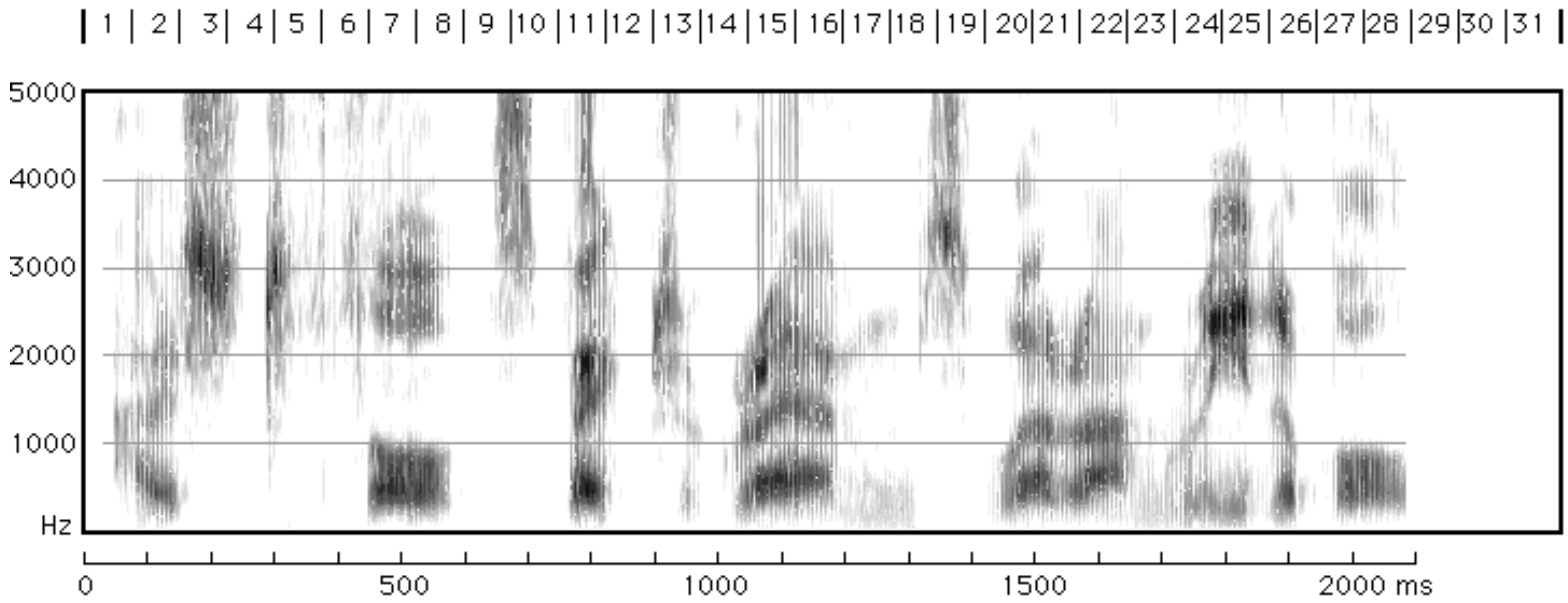


Spectrogram

Another way of examining sound

Spectrogram plots intensity and frequency with respect to time.

X-axis is time; Y-axis is frequency; darkness is intensity:



Fourier Series

- Fourier theorem states that a periodic function f(t) which is reasonably continuous may be expressed as the sum of a series of sine or cosine terms (called the Fourier series), each of which has specific amplitude and phase coefficients known as Fourier coefficients.

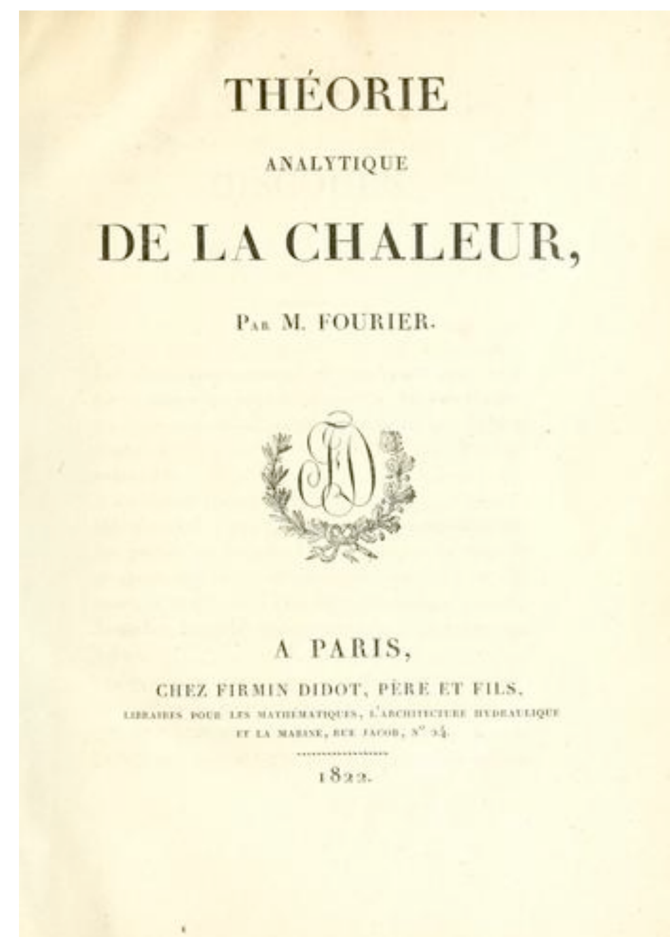
Jean Baptiste Joseph Fourier
1768-1830

- Linear Superimposition of Sinusoids to build complex waveforms

$$f(t) = a_0 + \sum_{1}^{\infty} A_n \cos(\omega_n t + \phi_n)$$

$$\omega_n = n\omega_1 = n\frac{2\pi}{T_1}$$

http://www.falstad.com/fourier/



THÉORIE

ANALYTIQUE

DE LA CHALEUR,

PAR M. FOURIER.

