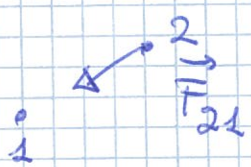
THEORY OF POTENTIAL

PLEASE, REFER TO BINNEY AND TREMAINE BOOK

You know that



$$\bar F_{21} = -\frac{G m_1 m_2}{r_{12}} \hat r$$

$\vec F$ is a conservative force as all "central" forces

$$\text{Work} = \int_A^B \bar F \, d\bar s$$

$$\ldots \hat r \cdot d\vec s$$

$$\ldots \hat r \, d\hat r \longrightarrow \int_{r_A}^{r_B}$$

W = potential energy

$$\Delta W = -\text{Work} = -\int \bar F \, d\bar s$$

$$W = -\frac{G m_1 m_2}{r_{12}} + \text{cost}$$

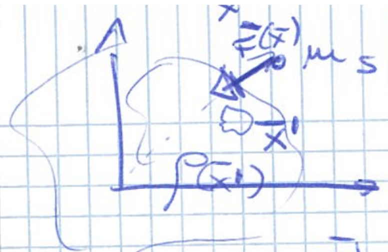
$\to 0$ as $r \to \infty$

· 1 · 2 · 3 ... particles

binding energy

$$W = \frac{-G m_1 m_2}{r_{12}} - \frac{G m_2 m_3}{r_{23}} - \frac{G m_1 m_3}{r_{13}} \ldots$$

Nell
Binney-Tremaine



$$\delta \bar F(\bar x) = G m_s \frac{\bar x' - \bar x}{|\bar x' - \bar x|^3}\, \delta m(\bar x') = G m_s \frac{\bar x' - \bar x}{|\bar x' - \bar x|^3}\, \rho(\bar x')\, d^3(x')$$

the overall force $\leftarrow$ integrating in the whole space

$$\bar F(\bar x) = \int \delta \bar F(\bar x)$$

$$\bar F(\bar x) = m_s\, \bar g(\bar x)$$

gravitational field

$$\boxed{\bar g(\bar x) \equiv G \int d^3\bar x' \frac{(\bar x' - \bar x)}{|\bar x' - \bar x|^3}\, \rho(x')} \qquad 2.2$$

We define the gravitational potential

$$\boxed{\phi(\bar x) \equiv - G \int d^3\bar x' \frac{\rho(\bar x')}{|\bar x' - \bar x|}} \qquad 2.3$$

$$\bar\nabla \phi(\bar x) = -G\, \bar\nabla_x \int d^3\bar x' \frac{\rho(\bar x')}{|\bar x' - \bar x|} =$$

the $\int$ does not depend on $\bar x$ ....

$$= -G \int d^3\bar x' \rho(\bar x')\, \bar\nabla_x \frac{1}{|\bar x' - \bar x|}$$

but $\bar\nabla_x \left(\frac{1}{|\bar x' - \bar x|}\right) = \frac{\bar x' - \bar x}{|\bar x' - \bar x|^3}$ See other papers

$$= \underbrace{-G \int d^3\bar x' \rho(\bar x') \frac{\bar x' - \bar x}{|\bar x' - \bar x|^3}}_{\bar g(\bar x)} \Longrightarrow$$

$$\boxed{g(\bar x) = -\bar\nabla \phi} \qquad 2.5$$

the potential is very useful ...

Now

$$\bar{\nabla}(\bar{g}(\bar{x})) = -\bar{\nabla}\bar{\nabla}\phi = -\nabla^2\phi$$

$$\bar{\nabla}(\bar{g}(\bar{x})) = G\int d^3\bar{x}'\, \bar{\nabla}_x\left(\frac{\bar{x}'-\bar{x}}{|\bar{x}'-\bar{x}|^3}\right)\rho(\bar{x}') \quad (2.6)$$

