Choice under risk and uncertainty

Introduction

Up until now, we have thought of the objects that our decision makers are choosing as being physical items

However, we can also think of cases where the outcomes of the choices we make are uncertain - we don’t know exactly what will happen when we do a particular choice. For example:

- You are deciding whether or not to invest in a business
- You are deciding whether or not to go skiing next month
- You are deciding whether or not to buy a house that straddles the San Andreas fault line

Risk and Uncertainty

In each case the outcomes are uncertain.

Here we are going to think about how to model a decision maker who is making such choices.

Economists tend to differentiate between two different types of ways in which we may not know for certain what will happen in the future: ***risk*** and ***uncertainty*** (sometimes called ***ambiguity***).

Risk: the probabilities of different outcomes are known, ***Uncertainty***: the probabilities of different outcomes are unknown

Now we consider models of choice under risk,

An example of choice under risk

For an amount of money ***£ x***, you can flip a coin. If you get heads, you get ***£10***. If you get tails, you get ***£0***.

Assume there is a 50% chance of heads and a 50% chance of tails.

For what price ***x*** would you choose to play the game?

i.e. you have a choice between the following two options.

- 1. Not play the game and get nothing
- 2. Play the game, and get -x for sure, plus a 50% chance of getting $10.

Figure out the expected value (or average pay-out) of playing the game, and see if it is bigger than 0. If it is, then play the game, if not, then don't.

With a 50% chance you will get £10- x,

With a 50% chance you will get -x.

Thus, the average payoff is:

$$0.5(10 - x) + 0.5(- x) = 5 - x$$

Thus the value of the game is £5 - x.

you should play the game if the cost of playing is less than *£ 5*.

Lotteries (or prospects)

Decision making under risk can be considered as a process of choosing between different lotteries.

A lottery (or prospect) consists of a number of possible outcomes with their associated probability

It can be described as:

$$\boldsymbol{q} = (x_1, p_1; x_2, p_2; \dots x_n, p_n)$$

where

x_i represents the i^{th} outcome and

p_i is its associated probability, $p_i \in [0,1]\ \forall i$ and $\sum_i p_i = 1$.

In the example the choice is between:

$$\boldsymbol{r} = (10 - x, 0.5; -x, 0.5)$$

$$\boldsymbol{s} = (0, 1)$$

in this last case we omit probability and we can write $\boldsymbol{s} = (0)$.

When an outcomes is for sure (i.e. its probability is 1) we write only the outcome.

$\boldsymbol{s} = (x)$ means that the outcome x is for sure

Sometime we can omit the zero outcomes, so the lottery $\boldsymbol{r} = (10, 0.5; 5, 0.3; 0, 0.2)$ can be written as $\boldsymbol{r} = (10, 0.5; 5, 0.3)$

Compound lotteries

Lotteries can be combined

From the previous example:

suppose you have the following lottery of lotteries:

$$\boldsymbol{c} = \left(\boldsymbol{r}, \frac{1}{2}; \boldsymbol{s}, \frac{1}{2}\right)$$

where

$\boldsymbol{r} = (10 - x, 0.5; -x, 0.5)$ and

$\boldsymbol{s} = (0, 1)$.

Then, the resulting lottery is:

$$\boldsymbol{c} = \left(10 - x, \frac{1}{4};\ -x, \frac{1}{4};\ 0, \frac{1}{2}\right)$$

More in general

Consider the two following lotteries

$\boldsymbol{r} = (x_1, p_1;\ \ldots x_n, p_n)$ and

$\boldsymbol{s} = (y_1, q_1;\ \ldots y_n, q_n)$,

then

$$\boldsymbol{c} = (\boldsymbol{r}, a; \boldsymbol{s}, 1 - a)$$
$$=$$
$$(x_1, ap_1;\ \ldots x_n, ap_n; y_1, (1 - a)q_1;\ \ldots y_n, (1 - a)q_n)$$

Choice under risk: the axioms of von Neumann and Morgenstern (vNM)

These axioms are related to the axioms on preferences and impose rationality to the individual's behaviour when individuals face choices among lotteries.

$\succcurlyeq$ satisfies:

a. Completeness

For all lotteries $\boldsymbol{q}$ and $\boldsymbol{r}$ we have that $\boldsymbol{q} \succcurlyeq \boldsymbol{r}$ *or* $\boldsymbol{r} \succcurlyeq \boldsymbol{q}$ (or both)

b. Transitivity

For any three lotteries $\boldsymbol{q}, \boldsymbol{r}, \boldsymbol{s}$ if $\boldsymbol{q} \succcurlyeq \boldsymbol{r}$ and $\boldsymbol{r} \succcurlyeq \boldsymbol{s}$, then $\boldsymbol{q} \succcurlyeq \boldsymbol{s}$

c. Continuity

For any three lotteries $\boldsymbol{q}, \boldsymbol{r}, \boldsymbol{s}$ where $\boldsymbol{q} \succcurlyeq \boldsymbol{r}$ and $\boldsymbol{r} \succcurlyeq \boldsymbol{s}$, there exists some probability p such that there is indifference between the middle ranked prospect $\boldsymbol{r}$ and the prospect $(\boldsymbol{q}, p; \boldsymbol{s}, 1-p)$, i.e.