A PARIS,

CHEZ FIRMIN DIDOT, PÈRE ET FILS,

LIBRAIRES POUR LES MATHÉMATIQUES, L'ARCHITECTURE HYDRAULIQUE ET LA MARINE, RUE JACOB, N° 24.

1822.

- Decompose our complex periodic waveform into a series of simple sinusoids
- Where

$$f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left[a_n \cos(n\omega t) + b_n \sin(n\omega t)\right]$$

$$a_0 = \frac{2}{T}\int_0^T f(t)dt$$

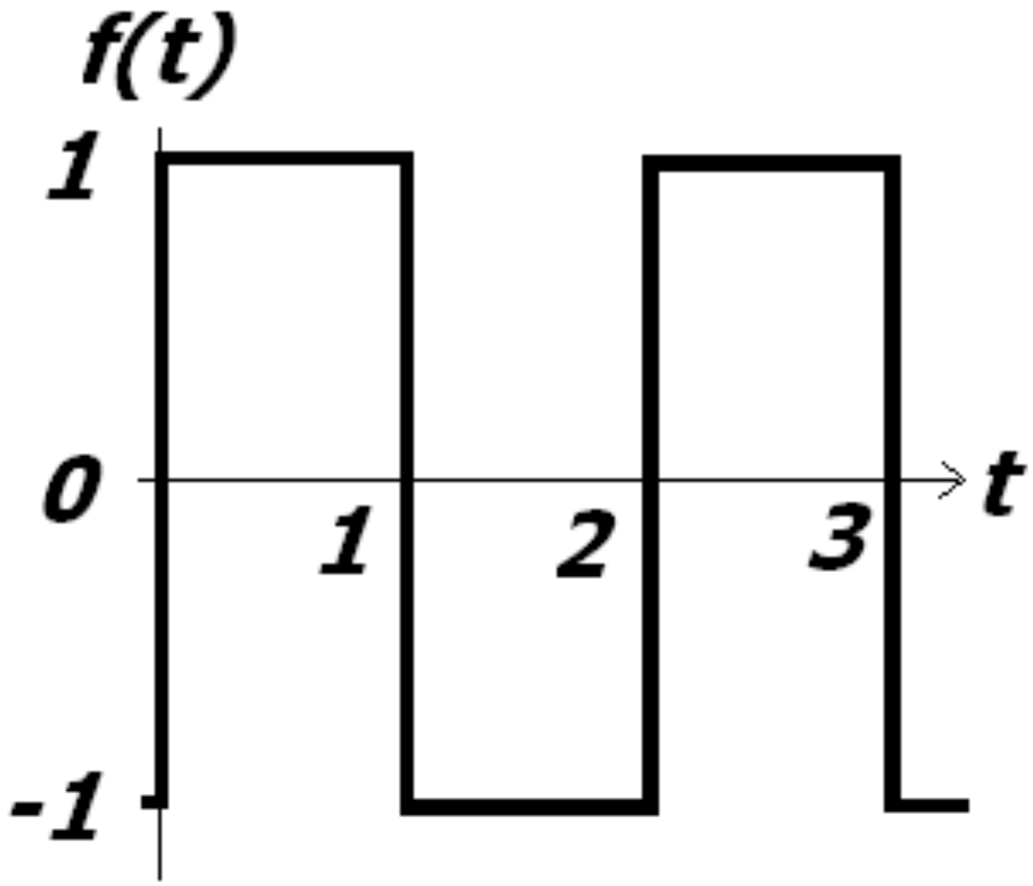
$$a_n = \frac{2}{T}\int_0^T f(t)\cos(n\omega t)dt$$

$$b_n = \frac{2}{T}\int_0^T f(t)\sin(n\omega t)dt$$

$$\omega = \frac{2\pi}{T}$$

Square Wave Example

- Consider



$$f(t) = 1 \qquad 0 < t < 1$$

$$= -1 \qquad 1 < t < 2$$

- Clearly the period is T=2 hence $\omega=\pi$
- When we integrate we need to do so over sections

  t= 0 to 1 and t=1 to 2

- So to find the series, we have to calculate coefficients a_0, a_n and b_n

$$a_0 = \int_0^2 f(t)dt$$

$$= \int_0^1 dt - \int_1^2 dt = 1 - 1 \Rightarrow a_0 = 0$$

$$a_n = \int_0^2 f(t)\cos(n\pi t)dt$$

$$= \int_0^1 \cos(n\pi t)dt - \int_1^2 \cos(n\pi t)dt$$

$$= \frac{1}{n\pi}\left[\sin(n\pi t)\Big|_0^1 - \sin(n\pi t)\Big|_1^2\right] \Rightarrow a_n = 0$$

When evaluating a_n note that the sin function is 0 when angle is every multiple of π