We know that

$$\bar{\nabla}_x\left(\frac{\bar{x}'-\bar{x}}{|\bar{x}'-\bar{x}|^3}\right) = -\frac{3}{|\bar{x}'-\bar{x}|^3} + \underbrace{\frac{3(\bar{x}'-\bar{x})(\bar{x}'-\bar{x})}{|\bar{x}'-\bar{x}|^5}}_{} \quad (2.7)$$

see the papers

then $\bar{x}'-\bar{x} \neq 0$ $\qquad \dfrac{3\cancel{(\bar{x}'-\bar{x})(\bar{x}'-\bar{x})}}{|\bar{x}'-\bar{x}|^{\cancel{5}\,3}}$

i.e

$$\bar{\nabla}_x\left(\frac{\bar{x}'-\bar{x}}{|\bar{x}'-\bar{x}|^3}\right) = 0 \qquad (\bar{x}'\neq\bar{x}) \quad (2.8)$$

So the ~~th~~ contribution to the integral (2.6) comes from the point $\bar{x}' = \bar{x}'$

around this point $\rho(\bar{x}') \sim$ const.

$$(2.6) \rightarrow \bar{\nabla}\bar{g}(\bar{x}) = G\rho(\bar{x})\int_{|\bar{x}'-\bar{x}|\leq h} d^3\bar{x}'\, \bar{\nabla}_x\left(\frac{\bar{x}'-\bar{x}}{|\bar{x}'-\bar{x}|^3}\right)$$

$\uparrow$ small

~~symmetric~~ changing variables

$$= -G\rho(\bar{x})\int_{|\bar{x}'-x|\leq h} d^3x'\, \bar{\nabla}_{x'}\left(\frac{\bar{x}'-\bar{x}}{|\bar{x}'-\bar{x}|^3}\right)$$

divergence theorem

$$\int_V \bar{\nabla}\bar{F}\, d^3\bar{x} = \int_S \bar{F}\, d^2\bar{S}$$

$$= -G\rho(\bar{x})\int_{|\bar{x}'-\bar{x}|= h} d^2S'\, \frac{(\bar{x}'-\bar{x})}{|\bar{x}'-\bar{x}|^3}$$

but on the shere $\rightarrow r^2 d^2\Omega$

$d^2S' = (\bar{x}'-\bar{x})\, h\, d^2\Omega$

oriented !

$$\bar{\nabla}\bar{g}(\bar{x}) = -G\rho(\bar{x})\int d^2\Omega = -4\pi G\rho(\bar{x})$$

+ eq. 2.5

$$\boxed{\nabla^2\phi = 4\pi G\rho}$$

POISSON EQ. (2.10) diff. eq.

$\rho \rightarrow \phi \rightarrow \bar{g}$

For isolated system $\phi \rightarrow 0$ as $|\bar{x}| \rightarrow \infty$

In the special case $\boxed{\rho = 0}$

Poisson eq $\rightarrow$

$\boxed{\nabla^2 \phi = 0}$ Laplace eq. 2.11

We integrate eq. 2.10

$$\int \nabla^2 \phi \, d^3\bar{x} = \int 4\pi G \rho(\bar{x}) \, d^3\bar{x}$$

$M = \int \rho(\bar{x}) d^3\bar{x}$

$\hookrightarrow$ mass

$$\int \bar{\nabla}(\bar{\nabla}\phi) \, d^3\bar{x} = 4\pi G M$$

e
use

divergence theor.

$$\bar{\nabla}\phi = -\vec{g}$$

the integral of the normal component on $\bar{S}$ $\rightarrow$ $4\pi G M$ contained in $\bar{S}$

$$\int d^2\bar{S} \, \bar{\nabla}\phi = 4\pi G M$$

or

$$\int \bar{g} \, d^2\bar{s} = -4\pi G M$$

(remember ? $\int_S \bar{E} \, d^2\bar{s} = \frac{Q}{\varepsilon_0}$ for $\bar{E}$ field)

$$\boxed{\int d^2\bar{S} \, \bar{\nabla}\phi = 4\pi G M}$$ 2.12

The Gauss theorem !

$\bar{g}$ comes from $\bar{\nabla}\phi$ so is conservative !

