2. Check if the following matrix is positive/negative semi-definite

a. $\begin{matrix} -4 & 0 & 1 \\ 1 & -2 & 0 \\ 0 & 2 & 1 \end{matrix}$

$|D_1| = -4 < 0$

$|D_2| = (-4)(-2) = 8 > 0$

$|D_3| = (-4)(-2)\cdot 1 + 1\cdot 1\cdot 2 = 10 > 0$

CONDITIONS FOR POSITIVE DEFINITENESS ARE STRICTLY VIOLATED

CONDITIONS FOR NEGATIVE DEFINITENESS ARE STRICTLY VIOLATED

THEN THE MATRIX IS INDEFINITE

3. Check if the following functions are concave/convex

   a. $\sqrt{xy}$

   b. $\ln x + \ln y$

a) JACOBIAN IS $J = \begin{pmatrix} \frac{1}{2}\sqrt{\frac{y}{x}} \\ \frac{1}{2}\sqrt{\frac{x}{y}} \end{pmatrix}$

the Hessian is $H = \begin{pmatrix} -\frac{1}{4} y^{\frac{1}{2}} x^{-\frac{3}{2}} & \frac{1}{4}\frac{1}{\sqrt{xy}} \\ \frac{1}{4\sqrt{xy}} & -\frac{1}{4} x^{\frac{1}{2}} y^{-\frac{3}{2}} \end{pmatrix}$

Consider $x, y \geq 0$. there are Two principal minors $|D_{1,1}|$ $|D_{1,2}|$

$|D_{11}| = -\frac{1}{4} y^{\frac{1}{2}} x^{-\frac{3}{2}} \leq 0$ $\quad |D_{12}| = -\frac{1}{2} x^{\frac{1}{2}} y^{-\frac{3}{2}} \leq 0 \quad \forall\, x y \geq 0$

$|D_2| = \frac{1}{16}\frac{1}{xy} - \frac{1}{16} \cdot \frac{1}{xy} = 0$

Hessian is negative semidefinite, Then the function is concave

b) $J = \begin{pmatrix} \frac{1}{x} \\ \frac{1}{y} \end{pmatrix}$ $\quad H = \begin{pmatrix} -\frac{1}{x^2} & 0 \\ 0 & -\frac{1}{y^2} \end{pmatrix}$

note that $x, y > 0$

$|D_1| = -\frac{1}{x^2} < 0$ $\quad |D_2| = \frac{1}{y^2 x^2} > 0 \quad \forall\, x, y > 0$

Hessian is negative definite, then the function is (strictly) concave

4. Find the all stationary points of the following functions. Then check the second order conditions and state if they are local/global maximizers/minimizers
   a. $x^3 - 12x + y^3 - 27y + z^3 - 3z$
   b. $-y^2 - 2x^2 + xy$
   c. $-y^2 - 2x^2 + 2xy$
   d. $e^x - xy$
   e. $\ln x + \ln y + \ln z - 4x - 5y - 6z$ for $z, x, y > 0$

a)
$$\begin{cases} 3x^2 - 12 = 0 & \rightarrow & x = 2 \quad x = -2 \\ 3y^2 - 27 = 0 & \rightarrow & y = 3 \quad y = -3 \\ 3z^2 - 3 = 0 & \rightarrow & z = 1 \quad z = -1 \end{cases}$$

STATIONARY POINTS ARE (x, y, z):
$(2,3,1)$ $(2,3-1)$ $(2,-3,1)$ $(-2,3,1)$ $(-2,-3,1)$ $(-2,3,-1)$ $(2,-3,-1)$ $(-2,-3,-1)$

$$H = \begin{pmatrix} 6x & 0 & 0 \\ 0 & 6y & 0 \\ 0 & 0 & 6z \end{pmatrix} \quad |D_1| = 6x \quad |D_2| = 36xy \quad |D_3| = 216xyz$$

GIVEN THAT x, y, z CAN BE ANY REAL NUMBER, THE SIGN OF THE PRINCIPAL LEADING MINORS CAN BE STRICTLY NEGATIVE AS STRICTLY POSITIVE. THEN WE CANNOT SAY THE FUNCTION IS CONCAVE OR CONVEX FOR ALL POSSIBLE VALUES OF x, y, z.
THEN WE CANNOT SAY THAT STATIONARY POINTS ARE GLOBAL MAX OR MIN.

