Unconstrained Optimization

UNCONSTRAINED OPTIMIZATION

We generalize the results for a single variable to the case of many variables

Consider the problem:

$$max_x \, f\,(x) \; subject\; to\; x \; \in S$$

where x is a vector

Proposition

Let *f* be a differentiable function of *n* variables defined on the set *S*. If the point *x* in the interior of *S* is a local or global maximizer or minimizer of *f* then

$$f_i'(x) = 0 \; for \; i = 1, \ldots, n.$$

Then the condition that all partial derivatives are equal to zero is a necessary condition for an interior optimum (and therefore for an optimum in an unconstrained optimization where each element of x could be any of the real numbers.

Conditions under which a stationary point is a local optimum

Let f be a function of n variables with continuous partial derivatives of first and second order, defined on the set S. Suppose that x^* is a stationary point of f in the interior of S (so that $f_i'(x^*) = 0$ for all i).

If $H(x^*)$ is negative definite then x^* is a local maximizer.
If x^* is a local maximizer then $H(x^*)$ is negative semidefinite.
If $H(x^*)$ is positive definite then x^* is a local minimizer.
If x^* is a local minimizer then $H(x^*)$ is positive semidefinite.

where $H(x)$ denotes the Hessian of f at x.

Conditions under which a stationary point is a global optimum

Suppose that the function f has continuous partial derivatives in a convex set S and x^* is a stationary point of f in the interior of S (so that $f_i'(x^*) = 0$ for all i).

1. if f is concave then x^* is a global maximizer of f in S if and only if it is a stationary point of f

2. if f is convex then x^* is a global minimizer of f in S if and only if it is a stationary point of f.

$H(z)$ is negative semidefinite for all $z \in S \Rightarrow$ [x is a global maximizer of f in S if and only if x is a stationary point of f]

$H(z)$ is positive semidefinite for all $z \in S \Rightarrow$ [x is a global minimizer of f in S if and only if x is a stationary point of f],

where $H(x)$ denotes the Hessian of f at x.

Example 1: Unconstrained Maximization with two variables

For example Utility = U(x, y) or Output = F(K, L)

Now try to find the values of x and y which maximise a function $f(x, y)$

Three steps:

1. Set **both** 1st order conditions equal to zero $f_x = 0$ and $f_y = 0$

(the slope of the function with respect to both variables must be simultaneously zero)

2. Solve the equations simultaneously for x and y

However this is a necessary but not sufficient condition (saddle points, points of inflection)

However this is a necessary but not sufficient condition (saddle points, points of inflection)

3. Second order conditions (for maximization)

$$H = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{pmatrix}$$

$$f_{xx} \leq 0, \;\; f_{yy} \leq 0 \;\; \text{and} \;\; f_{xx}f_{yy} - f^2_{xy} \geq 0$$

Note: Second order conditions (for minimization) are

$f_{xx} \geq 0$, $f_{yy} \geq 0$ and $f_{xx}f_{yy} - f^2_{xy} \geq 0$

$f(x,y) = 4x – 2x^2 + 2xy – y^2$

1. (i). $f_x = 4 – 4x + 2y = 0$

   (ii). $f_y = 2x – 2y = 0$

2. Solve: from (ii) we have $x = y$

   insert into (i) to get $4 – 4x + 2x = 0$ or

   $4 = 2x$ or $x = 2$

   so $y = x = 2$

3. $H = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{pmatrix} = \begin{pmatrix} -4 & 2 \\ 2 & -2 \end{pmatrix}$

The first order leading principal minor is $f_{xx} = -4 < 0$

The second order leading principal minor is

$$f_{xx}f_{yy} – f^2_{xy} = (-4)(-2) – (2)^2 = 4>0$$

Then the matrix H is negative definite

f is (strictly) concave, so we have a maximum point where $x = 2$ and $y = 2$

Example 2

Maximize f(x) = $-x_1^2 - 2x_2^2$

The first order conditions are:

$$\begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \end{pmatrix} = \begin{pmatrix} -2x_1 \\ -4x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Is this a maximum? – it will be if function is concave

1. H is,

$$H = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} \end{pmatrix} = \begin{pmatrix} -2 & 0 \\ 0 & -4 \end{pmatrix}$$

From H find the leading principal matrices by eliminating:

1. The last n-1 rows and columns – written as D_1 = (-2)
2. The last n-2 rows and columns – written as D_2 = H

Compute the determinants of these leading principal matrices.

1. $|D_1| = -2$
2. $|H| = 8$

Then the matrix H is negative definite

f is (strictly) concave

the values of x which satisfy FOC (0 and 0) give a maximum.

Example 3

Total revenue R = $12q_1 + 18q_2$

Total Cost = $2q_1^2 + q_1q_2 + 2q_2^2$

Find the values of q_1 and q_2 that maximise profit

Profit = revenue – cost = $12q_1 + 18q_2 - (2q_1^2 + q_1q_2 + 2q_2^2)$

The first order conditions are:

$$\begin{pmatrix} \frac{\partial \pi}{\partial q_1} \\ \frac{\partial \pi}{\partial q_2} \end{pmatrix} = \begin{pmatrix} 12 - 4q_1 - q_2 \\ 18 - q_1 - 4q_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Solving for q_1 and q_2 gives $q_1 = 2$ and $q_2 = 4$

Is this a maximum? –it will be if function is concave

The Hessian is

$$H = \begin{pmatrix} \frac{\partial^2 \pi}{\partial q_1^2} & \frac{\partial^2 \pi}{\partial q_1 \partial q_2} \\ \frac{\partial^2 \pi}{\partial q_2 \partial q_1} & \frac{\partial^2 \pi}{\partial q_2^2} \end{pmatrix} = \begin{pmatrix} -4 & -1 \\ -1 & -4 \end{pmatrix}$$

From H find the leading principal matrices by eliminating:

1. The last n-1 rows and columns – written as $D_1 = (-4)$
2. The last n-2 rows and columns – written as $D_2 = H$

Compute the determinants of these leading principal matrices.