Potential energy

a mass is already in place $\rho(\bar{x})$, $\phi(\bar{x})$
we bring an additional small mass δm
from ∞ to $\bar{x}$, the work done is
$\delta m\, \phi(\bar{x})$

So a small increment of density $\delta\rho(\bar{x})$
gives a change in the potential (a small change)

$$\delta W = \int d^3\bar{x}\; \delta\rho(\bar{x})\, \phi(\bar{x}) \qquad (2.13)$$

Poisson eq. $\nabla^2(\delta\phi) = 4\pi G(\delta\rho) \rightarrow$

$$\delta W = \int d^3\bar{x}\; \phi(\bar{x}) \frac{1}{4\pi G} \nabla^2(\delta\phi) =$$

$$= \frac{1}{4\pi G} \int d^3\bar{x}\, \phi(\bar{x})\, \nabla^2(\delta\phi) =$$

$$= \frac{1}{4\pi G} \int d^3\bar{x}\, \underbrace{\phi(x)}_{f}\, \bar{\nabla}\big(\underbrace{\bar{\nabla}(\delta\phi)}_{\bar{F}}\big)$$

"extended" theorem of divergence $\int_V f(\bar{\nabla}\bar{F})\, d^3\bar{x} = \int_S f\bar{F}\, d^2\bar{S} - \int_V (\bar{F}\cdot\bar{\nabla}) f\, d^3\bar{x}$
see Appendix of Binney and Tremaine

$$= \frac{1}{4\pi G} \int \phi\, \bar{\nabla}(\delta\phi)\, d^2\bar{S} - \frac{1}{4\pi G} \int d^3\bar{x}\; \bar{\nabla}(\delta\phi)\, \bar{\nabla}\phi(\bar{x}) \qquad (2.15)$$

$=0$ why $\phi \propto r^{-1}$, $|\bar{\nabla}(\delta\phi)| \propto r^{-2}$ as $r \to \infty$

Σ surface where $r \to \infty$

$\underbrace{\phi\bar{\nabla}(\delta\phi)}_{\text{integrand}} \propto r^{-3}$ while the area $\propto r^2$

Moreover, for the second term

$$\underbrace{\bar{\nabla}\phi\cdot\bar{\nabla}(\delta\phi) = \bar{\nabla}\phi\cdot\delta(\bar{\nabla}\phi)}$$

(In fact $\delta(\bar{\nabla}\phi) = \bar{\nabla}(\phi+\delta\phi) - \bar{\nabla}\phi =$
$= \bar{\nabla}\phi + \bar{\nabla}(\delta\phi) - \bar{\nabla}\phi = \bar{\nabla}(\delta\phi)$)

$$= \frac{1}{2}\,\delta(\bar{\nabla}\phi\cdot\bar{\nabla}\phi)$$

{ remember? In fact
For a vector $\bar{v}$ $\quad \frac{1}{2} d|\bar{v}|^2 = \frac{1}{2} d(\bar{v}\cdot\bar{v}) = \frac{1}{2}\, 2\bar{v}\, d\bar{v}$

$$= \frac{1}{2}\,\delta|(\bar{\nabla}\phi)|^2$$

2.15 $\longrightarrow$

$$\delta W = -\frac{1}{4\pi G}\int d^3\bar{x}\,\frac{1}{2}\,\delta|(\bar{\nabla}\phi)|^2 =$$

(sum of variations = variation of the sum)

$$= -\frac{1}{8\pi G}\,\delta\left(\int d^3\bar{x}\,|\bar{\nabla}\phi|^2\right) \qquad 2.16$$

Now we sum up all of the contributions δW

$$W = \sum \delta W$$

$$W = -\frac{1}{8\pi G}\int d^3\bar{x}\,|\bar{\nabla}\phi|^2 \qquad 2.17$$

To obtain an alternative expression for W

$$2.17 \rightarrow \quad W = -\frac{1}{8\pi G}\int d^3\bar{x}\; \bar{\nabla}\phi \cdot \bar{\nabla}\phi$$

but we know that

$$\bar{\nabla}(\phi\bar{\nabla}\phi) = \phi\bar{\nabla}(\bar{\nabla}\phi) + \bar{\nabla}\phi\cdot\bar{\nabla}\phi$$

$$\rightarrow \bar{\nabla}\phi\cdot\bar{\nabla}\phi = \bar{\nabla}(\phi\bar{\nabla}\phi) - \phi\nabla^2\phi$$

$$= -\frac{1}{8\pi G}\left[\cancel{\int d^3\bar{x}\,\bar{\nabla}(\phi\bar{\nabla}\phi)}_{=0} - \int d^3\bar{x}\,\phi\nabla^2\phi\right] =$$

div. theor.