$$(\boldsymbol{q}, p; \boldsymbol{s}, 1-p) \sim \boldsymbol{r}$$

Equivalently

there exist $a, b \in [0, 1]$ such that:

$$(\boldsymbol{q}, a; \boldsymbol{s}, 1-a) \succcurlyeq \boldsymbol{r} \succcurlyeq (\boldsymbol{q}, b; \boldsymbol{s}, 1-b)$$

d. Independence

Any state of the world that results in the same outcome regardless of one's choice can be ignored or cancelled

For any three lotteries $\boldsymbol{q}, \boldsymbol{r}, \boldsymbol{s}$ and any $p \in [0, 1]$

if $\boldsymbol{q} \succcurlyeq \boldsymbol{r}$

then $(\boldsymbol{q}, p; \boldsymbol{s}, 1-p) \succcurlyeq (\boldsymbol{r}, p; \boldsymbol{s}, 1-p)$

Example

If $\boldsymbol{q} = (3000)$, $\boldsymbol{r} = (4000, 0.8)$ and $\boldsymbol{q} \succcurlyeq \boldsymbol{r}$

then

$\boldsymbol{q}' = (3000, 0.25)$, $\boldsymbol{r}' = (4000, 0.2)$ and $\boldsymbol{q}' \succcurlyeq \boldsymbol{r}'$

Note that:

prospect $\boldsymbol{q}'$ is the compound lottery $\boldsymbol{q}' = (\boldsymbol{q}, 0.25; 0, 0.75\ )$ and

prospects $\boldsymbol{r}'$ is the compound lottery $\boldsymbol{r}' = (\boldsymbol{r}, 0.25; 0, 0.75\ )$

Directly related to independence is the axiom of betweenness.

If $q \succcurlyeq r$ then for any $a \in [0,1]$:

$$q \succcurlyeq (q, a; r, 1 - a)$$
$$r \preccurlyeq (q, a; r, 1 - a)$$

e. Monotonicity

a gamble which assigns a higher probability to a preferred outcome will be preferred to one which assigns a lower probability to a preferred outcome (as long as the other outcomes in the gambles remain unchanged)

Concept od stochastically dominance.

Stochastic dominance

Consider the following two lotteries:

$\boldsymbol{q} = (10, 0.01;\ 15, 0.02;\ 30, 0.01; 45, 0.06)$

$\boldsymbol{r} = (15, 0.03; 45, 0.07)$

r *dominates* ***q***, it is clear rewriting ***r*** as:

$\boldsymbol{r} = (10, 0.00;\ 15, 0.03; 30, 0.00;\ 45, 0.07)$

Consider two prospects ***q*** and ***r***

Let $x_1, x_2, \dots x_n$ the outcomes in ***q*** and ***r,*** ordered from the worst to the best.

Let be:

p_{qi} the probability of outcome *i* in prospect q

p_{ri} the probability of outcome *i* in prospect r

We say that prospect ***q*** stochastically dominates prospect ***r*** if:

$$\sum_{i=1}^{x} p_{qi} \leq \sum_{i=1}^{x} p_{ri} \ \forall x \in \{1, \dots, n\}$$

with strict inequality for at least one x

Expected Value

The expected value of prospect $\boldsymbol{r} = (x_1, p_1; \ldots x_n, p_n)$ is

$$E(\boldsymbol{r}) = \sum_i p_i \cdot x_i$$

Example

$\boldsymbol{r} = (1000, 0.25; 500, 0.75)$ and $u(x_i) = \sqrt{x_i}$

$E(\boldsymbol{r}) = 0.25 \cdot 1000 + 0.75 \cdot 500$

St. Petersburg paradox

- A fair coin is tossed repeatedly until a tail appears, ending the game.
- The pot starts at 2 dollars and is doubled every time a head appears.
- Prize is whatever is in the pot after the game ends:
  - 2 dollars if a tail appears on the first toss,
  - 4 dollars if a head appears on the first toss and a tail on the second,
  - 8 dollars if a head appears on the first two tosses and a tail on the third,
  - 16 dollars if a head appears on the first three tosses and a tail on the fourth, etc.
  - 2^k dollars if the coin is tossed k times until the first tail appears.

The expected value is ∞:

$$2 \cdot \frac{1}{2} + 4 \cdot \frac{1}{4} + 8 \cdot \frac{1}{8} + \cdots .. =$$

$$= \sum_{i=1}^{\infty} 2^i \cdot \frac{1}{2^i} =$$

$$= 1 + 1 + 1 + \cdots .. = \infty$$

The experimental evidence is that people are willing to pay only limited amount of money to play this lottery

Solution: the value that people attach to the first dollar of their wealth is larger tat the value they attach to the i^{th} dollar they earn.