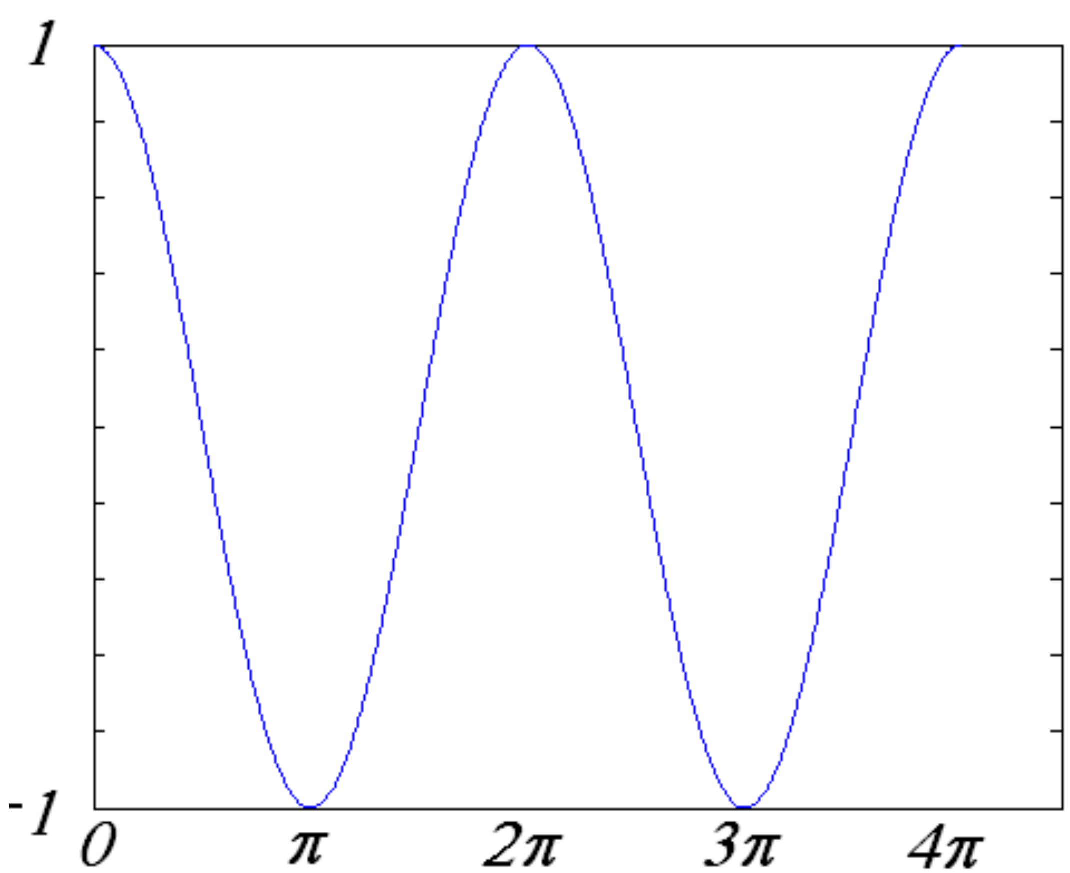
$$b_n = \int_0^2 f(t)\sin(n\pi t)dt = \int_0^1 \sin(n\pi t)dt - \int_1^2 \sin(n\pi t)dt$$

$$= -\frac{1}{n\pi}\left[\cos(n\pi t)\Big|_0^1 - \cos(n\pi t)\Big|_1^2\right]$$

$$= -\frac{1}{n\pi}\Big((\cos(n\pi) - 1) - (\cos(n\pi 2) - \cos(n\pi))\Big)$$

$$= -\frac{1}{n\pi}(2\cos(n\pi) - 1 - \cos(n2\pi))$$

- Knowing $$b_n = -\frac{1}{n\pi}(2\cos(n\pi) - 1 - \cos(n2\pi))$$
- We need to consider the cos function to determine values of b_n for n=1,2,3,.... etc



$$n = 1, \quad b_1 = -\frac{1}{n\pi}(2(-1) - 1 - 1) = \frac{4}{n\pi}$$

$$n = 2, \quad b_2 = -\frac{1}{n\pi}(2(1) - 1 - 1) = 0$$

$$n = 3, \quad b_3 = -\frac{1}{n\pi}(2(-1) - 1 - 1) = \frac{4}{n\pi}$$

$$n = 4, \quad b_4 = -\frac{1}{n\pi}(2(1) - 1 - 1) = 0$$

$$n = 5, \quad b_2 = -\frac{1}{n\pi}(2(-1) - 1 - 1) = \frac{4}{n\pi}$$

- We found coefficients to be

$$a_0 = 0 \qquad a_n = 0 \qquad b_n = \frac{4}{n\pi} \text{ when n} = 1,3,5...$$

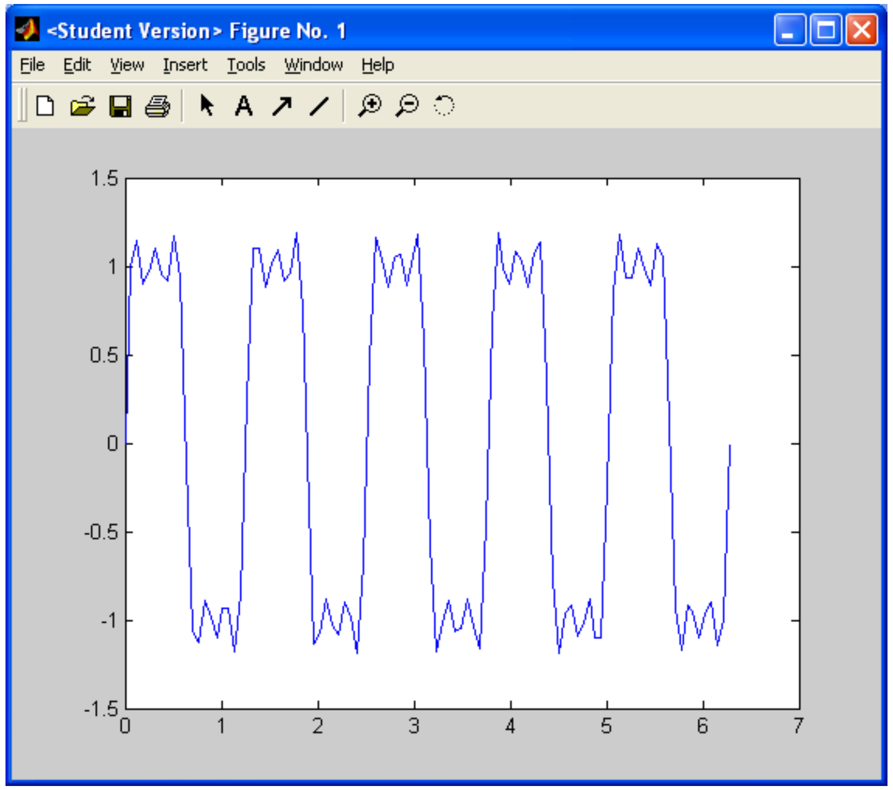
- Hence Fourier Series for a square wave is

$$f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left[a_n \cos(n\omega t) + b_n \sin(n\omega t)\right]$$

$$= \frac{4}{\pi}\,[\ \sin(\pi\ t) + \tfrac{1}{3}\sin(3\pi\ t) + \tfrac{1}{5}\sin(5\pi\ t) + \ldots\ ]$$