$$\int d^2\bar{s}\;\phi\bar{\nabla}\phi$$

$$\sim r^2 \quad \underbrace{r^{-1} \quad r^{-2}}_{r^{-3}}$$

I use eq. of Poisson

$$= \frac{1}{8\pi G}\int d^3\bar{x}\; 4\pi G\rho\phi$$

$$\boxed{W = \frac{1}{2}\int d^3\bar{x}\;\rho(\bar{x})\,\phi(\bar{x})} \quad 2.18$$

Potential energy for a body

Chandrasekhar

The potential - energy tensor

tensor $\overline{\overline{W}}$ $\begin{pmatrix} 11 & 12 & 13 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$ 3 x 3

ponent $\left[W_{jk} \equiv -\int d^3\bar{x}\, \rho(\bar{x})\, x_j \frac{\partial \phi}{\partial x_k} \right]$ 2.18

over all space

SS ep. dob.

2.3

$\phi = -G \int d^3\bar{x}' \frac{\rho(\bar{x}')}{|\bar{x}' - \bar{x}|}$

$$\cancel{W_{jk} = G \int d^3\bar{x} \int d^3\bar{x}' \left\{ \rho(\bar{x})\, x_j \frac{\partial}{\partial x_k} \right\}}$$

$$W_{jk} = G \int d^3\bar{x}\, \rho(\bar{x})\, x_j \frac{\partial}{\partial x_k} \int d^3\bar{x}' \frac{\rho(\bar{x}')}{|\bar{x}' - \bar{x}|} = \quad 2.20$$

does not depend on $\bar{x}$

$$W_{jk} = G \int d^3\bar{x} \int d^3\bar{x}'\, \rho(\bar{x})\, \rho(\bar{x}')\, \frac{x_j (x'_k - x_k)}{|\bar{x}' - \bar{x}|^3} \quad 2.21a$$

x and x' are dummy variables of integr.

$$W_{jk} = G \int d^3\bar{x}' \int d^3\bar{x}\, \rho(\bar{x}')\, \rho(\bar{x})\, \frac{x'_j (x_k - x'_k)}{|\bar{x} - \bar{x}'|^3} = \quad 2.21$$

change of the order of integration

$$= G \int d^3\bar{x} \int d^3\bar{x}'\, \rho(\bar{x})\, \rho(\bar{x}')\, \frac{(-x'_j)(x'_k - x_k)}{|\bar{x}' - \bar{x}|^3}$$

2.21a+b

$$2 W_{jk} = G \int d^3\bar{x} \int d^3\bar{x}'\, \rho(\bar{x})\, \rho(\bar{x}') \cdot (-1) \frac{(x'_j - x_j)(x'_k - x_k)}{|\bar{x}' - \bar{x}|^3}$$

$$W_{jk} = -\frac{1}{2} G \int d^3\bar{x} \int d^3\bar{x}'\, \rho(\bar{x})\, \rho(\bar{x}') \frac{(x'_j - x_j)(x'_k - x_k)}{|\bar{x}' - \bar{x}|^3} \quad 2.22$$

2.22

$\rightarrow \overline{\overline{W}}$ is symmetric

$W_{jk} = W_{kj}$

trace

$$\begin{pmatrix} \circ & & \\ & \circ & \\ & & \circ \end{pmatrix}$$

$$\text{trace}(\overline{\overline{W}}) = \sum_{j=1}^{3} W_{jj} =$$

$$= -\frac{1}{2} G \int d^3\bar{x}\, \rho(\bar{x}) \int d^3\bar{x}' \frac{\rho(\bar{x}')}{|\bar{x}' - \bar{x}|^3} \cancel{|\bar{x}' - \bar{x}|^2}$$

def. of ϕ

$$= \frac{1}{2} \int d^3\bar{x}\, \rho(\bar{x})\, \phi(\bar{x})$$ is eq. 2.18 for W 2.24

$$\boxed{\text{trace}(\overline{\overline{W}}) = W}$$

!