A decreasing marginal value can explain this paradox

Expected Utility and Expected Value

The expected utility of a prospect $\boldsymbol{r} = (x_1, p_1;\ \ldots x_n, p_n)$ is given by:

$$U(\boldsymbol{r}) = \sum_i p_i \cdot u(x_i)$$

Example

$\boldsymbol{r} = (1000, 0.25; 500, 0.75)$ and $u(x_i) = \sqrt{x_i}$

$U(\boldsymbol{r}) = 0.25\sqrt{1000} + 0.75\sqrt{500}$

St.Petersburg paradox when $u(x_i) = \sqrt{x_i}$

$$\sum_{i=1}^{\infty} 2^{\frac{i}{2}} \cdot \frac{1}{2^i} = \sum_{i=1}^{\infty} \left(\frac{1}{\sqrt{2}}\right)^i = \frac{1}{\sqrt{2} - 1} = 2.41$$

Representation theorem

Let be X the set of all possible lotteries.

A binary relation $\succcurlyeq$ satisfies vNM axioms if and only if there exists a function $u: X \rightarrow R$ such that:

$$q \succcurlyeq r$$

if and only if

$$u(q) \geq u(r)$$

Further assumptions

1) Asset integration

a prospect is acceptable if and only if the utility resulting from integrating the prospect with one's assets exceeds the utility of those assets alone.

Lottery $\boldsymbol{r} = (x_1, p_1;\ \ldots x_n, p_n)$ is acceptable at asset position *w* if

$$U(x_1 + w, p_1;\ \ldots x_n + w, p_n) \geq U(w)$$

2. Risk aversion

a person is said to be risk averse if he prefers the certain prospect (x) to any risky prospect with expected value equal to x.

Risk aversion is caused by the concavity in the utility function

More in general we can talk of *Risk Attitudes*

Risk Attitudes

A decision maker is ***risk neutral*** if he is indifferent between receiving a lottery's expected value and playing the lottery.

Consider $\boldsymbol{r} = (x_1, p_1;\ \ldots x_n, p_n)$ then:

$$u\left(\sum_i p_i \cdot u(x_i)\right) = \sum_i p_i \cdot u(x_i)$$

A decision maker is risk neutral if its utility function is linear, i.e. $u(x) = a + b\,x$

A decision maker is ***risk averse*** if he prefers receiving the lottery's expected value instead of playing the lottery.

Consider $\boldsymbol{r} = (x_1, p_1;\ \ldots x_n, p_n)$ then:

$$u\left(\sum_i p_i \cdot u(x_i)\right) > \sum_i p_i \cdot u(x_i)$$

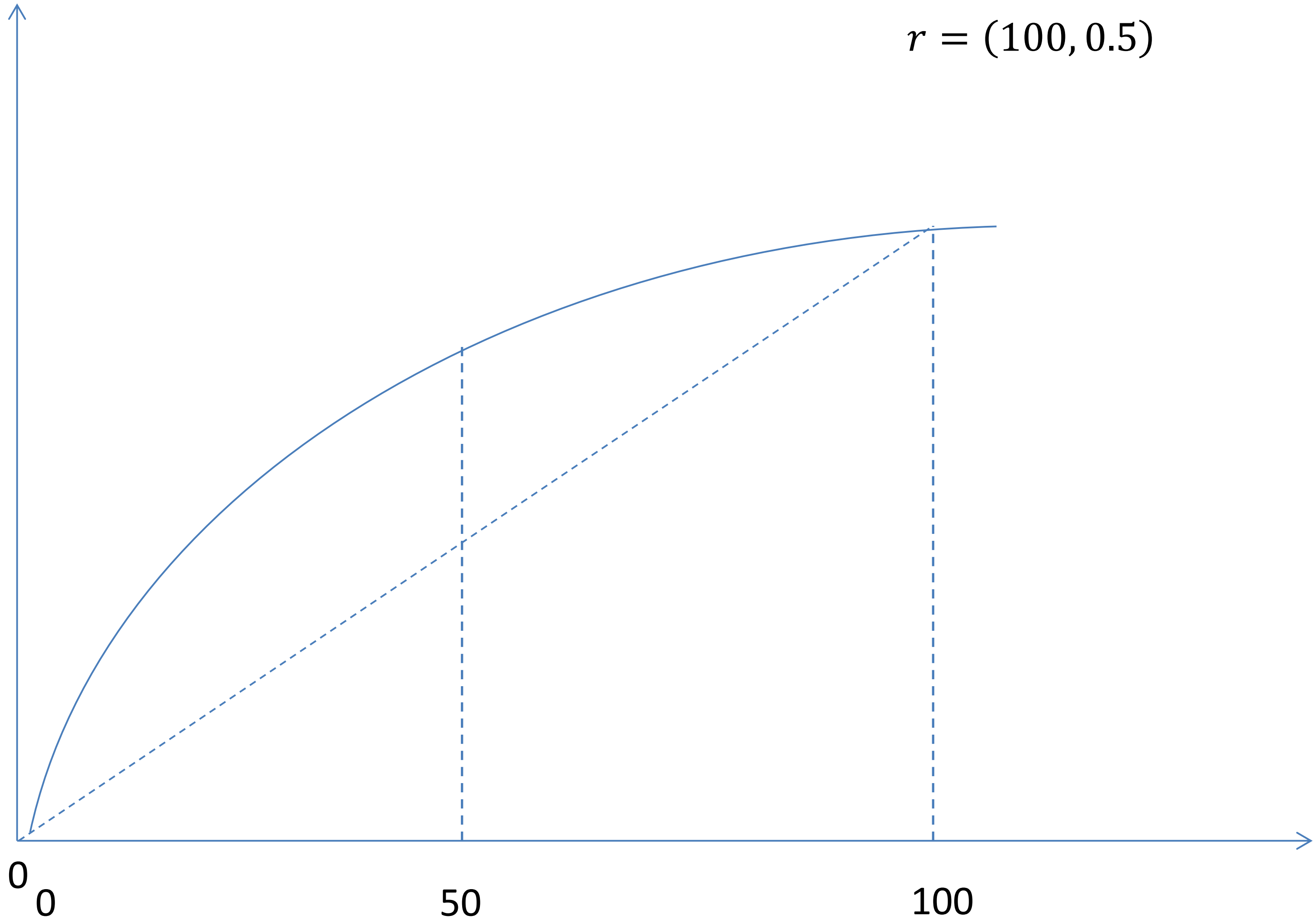
A decision maker is risk averse if its utility function is strictly concave, i.e. $u''(x) < 0$