1. $|D_1| = -4$
2. $|H| = (-4) * (-4) - 1 = 15$

So H is negative definite, then f is (strictly) concave and the values for q_1 and q_2 maximise profits

Example with three variables

Maximize $f(x) = -x_1{}^2 - 2\,x_2{}^2 - x_3{}^2$

The first order conditions are:

$$\begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \\ \frac{\partial f}{\partial x_3} \end{pmatrix} = \begin{pmatrix} -2x_1 \\ -4x_2 \\ -2x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The Hessian is:

$$H = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_1 \partial x_3} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \frac{\partial^2 f}{\partial x_2 \partial x_3} \\ \frac{\partial^2 f}{\partial x_3 \partial x_1} & \frac{\partial^2 f}{\partial x_3 \partial x_2} & \frac{\partial^2 f}{\partial x_3^2} \end{pmatrix} = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

From H find the leading principal matrices by eliminating:

1. The last n-1 rows and columns – $D_1 = (-2)$
2. The last n-2 rows and columns – $D_2 = \begin{pmatrix} -2 & 0 \\ 0 & -4 \end{pmatrix}$
3. The last 0 rows and columns – $D_3 = H$

1. Compute the determinants of these leading principal matrices.
   1. $|D_1| = -2,$
   2. $|D_2| = 8$
   3. $|H| = -16$

H is negative definite, then f is (strictly) concave

Summing up – two variable maximization

1. Differentiate f(x) and solve the first order conditions are:

$$\begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

2. Check concavity of f to see if the conditions represent a maximum.

   a. We compute the Hessian
   b. We check if it is negative definite
   c. i.e. check if, for all x_1 and x_2,

$$\frac{\partial^2 f}{\partial x_1^2} < 0 \quad \text{and} \quad \begin{vmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} \end{vmatrix} \; or \begin{vmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{vmatrix} = f_{11} f_{22} - f_{21} f_{12} > 0$$

3. If these conditions hold, H is negative definite, f is strictly concave and the stationary point is a maximum

4. If these conditions are violated by equality, i.e. are equal to zero, check the conditions for semi definiteness

$$\frac{\partial^2 f}{\partial x_1^2} \leq 0 \qquad \frac{\partial^2 f}{\partial x_2^2} \leq 0 \qquad \begin{vmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{vmatrix} = f_{11} f_{22} - f_{21} f_{12} \geq 0$$

5. If these conditions hold, H is negative semidefinite, f is concave and the stationary point is a maximum

6. If these conditions are violated, we need further investigation

Summing up – 3 variable maximization

1. Differentiate f(x) and solve the the first order conditions are:

$$\begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \\ \frac{\partial f}{\partial x_3} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

2. Check concavity of f to see if the conditions represent a maximum.

    a. We compute the Hessian

    b. We check if it is negative definite

b.We check if it is negative definite

$$\frac{\partial^2 f}{\partial x_1^2} < 0$$

$$\begin{vmatrix} \dfrac{\partial^2 f}{\partial x_1^2} & \dfrac{\partial^2 f}{\partial x_1 \partial x_2} \\ \dfrac{\partial^2 f}{\partial x_2 \partial x_1} & \dfrac{\partial^2 f}{\partial x_2^2} \end{vmatrix} or \begin{vmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{vmatrix} = f_{11} f_{22} - f_{21} f_{12} > 0$$

$$\begin{vmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{vmatrix} = f_{11} \begin{vmatrix} f_{22} & f_{23} \\ f_{32} & f_{33} \end{vmatrix} - f_{12} \begin{vmatrix} f_{21} & f_{23} \\ f_{31} & f_{33} \end{vmatrix} + f_{13} \begin{vmatrix} f_{21} & f_{22} \\ f_{31} & f_{32} \end{vmatrix} < 0$$

3. If these conditions hold, H is negative definite, f is strictly concave and the stationary point is a maximum
4. If these conditions are violated by equality, i.e. are equal to zero, check the conditions for semi definiteness

$$f_{11} \le 0, f_{22} \le 0\ f_{33} \le 0$$

$$\begin{vmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{vmatrix} \ge 0 \begin{vmatrix} f_{11} & f_{13} \\ f_{31} & f_{33} \end{vmatrix} \ge 0 \begin{vmatrix} f_{22} & f_{23} \\ f_{32} & f_{33} \end{vmatrix} \ge 0$$

$$\begin{vmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{vmatrix} \le 0$$

5. If these conditions hold, H is negative semidefinite, f is concave and the stationary point is a maximum
6. If these conditions are violated, we need further investigation

Important properties

Consider the problem

$$max_x \ f\ (x) \ subject \ to \ x \ \in S$$

and let x^* be its solution

1) x^* is the solution of the following problem:

$$max_x \ g(f\ (x)) \ subject \ to \ x \ \in S$$

where $g(.)$ is a no decreasing function

2) The following problem is equivalent

$$min_x \ -f\ (x) \ subject \ to \ x \ \in S$$