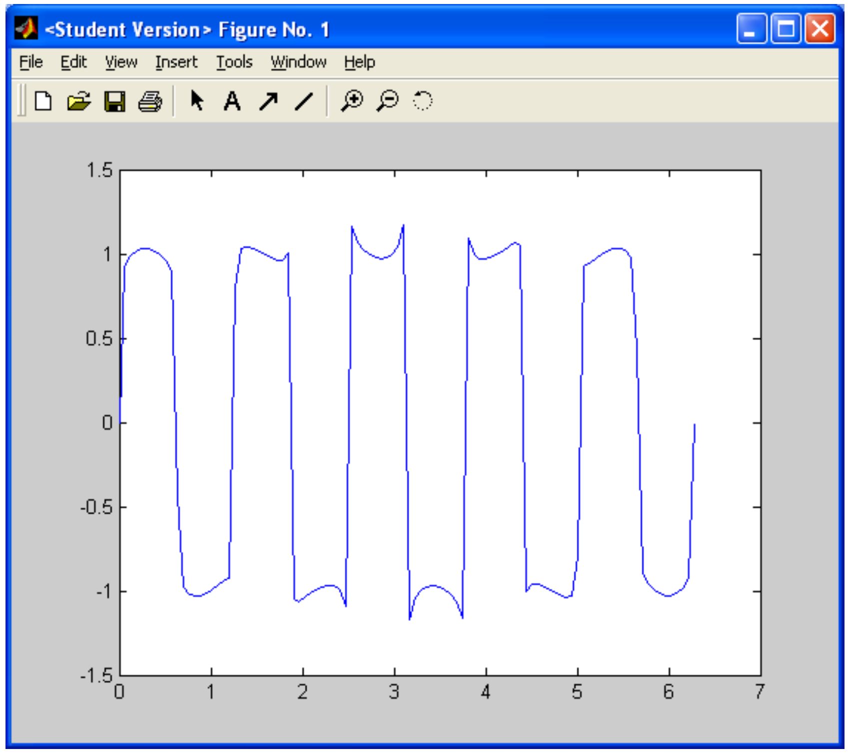
Let's try implementing series in Matlab

```
f=100;
w=2*pi*f;
t=linspace(0,1/f*5,100);
y1=4/pi*(sin(w*t)+1/3*sin(3*w*t)
+1/5*sin(5*w*t));
plot(t,y1)
```

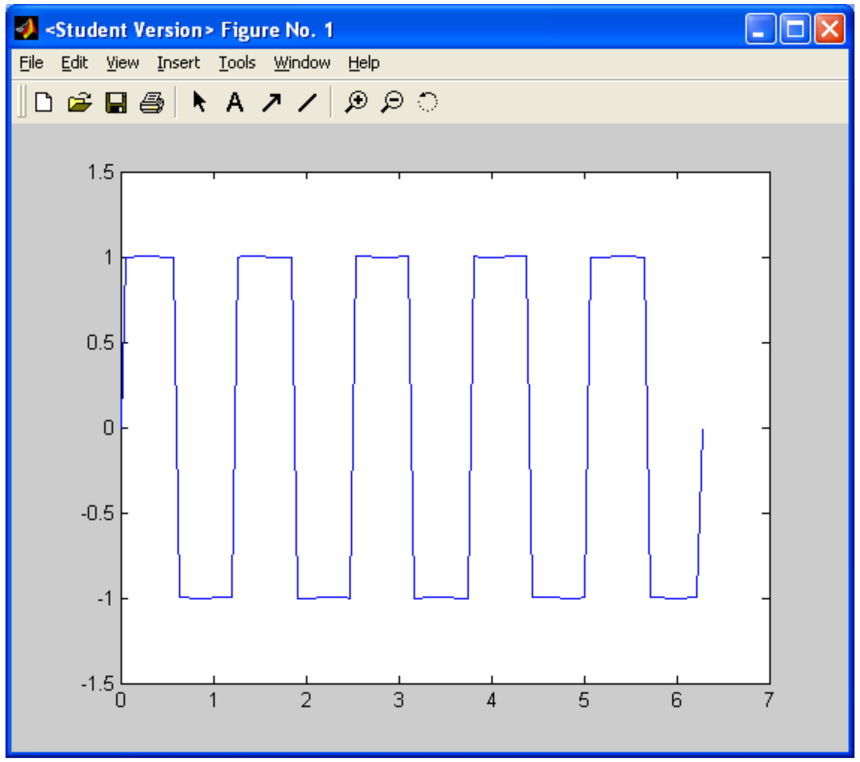


We can see it's not quite a square wave (Gibbs phenomenon) given series should be infinite. However with a for loop we can add more to the series