A decision maker is ***risk seeking*** if he prefers playing the lottery instead of receiving its expected value.

Consider $\boldsymbol{r} = (x_1, p_1;\ \dots x_n, p_n)$ then:

$$u\left(\sum_i p_i \cdot u(x_i)\right) < \sum_i p_i \cdot u(x_i)$$

A decision maker is risk seeking if its utility function is strictly convex, i.e. $u"(x) > 0$


$r = (100, 0.5)$
0
0
50
100

All these results are proved by Jensen's Inequality

Let x be a random variable where $E(x)$ is its expected value and $f(x)$ is a concave function then:

$$f\big(E(x)\big) \geq E\big(f(x)\big)$$

$f(x)$ is a convex function then:

$$f\big(E(x)\big) \leq E\big(f(x)\big)$$

Measures of risk aversion

For of a lottery $\boldsymbol{q}$, the risk premium $R(\boldsymbol{q})$ is defined as

$$R(\boldsymbol{q}) = E(\boldsymbol{q}) - CE(\boldsymbol{q})$$

where $CE(\boldsymbol{q})$ is the *certainty equivalent wealth* defined as

$$U\big(CE(\boldsymbol{q})\big) = U(\boldsymbol{q})$$

Interpretation:

the risk premium $R(\boldsymbol{q})$ is the amount of money that an agent is willing to pay to avoid a lottery.

Example.

Person A has to “play” the following lottery $\boldsymbol{q} = (100, 0.5; 64, 0.5)$. Assume that his utility function is $u(x) = \sqrt{x}$

Compute the risk premium.

$$U\big(CE(\boldsymbol{q})\big) = U(\boldsymbol{q}) \rightarrow \sqrt{CE(\boldsymbol{q})} = 0.5\sqrt{100} + 0.5\sqrt{64} \rightarrow$$
$$CE(\boldsymbol{q}) = 81$$
$$R(\boldsymbol{q}) = E(\boldsymbol{q}) - CE(\boldsymbol{q}) = 100 \cdot 0.5 + 64 \cdot 0.5 - 81 = 1$$

Person B utility function is $u(x) = x$. He proposes to person A to buy the lottery. Which is the minimum price that person A will accept?

Answer: 81

Is convenient for person B?

Answer: yes

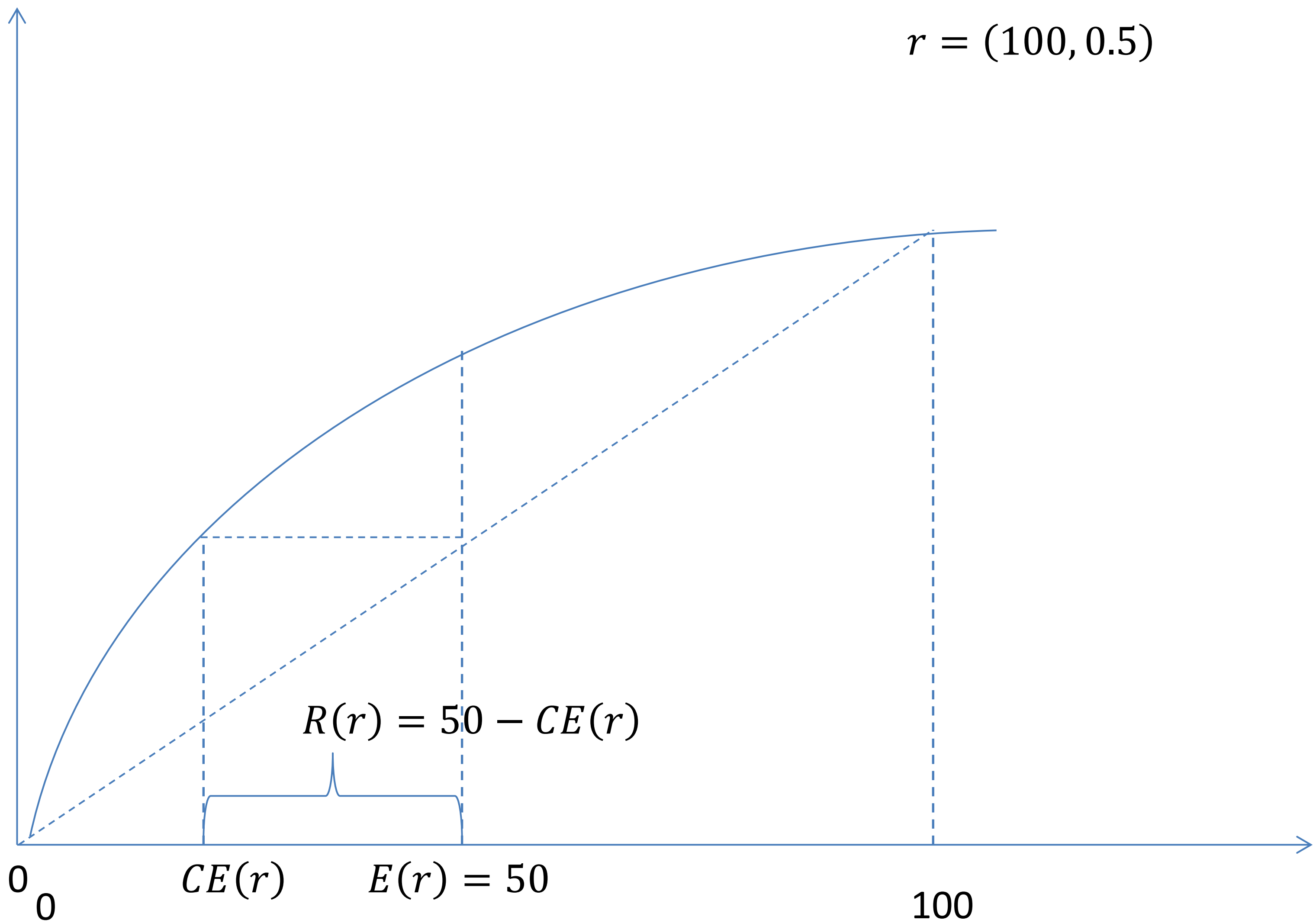
Person A has to “play” the following lottery $\boldsymbol{q} = (100, 0.5; 64, 0.5)$. Assume that his utility function is $u(x) = \sqrt{x}$