If we make directly the trace ~~of~~ using 2.18

$$\sum_{j=1}^{3} W_{jj} = -\sum_{jj} \int d^3\bar{x}\, \rho(\bar{x})\, x_j \frac{\partial \phi}{\partial x_j} =$$

$$= -\int d^3\bar{x}\, \rho(\bar{x})\, \bar{x} \cdot \bar{\nabla}\phi$$

So

$$W = -\int d^3\bar{x}\, \rho\, \bar{x}\, \bar{\nabla}\phi$$ 2.24

alternative expression for the potential energy of a body

Spherical systems

Newton's theorems $\rightarrow$
(page 60)*

Theoreme COROLLARY

gravitational potential inside
an empty shell is constant
since

$$\bar{\nabla}\phi = -\bar{F} = 0 \quad \Rightarrow \quad \phi = \text{const.}$$

$$\phi(r) = ? \quad \longleftarrow \quad \text{from eq. 2.3}$$

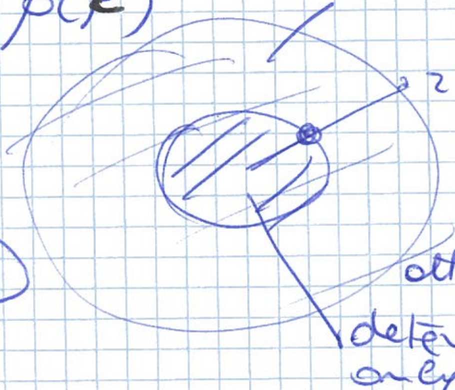
def. of gravit. potent.

$$\phi = -G\int d^3x' \frac{\rho(x')}{|\bar{x}' - \bar{x}|}$$

$$\hookrightarrow \phi = -\frac{GM}{R} \qquad (2.25)$$

FOR 1 shell!

$$\text{also } \phi = -\frac{G}{R}\int dV \rho(r')$$



From I and II Theorems

$$\vec{F}(r) = -\frac{GM(r)}{r^2}\hat{e}_r \qquad 2.27$$

where

$$M(r) = 4\pi\int_0^r dr' \, r'^2 \rho(r')$$

Total gravitational potential
is given by the sum of the potential
of the shells of mass

$$dM(r) = 4\pi\rho(r)\, r^2 dr$$

$$\phi(r) = -\frac{G}{r}\underbrace{\int_0^r dM(r')}_{\text{internal shells}} - G\underbrace{\int_r^\infty \frac{dM(r')}{r}}_{\text{external}}$$

$$dM(r') = 4\pi r'^2 \rho(r')\, dr'$$

$$\vec{\phi}(r) = -\frac{G}{r}\int_0^r 4\pi r'^2 \rho(r') - G\int_r^\infty \frac{4\pi r'^2 \rho(r')\, dr'}{r'}$$

$$= -4\pi G\left[\underbrace{\frac{1}{r}\int_0^r dr'\, r'^2 \rho(r')}_{\text{internal shells}} + \underbrace{\int_r^\infty dr'\, r'\, \rho(r')}_{\text{external shells}}\right] \quad (2.28)$$

def.
circular speed = the speed of a particle of negligible mass (= test particle without mass) in a circular orbit at radius r