To n=10



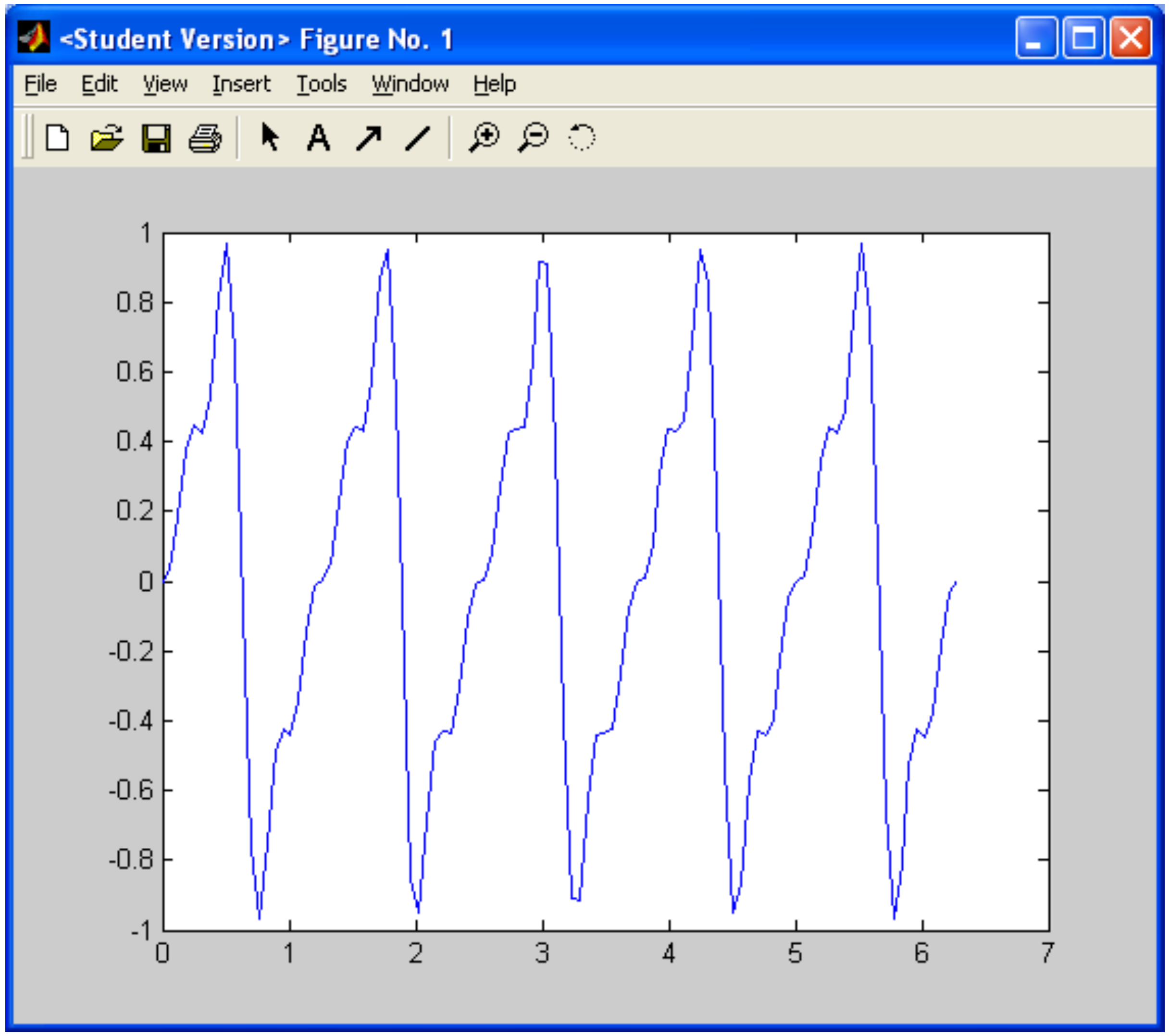
To n=1000



Saw tooth example

- Evaluating Fourier series for saw tooth produces

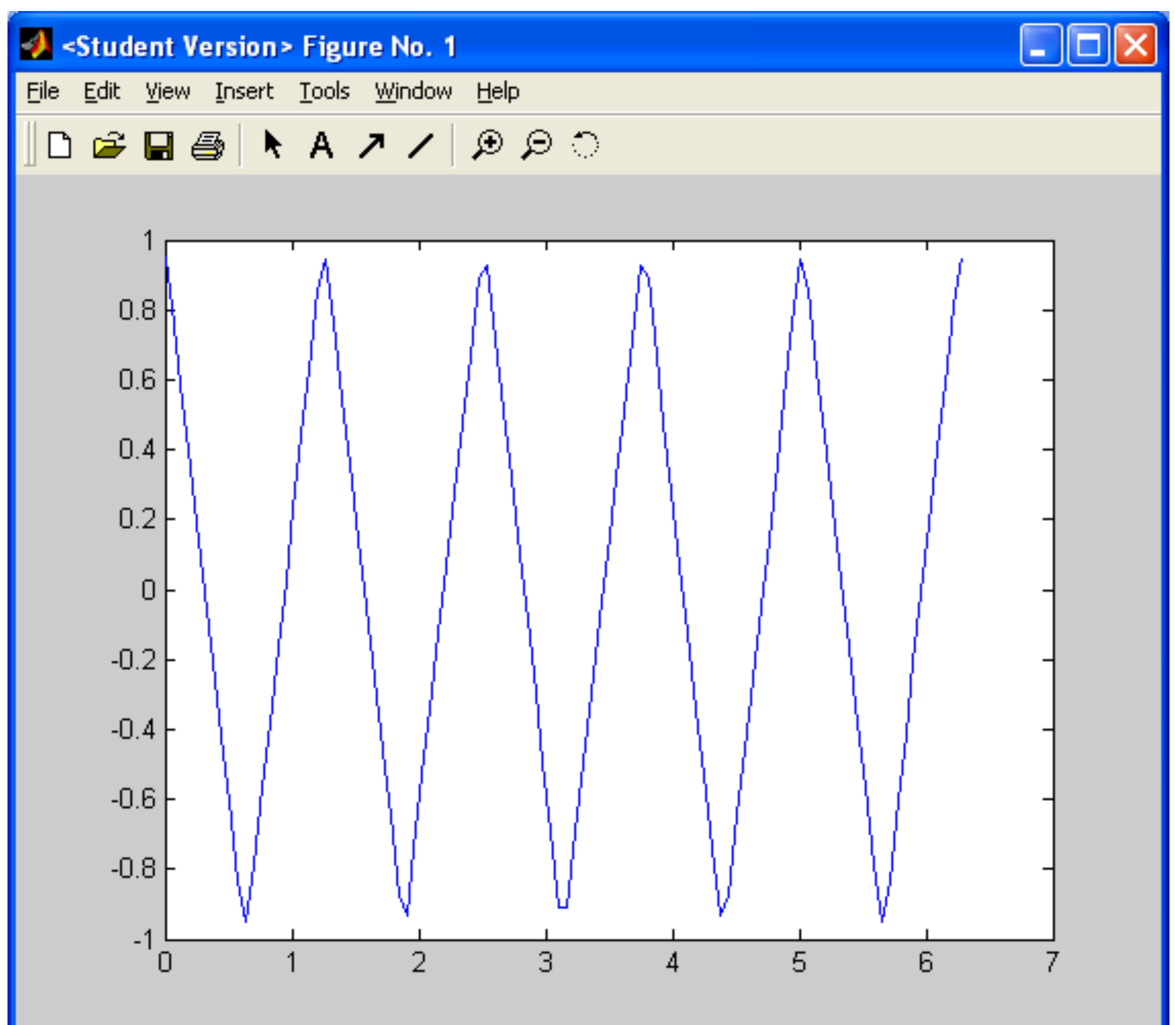
$$f(t) = \frac{2}{\pi}\left(\sin(\omega t) - \frac{1}{2}\sin(2\omega t) + \frac{1}{3}\sin(3\omega t) - \frac{1}{4}\sin(4\omega t) + \ldots\right.$$