We have computed that $R(\boldsymbol{q}) = 1$

Selling the lottery at p=81 is equivalent to hold the lottery and pay 19 when lottery’s outcome is 100 and to receive 17 when lottery’s outcome is 64.

In expected terms Person A pays 1

$(-19 \cdot 0.5 + 17 \cdot 0.5 = -1)$


$r = (100, 0.5)$
$R(r) = 50 - CE(r)$
0
0
$CE(r)$
$E(r) = 50$
100

1. Arrow-Pratt measure of absolute risk-aversion:

$$A(c) = -\frac{u''(c)}{u'(c)}$$

2. Arrow-Pratt-De Finetti measure of relative risk-aversion or coefficient of relative risk aversio*n*

$$R(c) = -\frac{c \cdot u''(c)}{u'(c)}$$

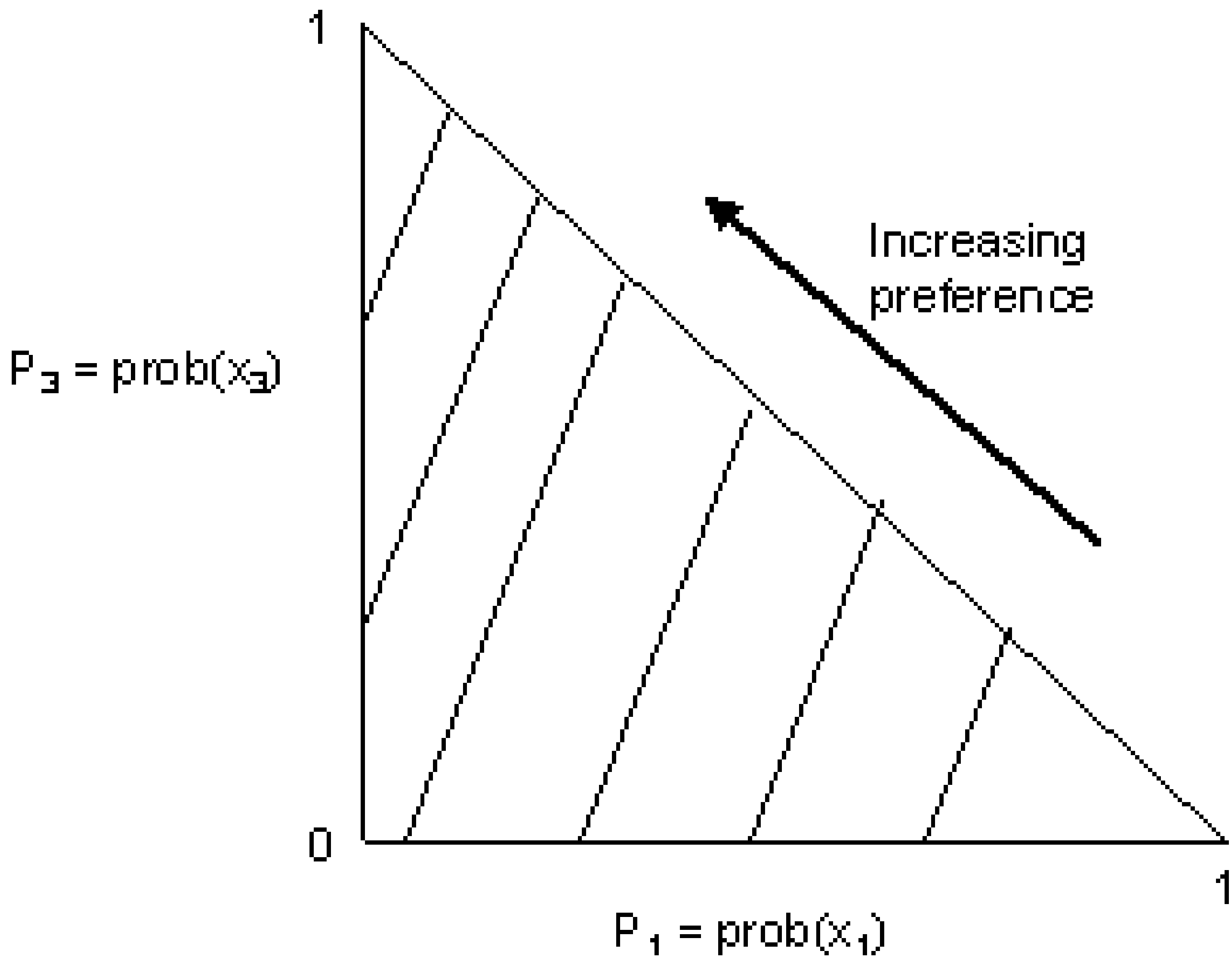
Type of Risk-Aversion	Example of utility functions
Increasing absolute risk-aversion	$u(w) = w^{-c \cdot w^2}$
Constant absolute risk-aversion	$u(w) = -e^{-c \cdot w}$
Decreasing absolute risk-aversion	$u(w) = \ln w$