$|F_g| = |F_{centripetal}|$

$$v_c^2 = r\,|F_g| = r\frac{d\phi}{dr} = \frac{GM(r)}{r} \quad (2.29)$$

circular frequency

$$\Omega \equiv \frac{v_c}{r} = \sqrt{\frac{GM(r)}{r^3}} \quad (2.30)$$

escape speed $(T+W=0)$

$$v_e(r) \equiv \sqrt{2|\phi(r)|} \quad (2.31)$$

Potential energy

$W = \int d^3\bar{x}\, \rho\, \bar{x}\, \bar{\nabla}\phi$

in spherical coord.
$\bar{x} = r\,\hat{e}_r$
$d^3\bar{x} = 4\pi r^2\, dr$

$$W = \int 4\pi r^3 dr\, \rho\, \hat{e}_r \cdot (\bar{\nabla}\phi) =$$

$$= -4\pi\int dr\, r^3 \rho\left(\hat{e}_r \frac{GM(r)}{r^2}\hat{e}_r\right) = -4\pi G\int dr\, r\, \rho(r) M(r) \quad 2.32$$

$\bar{\bar{W}}$ is diagonal
and $W_{jk} = 0$ $j \neq k$, $W_{jk} = \frac{1}{3} W \delta_{jk}$ $\bar{\bar{W}}$ is isotropic

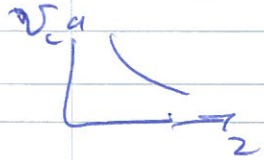
2. DI POISSON

$$\nabla^2\phi = \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\phi}{dr}\right) = 4\pi\rho G$$

Simple systems

a) Point-mass (Keplerian potential)

$\phi(r) = -\frac{GM}{r}$ $\qquad v_c = \sqrt{\frac{GM}{r}}$



$v_e(r) = \sqrt{\frac{2GM}{r}}$

2.34

b) Homogeneous sphere $\rho = \text{const.}$

$M(r) = \frac{4}{3}\pi r^3\rho$

$$v_c^2 = \frac{GM(r)}{r} = \frac{G\frac{4\pi}{3}r^{\cancel{3}2}\rho}{\cancel{r}} \to v_c = \sqrt{\frac{4\pi G\rho}{3}}\,r$$

typical time $T = \frac{2\pi r}{v_c} = \frac{2\pi}{\Omega}$

$$= \frac{\cancel{2}\pi\cancel{r}}{\sqrt{\frac{4\pi G\rho}{3}}\,\cancel{r}} = \sqrt{\frac{3\pi}{G\rho}} \propto \sqrt{3\pi}\,(G\rho)^{-1}$$

dynamical time

gravitational radius

$r_g \equiv \frac{GM^2}{|W|}$ $\qquad$ homogeneous sphere a $\qquad r_g = \frac{5}{3}a$

2.28 $\longrightarrow$ $\phi(r)$

$$r<a \quad \phi(r) = -4\pi G\left[\frac{1}{r}\int_0^r dr'\,r'^2\rho(r') + \int_r^a dr'\,r'\rho(r')\right] =$$

$$= -4\pi G\rho\left\{\frac{1}{r}\left[\frac{r'^3}{3}\right]_0^r + \left[\frac{r'^2}{2}\right]_r^a\right\} =$$

$$= -4\pi G\rho\left\{\frac{1}{\cancel{r}}\frac{r^{\cancel{3}2}}{3} + \frac{a^2}{2} - \frac{r^2}{2}\right\} =$$

$$= -4\pi G\rho\left\{\frac{a^2}{2} - \frac{r^2}{6}\right\} = -2\pi G\rho\left(a^2 - \frac{r^2}{3}\right)$$

$r > a$

$$\phi(r) = -4\pi G\left[\frac{1}{r}\int_0^a dr'\, r'^2 \rho(r')\right] = \frac{-4\pi G}{r}\rho\left[\frac{r^3}{3}\right]_0^a = $$

$$= -\frac{4\pi G \rho a^3}{3r}$$

↳ 1 punto massa

on $M = \frac{4}{3}\pi a^3$ $\quad \phi(r) = \frac{-GM}{r}$

2.43

$$\phi(r) = \begin{cases} -2\pi G\rho\left(a^2 - \frac{1}{3}r^2\right) & (r < a) \\ -\frac{4\pi G\rho a^3}{3r} & (r > a) \end{cases}$$

FOR

is a "box" potential
NON-PHYSICAL

Real system: efforts of observers to fit observed central regions + efforts of theorist to avoid that $M \underset{r\to\infty}{\to} \infty$

c) Plummer model (for globular clusters!)