Triangle Example

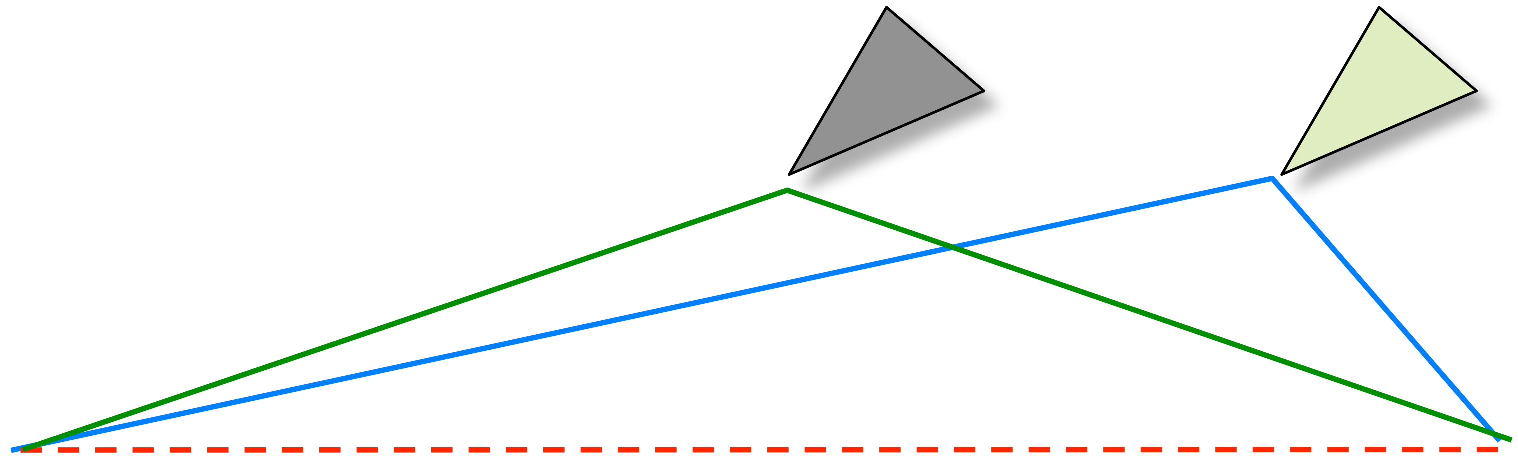
- Evaluating Fourier series for triangle wave

$$f(t) = \frac{8}{\pi^2}\left(\cos(\omega t) + \frac{1}{9}\cos(3\omega t) + \frac{1}{25}\cos(5\omega t) + \frac{1}{49}\cos(7\omega t) + \ldots\right)$$



Plucked string

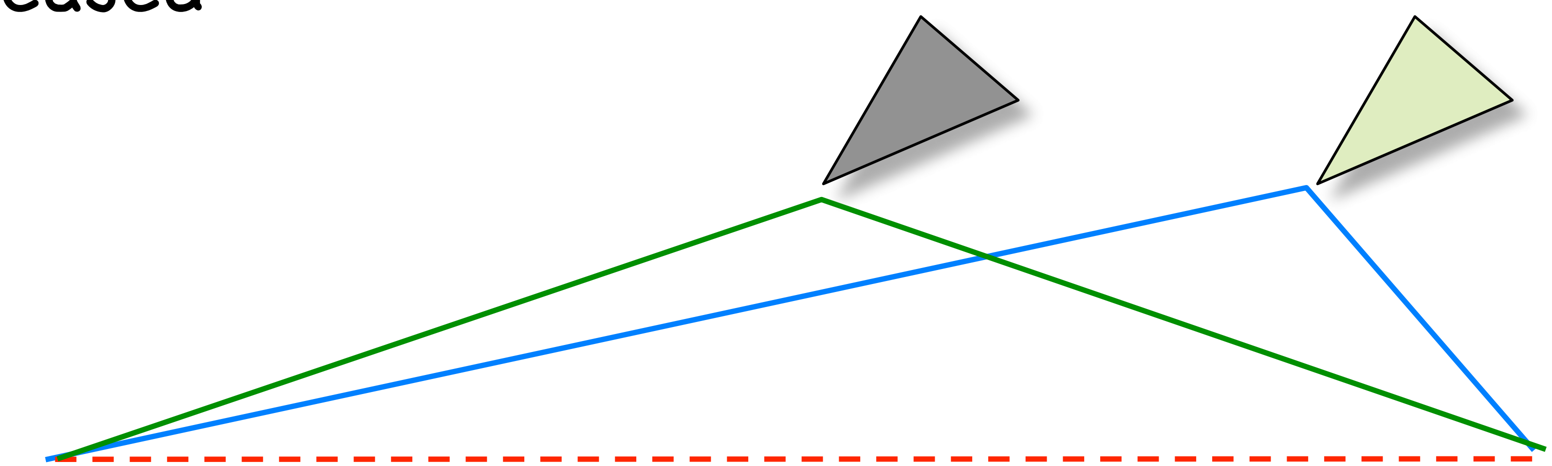
- Can one predict the amplitude of each mode (overtone/harmonic?) following plucking?