Type of Risk-Aversion	Example of utility functions
Increasing relative risk-aversion	$w - cw^2$
Constant relative risk-aversion	$\ln(w)$
Decreasing relative risk-aversion	$-e^{2 \cdot w^{-\frac{1}{2}}}$

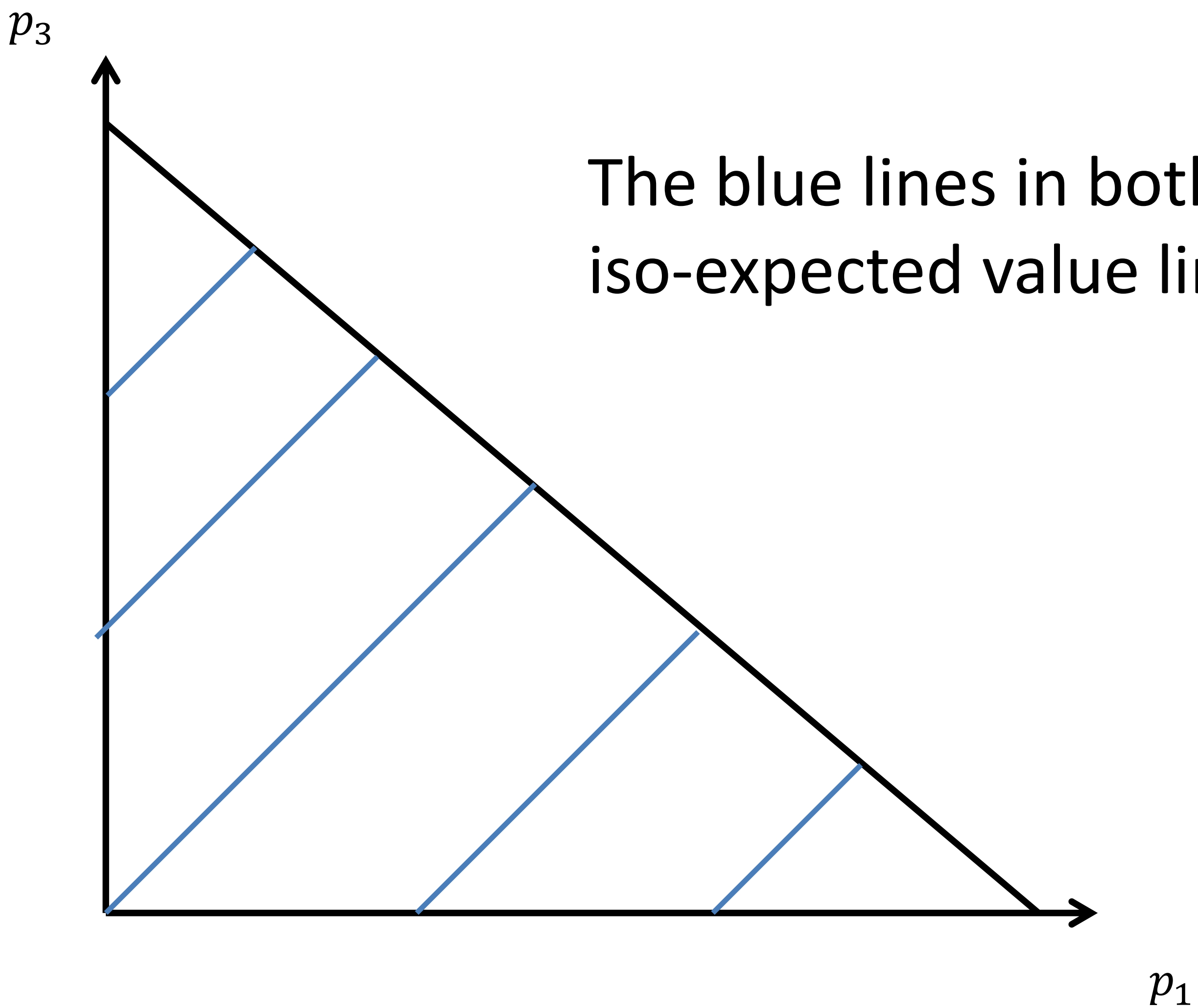
The Machina triangle

- two-dimensional representation
- 3 possible outcomes, x_1, x_2, and x_3, and $x_3 \succcurlyeq x_2 \succcurlyeq x_1$.
- they occur with probabilities p_1, p_2, and p_3 respectively, where $\sum_i p_i = 1$
- since $p_2 = 1 - p_1 - p_3$, we can represent these lotteries by points in a unit triangle in the (p_1, p_3) plane, known as the Machina triangle
- Example
  - $r = (x_1, p_1, x_2, p_2; x_3, p_3,)$
  - $U(r) = p_1 \cdot u(x_1) + p_2 \cdot u(x_2) + p_3 \cdot u(x_3)$
  - $U(r) = p_1 \cdot u(x_1) + (1 - p_1 - p_3) \cdot u(x_2) + p_3 \cdot u(x_3)$

- Example
  - $r = (x_1, p_1, x_2, p_2; x_3, p_3,)$
  - $U(r) = p_1 \cdot u(x_1) + p_2 \cdot u(x_2) + p_3 \cdot u(x_3)$
  - Replace $p_2 = 1 - p_1 - p_3$
  - $U(r) = p_1 \cdot u(x_1) + (1 - p_1 - p_3) \cdot u(x_2) + p_3 \cdot u(x_3)$
  - Hold the utility constant at a level $\overline{U}$ and solve by p_3
  - $p_3 = \frac{\overline{U} - u(x_2)}{u(x_3) - u(x_2)} + \frac{\left(u(x_2) - u(x_1)\right)}{u(x_3) - u(x_2)} \cdot \ p_1$
  - Slope is positive, intercept could be either positive or negative