density ~ const center
and then $\rho \underset{r\to\infty}{\to} 0$

so center $\quad \phi \propto r^2 + \text{const}$
external $\quad \phi \propto r^{-1}$

$$\phi = \frac{-GM}{\sqrt{r^2+b^2}} \qquad 2.44$$

↳ Plummer scale length b
M = total mass

$$\nabla^2\phi = \frac{3GMb^2}{(r^2+b^2)^{5/2}} = 4\pi G\rho$$

$$\boxed{\rho \underset{r \gg b}{\longrightarrow} r^{-5}}$$

f) power law density (for galaxies)

$$\boxed{3-\alpha<0} \quad \rho(r) = \rho_0 \left(\frac{r_0}{r}\right)^{\alpha} \qquad (2.58)$$

$\boxed{-\alpha<0} \rightarrow \alpha < 3$ to have a finite mass at $r \to 0$

$$M = \int dr' \, 4\pi r'^2 \rho(r') = 4\pi \int r'^2 \rho_0 \left(\frac{r_0}{r'}\right)^{\alpha} dr' =$$

$$= 4\pi \rho_0 r_0^{\alpha} \frac{r^{3-\alpha}}{3-\alpha} \qquad \left(\frac{r^{3-\alpha}}{3-\alpha} \xrightarrow[r\to 0]{} 0\right)$$

$$v_c^2 = \frac{GM(r)}{r} = \frac{4\pi G \rho_0 r_0^{\alpha}}{3-\alpha} r^{2-\alpha} \qquad (2.61)$$

observ. $v_c \sim$ flat !
of rotation curves

$$\Longrightarrow \boxed{\alpha \sim 2}$$

$$\rho(r) = \rho_0 \left(\frac{r_0}{r}\right)^2$$

singular isothermal sphere

However M diverges at $r \to \infty$ for all $\alpha \leq 3$

but this model is useful since for the 1st Newton theorem the mass exterior a radius r does not affect the dynamics interior to r.

useful to fit internal regions (see also G-L fit)

(8)

Two-power density models

$$\rho(r) = \frac{\rho_0}{(r/a)^{\alpha}\left(1+\frac{r}{a}\right)^{\beta-\alpha}} \qquad (2.64)$$

Jaffe 1983 $\boxed{\alpha=2 \;\; \beta=4}$ $\quad \dfrac{\rho_0}{\left(\frac{r}{a}\right)^2\left(1+\frac{r}{a}\right)^2} \xrightarrow[r\to 0]{} r^{-2}(r+a)^{-2}$

$\boxed{\alpha=1 \;\; \beta=4}$

Hernquist 1990 $\quad \dfrac{\rho_0}{\left(\frac{r}{a}\right)\left(1+\frac{r}{a}\right)^3} \xrightarrow[r\to 0]{} r^{-1}(r+a)^{-3}$

Navarro
Frenk
White 1995 $\boxed{\alpha=1 \;\; \beta=3}$ $\quad \dfrac{\rho_0}{\left(\frac{r}{a}\right)\left(1+\frac{r}{a}\right)^2} \xrightarrow[r\to 0]{} r^{-1}(r+a)^{-2}$

NFW model

From simulations ρ_0 and a are correlated

FOR ALL $\rho \xrightarrow[r\to 0]{} \infty$ NO GOOD!

r_{200} where the mean density is $200\rho_c$

$M_{200} = 200\rho_c \frac{4\pi}{3} r_{200}^3$

$c = \dfrac{r_{200}}{a}$

$M(r) = \int_0^{r} dr' \, 4\pi r'^2 \rho(r') =$

$= \dots$

For J e H models
$M \xrightarrow[r\to\infty]{}$ finite value

For NFW
$M \xrightarrow[r\to\infty]{} \infty$

Sersic - 3D (Einasto model)

$$\rho(r) = \rho_0 \exp\left[-\left(\frac{r}{a}\right)^{1/m}\right] \qquad m \sim 6$$

formally better than NFW

Einasto 1969

$$\rho \xrightarrow[r \to 0]{} \text{finite}$$

$$M \xrightarrow[r \to \infty]{} \text{finite}$$

Gary

figure of Mamon

COURSE

CREDIT TO GARY MAMON
http://www2.iap.fr/users/gam/M2/index.html

Voir Łokas & Mamon (2001) pour les détails du modèle.