NOW WE CHECK THE DEFINITENESS OF H COMPUTED FOR EACH STATIONARY POINT.

$(2,3,1) \rightarrow |D_1| > 0 \quad |D_2| > 0 \quad |D_3| > 0 \rightarrow$ LOCAL MIN
$(2,3,-1) \rightarrow |D_1| > 0 \quad |D_2| > 0 \quad |D_3| < 0 \rightarrow$ NO MAX NO MIN
$(2,-3,1) \rightarrow |D_1| > 0 \quad |D_2| < 0 \quad |D_3| < 0 \rightarrow$ NO MAX NO MIN
$(-2,3,1) \rightarrow |D_1| < 0 \quad |D_2| < 0 \quad |D_3| < 0 \rightarrow$ NO MAX NO MIN
$(-2,-3,1) \rightarrow |D_1| < 0 \quad |D_2| > 0 \quad |D_3| > 0 \rightarrow$ NO MAX NO MIN
$(-2,3,-1) \rightarrow |D_1| < 0 \quad |D_2| < 0 \quad |D_3| > 0 \rightarrow$ NO MAX NO MIN
$(2,-3,-1) \rightarrow |D_1| > 0 \quad |D_2| < 0 \quad |D_3| > 0 \rightarrow$ NO MAX NO MIN
$(-2,-3,-1) \rightarrow |D_1| < 0 \quad |D_2| > 0 \quad |D_3| < 0 \rightarrow$ LOCAL MAX

b) $-y^2-2x^2+xy$

$J=\begin{pmatrix}-4x+y\\-2y+x\end{pmatrix}$

$\begin{cases}-4X+Y=0\\-2Y+X=0\end{cases}$ $\quad Y=4X \quad -7X=0 \rightarrow X=0 \rightarrow Y=0$

STATIONARY POINT $(x,y)=(0,0)$

$H=\begin{pmatrix}-4 & 1\\ 1 & -2\end{pmatrix}$

$|D_1|=-4<0 \quad |D_2|=(-4)(-2)-1=7>0$

FUNCTION IS CONCAVE, THEN THE STATIONARY POINT IS A LOCAL AND GLOBAL MAX

c) $-y^2-2x^2+2xy$

$\begin{cases}-4x+2y=0\\-2y+2x=0\end{cases} \rightarrow x=y$

$-4y+2y=0 \rightarrow y=0 \rightarrow x=0$

$H=\begin{bmatrix}-4 & +2\\ +2 & -2\end{bmatrix}$

$|D_1|=-4<0 \quad |D_2|=8-4=4>0$

$(x,y)=(0,0)$ IT IS A LOCAL AND GLOBAL MAXIMUM

d) $e^x - xy$

$$\begin{cases} e^x - y = 0 \\ -x = 0 \rightarrow \quad x = 0 \qquad y = 1 \end{cases}$$

$(x, y) \rightarrow (0, 1)$ is the unique stationary point

$$H = \begin{bmatrix} e^x & -1 \\ -1 & 0 \end{bmatrix}$$

$|D_1| = e^x > 0 \quad \forall x$

$|D_2| = -1 < 0$

THEN THE HESSIAN IS INDEFINITE, THE FUNCTION IS NOT CONVEX NOT CONCAVE THEN THE STATIONARY POINT IS NOT GLOBAL MIN OR MAX

COMPUTING THE HISSIAN AT THE STATIONARY POINT WE GET

$$H = \begin{bmatrix} 1 & -1 \\ -1 & 0 \end{bmatrix}$$

AGAIN IT IS INDEFINITE

THEN $(0, 1)$ IS NOT A LOCAL MAX OR A LOCAL MIN

e) $\ln x + \ln y + \ln z - 4x - 5y - 6z$ NOTE IT MUST BE: $x, y, z > 0$

$\frac{1}{x} - 4 = 0 \rightarrow x = \frac{1}{4}$

$\frac{1}{y} - 5 = 0 \rightarrow y = \frac{1}{5}$

$\frac{1}{z} - 6 = 0 \rightarrow z = \frac{1}{6}$

$$H = \begin{bmatrix} -\frac{1}{x^2} & 0 & 0 \\ 0 & -\frac{1}{y^2} & 0 \\ 0 & 0 & -\frac{1}{z^2} \end{bmatrix}$$