- Using the procedure to measure the Fourier coefficients it is possible to predict the amplitude of each harmonic tone.

Plucked string

•You know the shape just before it is plucked.
•You know that each mode moves at its own frequency
•The shape when released
•We rewrite this as



$$f(x, t = 0)$$

$$f(x, t = 0) = \sum_{m=0}^{\infty} A_m \sin \frac{2\pi m x}{2L}$$

Plucked string

Each harmonic has its own frequency of oscillation, the m-th harmonic moves at a frequency $f_m = mf_0$ or m times that of the fundamental mode.

$$f(x, t=0) = \sum_{m=0}^{\infty} A_m \sin \frac{2\pi mx}{2L} \quad \text{initial condition}$$

$$f(x,t) = \sum_{m=0}^{\infty} A_m \sin \frac{2\pi mx}{2L} \cos 2\pi m f_0 t$$

http://www.falstad.com/loadedstring/

Do we hear phase?

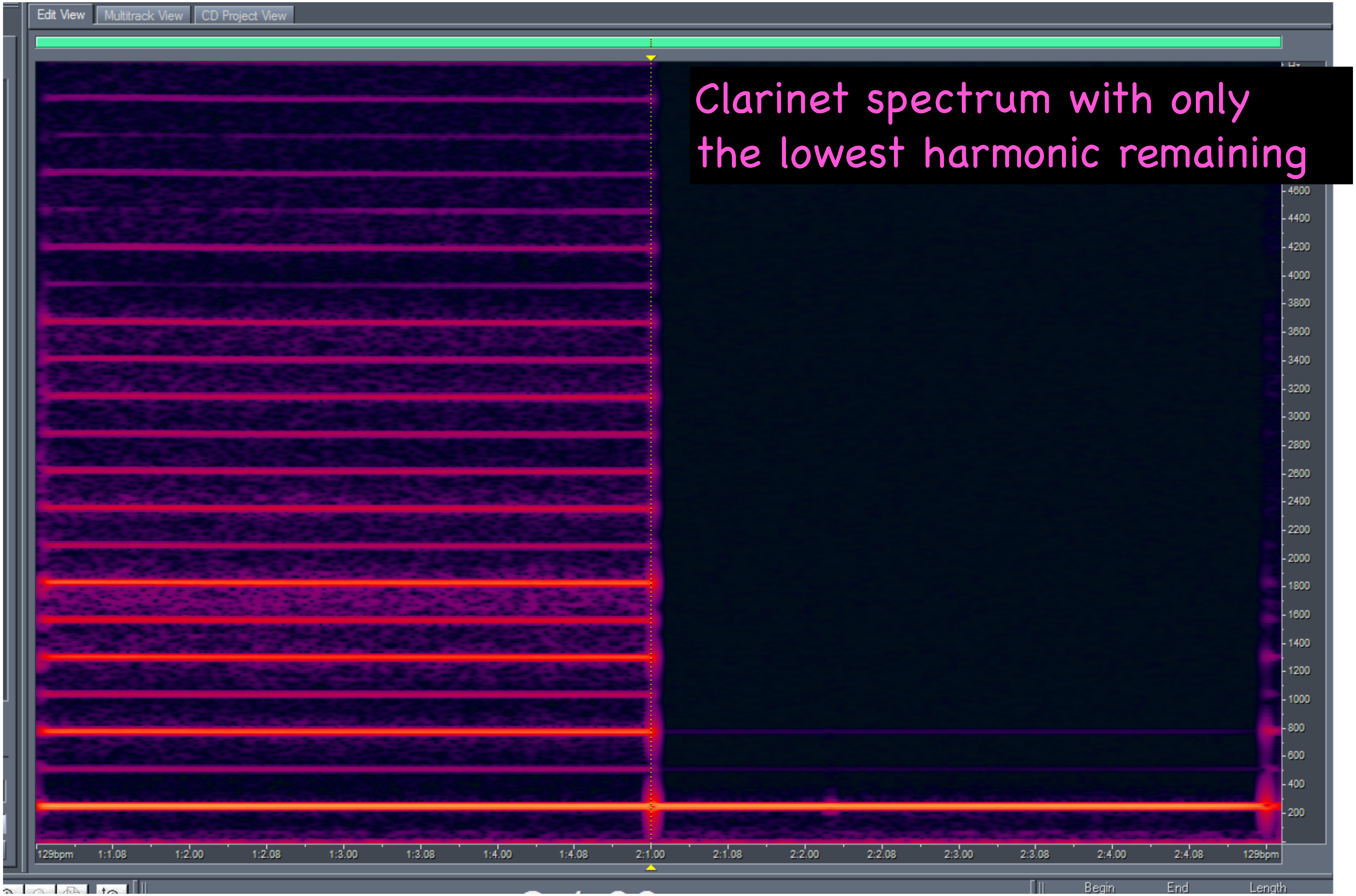
Helmholtz and Ohm argued that our ear and brain are only sensitive to the frequencies of sounds. Timbre is a result of frequency mix.

These two are sums with the same amplitude sine waves components,
however the phases of the sine waves differ.

This sound file has varying phases of its frequencies. Do we hear any difference in time?