D'autres (Fukushige & Makino 1997; Moore et al. 1998) ont trouvé dans leurs simulations cosmologiques à N corps des profils de structures encore plus pentus au centre avec $\rho \propto r^{-3/2}$ aux petits rayons.

Sérsic-3D Récemment, Navarro et al. (2004) ont montré que les structures des très grosses simulations cosmologiques à N corps sont encore mieux ajustés par l'analogue 3D du modèle de Sérsic (qui ajuste très bien les profils projetés des galaxies elliptiques)

$$\boxed{\rho(r) = \rho_0 \exp\left[-(r/a)^{1/m}\right]}, \qquad \text{(I-38)}$$

avec $m \simeq 6$. Le Sérsic-3D est plus esthétique que le NFW car il possède une densité centrale, ρ_0, finie et une masse totale qui converge. Son potentiel gravitationnel est donné par Cardone, Piedipalumbo & Tortora (2005). Il est parfois appelé profil d'Einasto, qui l'avait suggéré dans Einasto (1969) et avant dans un article de l'institut d'astrophysique d'Alma-Ata en 1965.

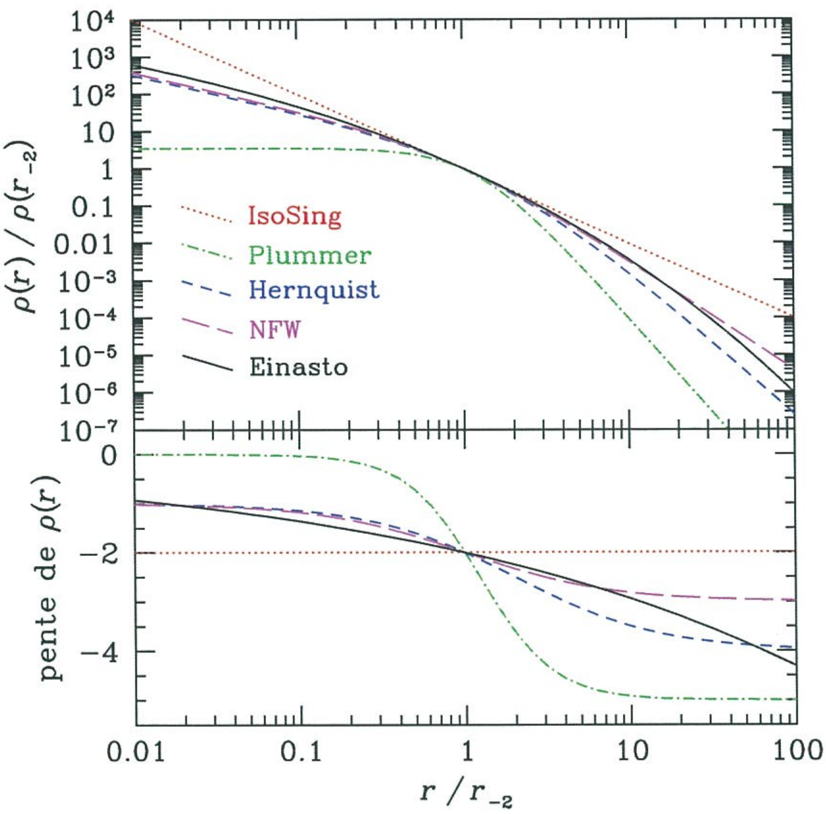


Figure I-1: Profils de densité (*haut*) et pentes associées (*bas*) des modèles couramment employés, où r_{-2} est le rayon où la pente est -2.

La figure I-1 illustre les profils de densité couramment employés, ainsi que leurs pentes associées. Les rayons maximaux où les structures sont en équilibre dynamique sont envi-