Solutions Problem set 5

1. Solve the following problem
$$\max x^2 + y^2 + z^2 \quad s.t. \;\; x + y + z = 10$$
Say if the stationary point(s) is (are) global max/min

Solution

The Lagrangean is
$$\mathrm{L} = x^2 + y^2 + z^2 - \lambda(x + y + z - 10)$$
the stationary points are given by the solution ofthe following system of equation:
$$\begin{cases} 2x - \lambda = 0 \\ 2y - \lambda = 0 \\ 2z - \lambda = 0 \\ x + y + z = 10 \end{cases}$$
The solution is
$$x = y = z = \frac{10}{3}$$
$$\lambda = \frac{20}{3}$$
To check concavity/convexity of the Lagrangean is sufficient to check the objective function because the constraint is linear.

The Hessian of the objective function is:
$$H = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

The leading principal minors are:
$$D_1 = 2, D_2 = \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 4, D_3 = \begin{vmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{vmatrix} = 8$$
Then all leading principal minors are strictly convex, then the objective function is strictly convex. As a consequence the Lagrangean is convex and the stationary point is a global minimum.

2. A consumer is characterized by an utility function $U(x,y) = x + \ln y$. He faces prices $p_x = 1$ and $p_y > 1$. Moreover he faces a budget constraints of $B > 0$. Note the function is defined for all $x \geq 0$ and $y > 0$.
    a. Find the values of x and y that maximize the consumer's utility
    b. Compute the value of relaxing constrain B
    c. Compute the effect on the consumer of an increases of p_y

Solution

The consumer problem is:

$$\max_{\{x,y\}} x + \ln y \quad s.t. \quad p_y\, y + x = B$$

a. Lagrangean is

$$L = x + \ln y - \lambda\,(p_y\, y + x - B)$$

the stationary points are given by the solution ofthe following system of equation:

$$\begin{cases} 1 - \lambda = 0 \\ \dfrac{1}{y} - \lambda p_y = 0 \\ p_y\, y + x - B = 0 \end{cases}$$

From the first we have $\lambda = 1$.
Then replacing $\lambda = 1$ in the second we have that $y = \frac{1}{p_y}$.
Finally replacing y in the last we have that $x = B - 1$
Is this stationary point a maximum? Given the constraint is linear, it depends if the objective function is concave.
The Hessian of the objective function is:

$$H = \begin{pmatrix} 0 & 0 \\ 0 & -\dfrac{1}{y^2} \end{pmatrix}$$

The two principal minors of order 1 are 0 and $-\frac{1}{y^2}$.
The principal minor of order 2 is equal to 0.
Then H is semidefinite negative, so the objective function is concave. We can conclude that the Lagrangean is concave so that the stationary point is a global maximum.

b. The value of objective function in the maximum is:
$U(x^*, y^*) = B - 1 + \ln\frac{1}{p_y} = B - 1 - \ln p_y$
The value of relaxing constraint B is the derivative of $U(x^*, y^*)$ with respect to B

$$\frac{d\, U(x^*, y^*)}{d\, B} = 1$$

c. The effect of an increases of p_y is the derivative of $U(x^*, y^*)$ with respect to p_y:

$$\frac{d\, U(x^*, y^*)}{d\, p_y} = -\frac{1}{p_y}$$

3. Consider the following problem

$$\max_{\{K,L\}} aK + \ln L \quad s.t.\ K + bL = M$$
$$where\ a > 0, b > 0, M > 0$$

a. Write the Lagrangian
b. Write the first order conditions
c. Find the quantities of K and L that satisfy the first order conditions
d. Write the bordered Hessian matrix
e. Prove that this result is a local maximum (second order condition)
f. Prove that this result is a global maximum
g. Find the marginal effect of a change of parameter b on the maximized value
h. Find the marginal effect of a change of parameter *a* on the maximized value
i. Find the marginal effect of a change of parameter M on the maximized value

Solution

a. $L = aK + \ln L - \lambda(K + bL - M)$

b. $\begin{cases} a - \lambda = 0 \\ \frac{1}{L} - \lambda b = 0 \\ K + bL = M \end{cases}$

c. $\begin{cases} \lambda = a \\ L = \frac{1}{ab} \\ K = M - \frac{1}{a} \end{cases}$

d. The bordered Hessian of the Lagrangean is: $H_b = \begin{pmatrix} 0 & a-\lambda & \frac{1}{L} - \lambda b \\ a - \lambda & 0 & 0 \\ \frac{1}{L} - \lambda b & 0 & -\frac{1}{L^2} \end{pmatrix}$

e. The bordered Hessian in the stationary point is:

$$H_b = \begin{pmatrix} 0 & 0 & \frac{1}{ab} - ab \\ 0 & 0 & 0 \\ \frac{1}{ab} - ab & 0 & -\frac{1}{(ab)^2} \end{pmatrix}$$

Its determinant is equal to 0, then sufficient conditions for a local max/min are not satisfied. Note that are not satisfied conditions that are sufficient (not necessary). Then it could be a local max or min.

f. The constrained set is convex, then for a global maximum it is enough that the function $aK + \ln L$ is quasiconvex

The bordered Hessian of $aK + \ln L$ is: $H_b = \begin{pmatrix} 0 & a & \frac{1}{L} \\ a & 0 & 0 \\ \frac{1}{L} & 0 & -\frac{1}{L^2} \end{pmatrix}$

$$B_1 = \begin{vmatrix} 0 & a \\ a & 0 \end{vmatrix} = -a^2 < 0$$

$$B_2 = \begin{vmatrix} 0 & a & \frac{1}{L} \\ a & 0 & 0 \\ \frac{1}{L} & 0 & -\frac{1}{L^2} \end{vmatrix} = \frac{a^2}{L^2} > 0$$

Then the objective function is strictly quasi concave, so the stationary point is a global maximum.

g. The value of the objective function in the maximum is

$$f(K^*, L^*) = aM - 1 + \ln\frac{1}{ab} = aM - 1 - \ln ab$$

The marginal effect of a change of parameter b is given by:

$$\frac{\partial f(K^*, L^*)}{\partial b} = -\frac{1}{b}$$

h. The value of the objective function in the maximum is

$$f(K^*, L^*) = aM - 1 + \ln\frac{1}{ab} = aM - 1 - \ln ab$$

The marginal effect of a change of parameter a is given by:

$$\frac{\partial f(K^*, L^*)}{\partial a} = M - \frac{1}{a}$$

i. The value of the objective function in the maximum is

$$f(K^*, L^*) = aM - 1 + \ln\frac{1}{ab} = aM - 1 - \ln ab$$

The marginal effect of a change of parameter M is given by:

$$\frac{\partial f(K^*, L^*)}{\partial M} = a$$

4. Define the function f by $f(x, r) = x^{1/2} - rx$, where $x \geq 0$. On a graph with r on the horizontal axis, sketch the function for several values of x *(for example x=0.5, x=1, x=2)*. Sketch, in addition, the value function f^*, where $f^*(r)$ is the maximal value of $f(x, r)$ for each given value of r.

Solution

Solve

$$\max\ x^{1/2} - rx$$

FOC are:

$$0.5x^{-0.5} - r = 0$$

$$x = \frac{1}{4r^2}$$

$$f^*(r) = \frac{1}{2r} - \frac{1}{4r} = \frac{1}{4r}$$