$D_1 = -\frac{1}{x^2} < 0 \quad D_2 = \frac{1}{x^2}\frac{1}{y^2} > 0$

$\forall x, y, z > 0$

$D_3 = -\frac{1}{x^2y^2z^2} < 0$

$\Longrightarrow$ concave function

$(x, y, z) = \left(\frac{1}{4}, \frac{1}{5}, \frac{1}{6}\right)$

is a local / global maximum

5. Suppose that a firm that uses 2 inputs has the production function $f(x, y) = 12x^{\frac{1}{3}}y^{\frac{1}{2}}$ and faces the input prices (p_x, p_y) and the output price q.
   a. Show that f is concave, so that the firm's profit is concave.
   b. Find a global maximum of the firm's profit (and give the input combination that achieves this maximum).

a)

$$J = \begin{pmatrix} 4x^{-\frac{2}{3}} y^{\frac{1}{2}} \\ 6x^{\frac{1}{3}} y^{-\frac{1}{2}} \end{pmatrix} \qquad H = \begin{pmatrix} -\frac{8}{3}x^{-\frac{5}{3}} y^{\frac{1}{2}} & 2x^{-\frac{2}{3}} y^{-\frac{1}{2}} \\ 2x^{-\frac{2}{3}} y^{-\frac{1}{2}} & -3x^{\frac{1}{3}} y^{-\frac{3}{2}} \end{pmatrix}$$

THE PRINCIPAL MINORS OF ORDER 1 ARE

$$|D_{1,1}| = -\frac{8}{3}x^{-\frac{5}{3}} y^{\frac{1}{2}} \leq 0 \qquad |D_{2,2}| = -3x^{\frac{1}{3}} y^{-\frac{3}{2}} \leq 0 \quad \forall x, y \geq 0$$

$$|D_2| = 4x^{-\frac{4}{3}} y^{-1} \geq 0 \quad \forall x, y \geq 0$$

NOTE WE CONSIDER ONLY $x, y \geq 0$ AS INPUTS CANNOT BE NEGATIVE

THEN THE FUNCTION $f(x, y)$ IS CONCAVE

b) THE PROFIT FUNCTION IS:

$$\Pi = q\, 12 x^{\frac{1}{3}} y^{\frac{1}{2}} - P_x x - P_y y$$

$$\begin{cases} 4x^{-\frac{2}{3}} y^{\frac{1}{2}} - P_x = 0 \\ 6x^{\frac{1}{3}} y^{-\frac{1}{2}} - P_y = 0 \end{cases}$$

THE SOLUTION OF THE FIRST ORDER CONDITIONS GIVES US THE GLOBAL MAXIMUM BECAUSE THE PROFIT FUNCTION IS CONCAVE

$$\begin{cases} x = \left(\dfrac{24q^2}{P_x P_y}\right)^3 \\[2ex] y = \dfrac{P_x^2}{16q^2}\left(\dfrac{24q^2}{P_x P_y}\right)^4 \end{cases}$$

IT IS THE GLOBAL MAX

Problem set 4

1. Read the section 3.4 Quasiconcavity and quasiconvexity in the math tutorial of Martin Osborne (http://www.economics.utoronto.ca/osborne/MathTutorial/QCCF.HTM) and chck quasiconcavity for the following functions:
   a. $y + \ln x \; for \; x, y > 0$
   b. $4xy + 4x \; for \; x, y < 0$

a) THE HESSIAN IS $H = \begin{pmatrix} -\frac{1}{x^2} & 0 \\ 0 & 0 \end{pmatrix}$

THE BORDERED HESSIAN IS $H_b = \begin{pmatrix} 0 & \frac{1}{x} & 1 \\ \frac{1}{x} & -\frac{1}{x^2} & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$D_1 = -\frac{1}{x^2} < 0 \; \forall x \quad D_2 = \frac{1}{x^2} > 0 \; \forall x$

$\rightarrow$ quasiconcave

b) THE HESSIAN IS $H = \begin{pmatrix} 0 & 4 \\ 4 & 0 \end{pmatrix}$

$$H_b = \begin{pmatrix} 0 & 4(y+1) & 4x \\ 4(y+1) & 0 & 4 \\ 4x & 4 & 0 \end{pmatrix}$$

$D_1 = -16(y+1)^2 < 0 \quad \forall x$

$D_2 = 128(y+1)x$ that is > 0 if $y < -1$, < 0 if $y > -1$

$\rightarrow$ no quasiconcave no quasiconvex