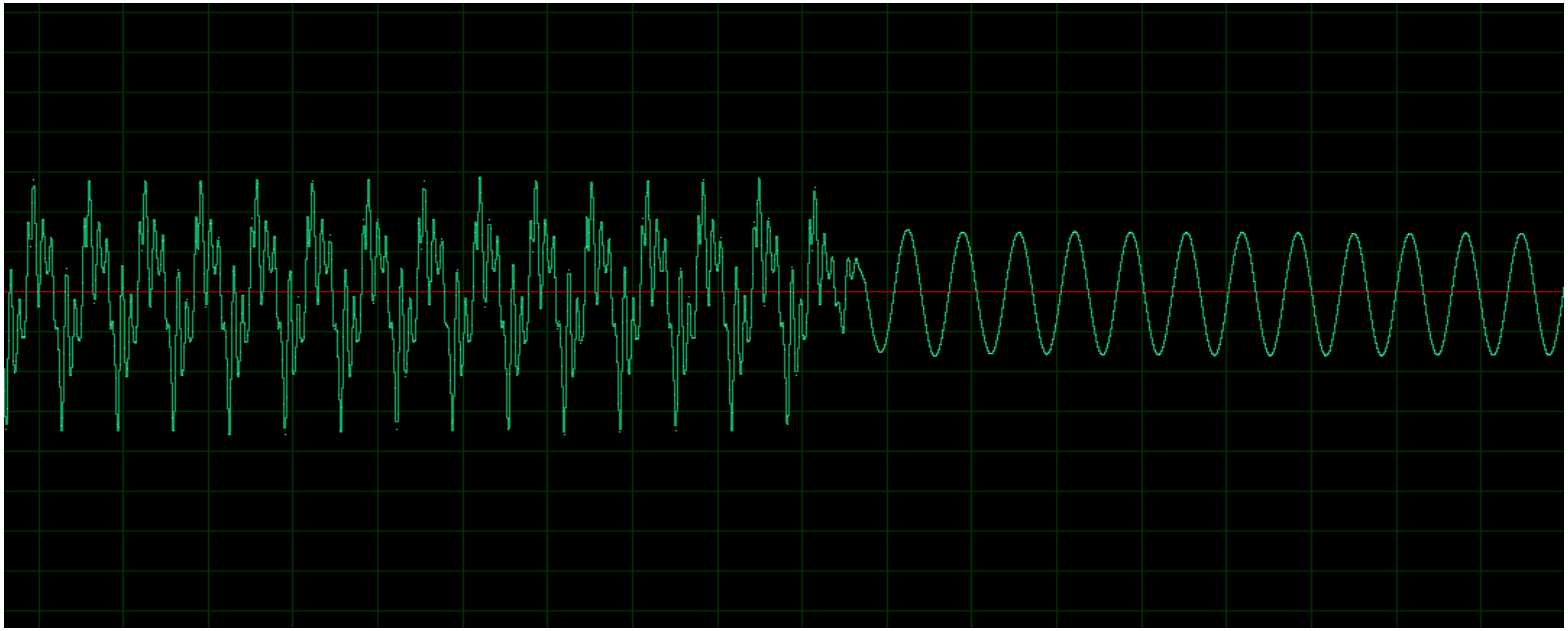
Clarinet spectrum

Spectral view



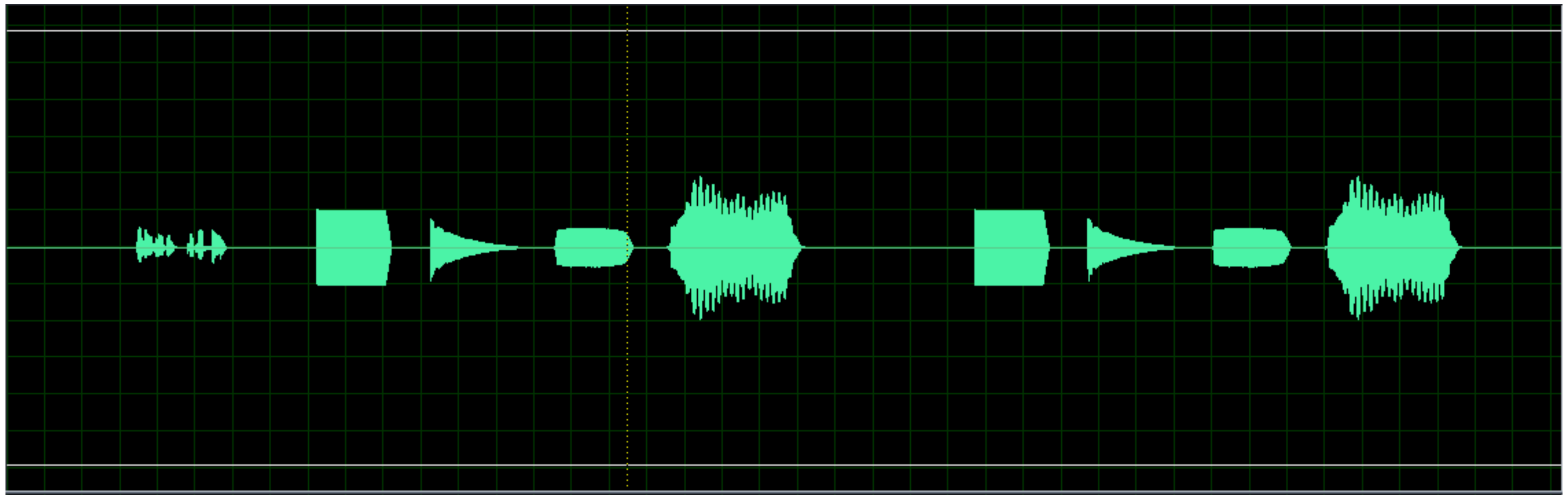
Waveform view

Full sound   Only lowest harmonic



Four complex tones in which all partials have been removed by filtering

One is a French horn, one is a violin, one is a pure sine, one is a piano (but out of order)



It's hard to identify the instruments. However clues remain (attack, vibrato, decay)