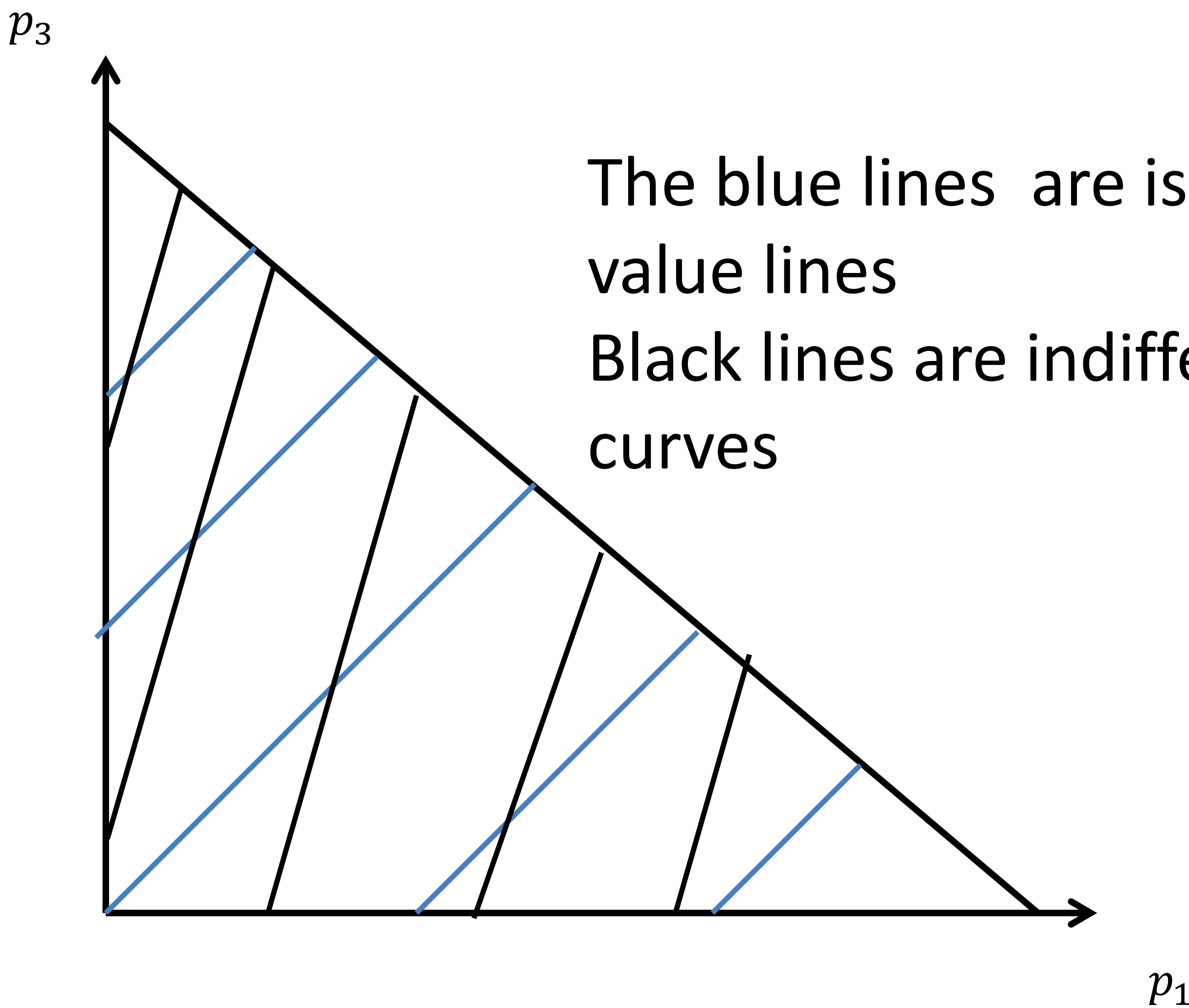

1
Increasing
preference
P3 = prob(x3)
0
1
P1 = prob(x1)

Representing risk attitudes using indifference curves



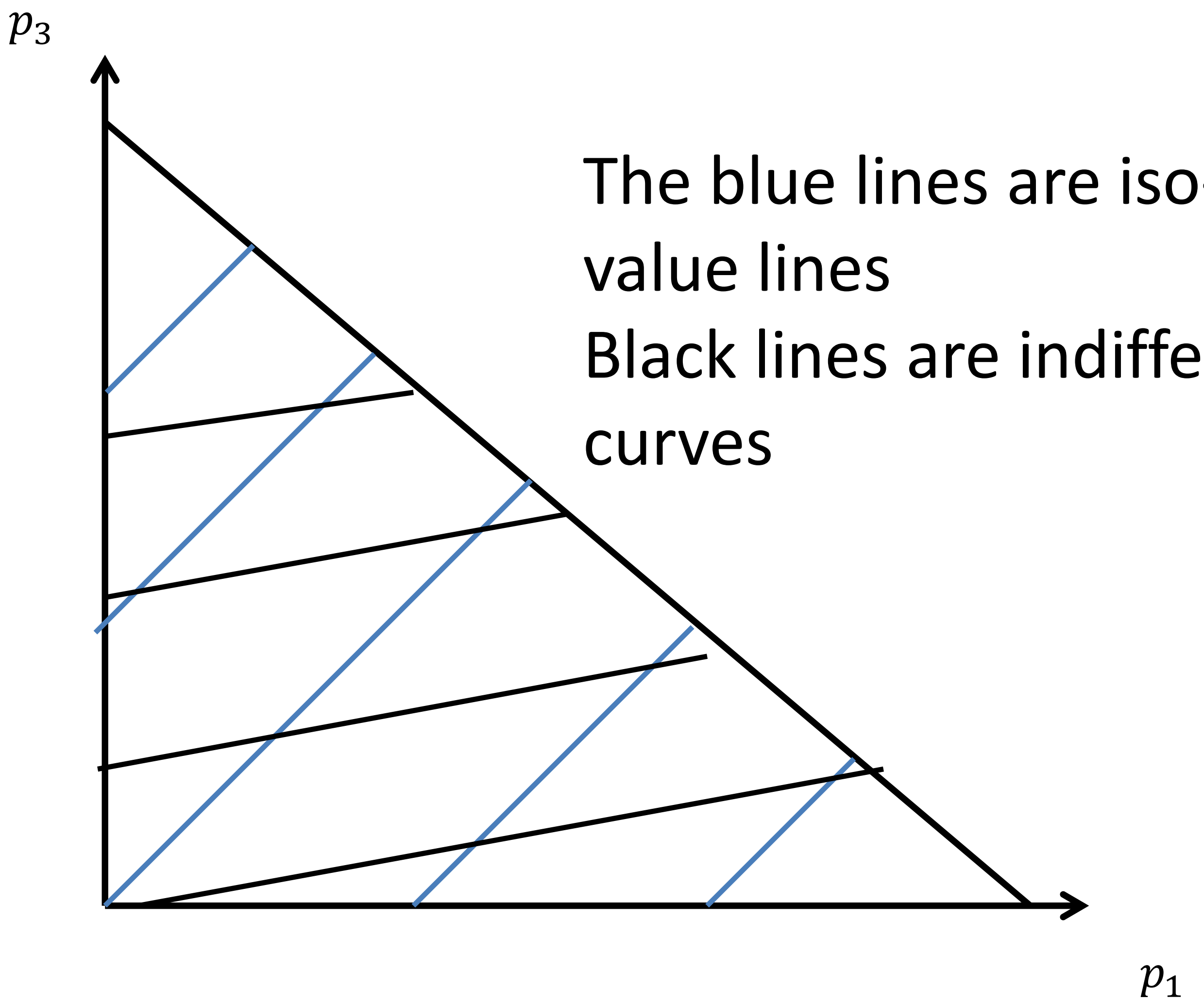
The blue lines in both panels are iso-expected value lines

Risk averse



The blue lines are iso-expected value lines
Black lines are indifference curves

Risk seeking



The blue lines are iso-expected value lines

Black lines are indifference curves