Golden eggs model

Introduction

In economics, **dynamic inconsistency**, or **time inconsistency**, describes a situation where a decision-maker's preferences change over time, such that what is preferred at one point in time is inconsistent with what is preferred at another point in time.

It is often easiest to think about preferences over time in this context by thinking of decision-makers as being made up of many different "selves", with each self representing the decision-maker at a different point in time. So, for example, there is my today self, my tomorrow self, my next Tuesday self, my year from now self, etc.

The inconsistency will occur when somehow the preferences of some of the selves are not aligned with each other; there is disagreement between a decision-maker's different selves about what actions should be taken.

When individuals discount future values differently from the exponential discount, they display time inconsistent preference.

A common case of discounting that is not exponential is the quasi – hyperbolic discount that is given by the following functional form of the discounting function:

$$D(k) = \begin{cases} 1\ if\ k = 0 \\ \beta\delta^k\ if\ k \geq 1 \end{cases} \quad 0 \leq \beta \leq 1, 0 \leq \delta \leq 1$$

<u>Example of time consistent preferences (exponential discount at constant rate)</u>

Consider the following problem:

$$\begin{cases} max_{\forall c_t} \sum_{t=1}^{\infty} \delta^{t-1} \log c_t \\ \text{such that:} \\ R\ x_t = x_{t+1} + c_{t+1} \\ c_t \geq 0 \\ x_t \geq 0 \\ x_0\ \text{ given} \end{cases}$$

We know, from previous example, the result is:

$$c_1 = R(1-\delta)x_0$$

$$c_t = R\ \delta c_{t-1}\ \ \forall t \geq 2$$

Using the relation $Rx_t - c_t = x_{t+1}$ we can re-write the solution in the following way:

$$x_t = R\ \delta x_{t-1} = (R\ \delta)^t x_0$$

$$c_t = R(1-\delta)x_{t-1}\ \forall t \geq 1$$

In each period *t* this individual will consume a share of $R(1-\delta)$ of the endowment (at time t).

It is directly verifiable that this individual will confirm his decisions taken at time $t = 0$ in any future time ($t \geq 1$). Indeed in the current period he consumes $R(1-\delta)$ of the (current) endowment and plans to consume an identical share in the future periods (of the future endowments). If he repeats the maximization procedure in a future period the results will be to confirm previous decisions (you can try to solve the problem above starting from a period $\bar{t} > 0$ and endowment $x_{\bar{t}}$).

<u>Example of time inconsistent preferences (quasi-hyperbolic discounting)</u>

Assume an individual with a discounting function as:

$$D(k) = \begin{cases} 1 \; if \; k = 0 \\ \beta\delta^k \; if \; k \geq 1 \end{cases} \quad 0 \leq \beta \leq 1, 0 \leq \delta \leq 1$$

The problem to decide the optimal levels of consumption now is:

$$\begin{cases} max_{\forall c_t} c_1 + \beta \sum_{t=2}^{\infty} \delta^t \log c_t \\ \text{such that:} \\ R\, x_t = x_{t+1} + c_t \\ c_t \geq 0 \\ x_t \geq 0 \\ x_1 \text{ given} \end{cases}$$

The first order conditions are:

$$\begin{cases} -\frac{1}{c_1} + R\beta\,\delta\frac{1}{c_2} = 0 \\ -\frac{1}{c_{t-1}} + R\,\delta\frac{1}{c_t} = 0 \;\forall t \geq 3 \end{cases}$$

Therefore we have that solution has to satisfy:

$$\begin{cases} c_2 = R\beta\delta c_1 \\ c_t = R\delta c_{t-1} \;\forall t \geq 3 \end{cases}$$

Using the relation $\sum_{t=1}^{\infty} \frac{c_t}{R^t} = x_0$ the solution is:

$$c_1 = \frac{R(1-\delta)}{\beta} x_0$$

$$c_1 = R\beta\delta c_0$$

$$c_t = R\delta c_{t-1} \;\forall t \geq 2$$

We can arrange the solution in the following way:

$$c_1 = \frac{R(1-\delta)}{\beta} x_0$$

$$c_t = R(1-\delta) x_t \;\forall t \geq 1$$

This individual consumes a fraction of the endowment equal to $\frac{R(1-\delta)}{\beta}$ in the current period and plans to consume a constant fraction of his endowment equal to $R(1-\delta)$ in all future periods.

It easy to check that this individual has time inconsistent preferences: If he repeats the maximization problem at time t=1 he obtains:

$$c_1 = \frac{R(1-\delta)}{\beta} x_0$$

while in the plan chosen at time t=0 $c_1 = R(1-\delta)x_0$. That is, when time 1 arrives he changes the previous plan and decides to consume more.

From the point of view of time 0, this change represents a deviation from the optimal consumption plan, reducing the total expected utility computed at time 0 (if at time 0 this individual is able to forecast the future behavior).

Golden eggs model: a simplified version

Laibson, 1997, Quarterly Journal of Economics.

There exist two kind of assets: liquid asset, denoted by y, and illiquid asset, denoted by z.

A sale of asset z have to be initiated one period before the actual proceeds are received.

Discrete time $t \in \{1,2,....T\}$

Consumer begins life with exogenous endowment y_0, z_0

Each period t is divided in 4 sub-periods:

1. Both assets produce a gross return of R=1+r (both, liquid and illiquid).
2. consumer gets access to her liquid saving Ry_t
3. Consumer decides current consumption: $c_t \leq Ry_t$
4. Consumer decides new asset allocation s.t.: $R(y_t + z_t) - c_t = z_{t+1} + y_{t+1}$ $\qquad y_t, z_t \geq 0$

The discounting function is:

$$\mathrm{D(k)} = \begin{cases} 1 \text{ if } \mathrm{k} = 0 \\ \beta\delta^{\mathrm{k}} \text{ if } \mathrm{k} \geq 1 \end{cases} \quad 0 \leq \beta \leq 1, 0 \leq \delta \leq 1$$

Therefore this individual has time inconsistent preferences if $\beta < 1$; if $\beta = 1$ we are in the case of exponential discount at constant rates, so individual has time consistent preferences. In the following we assume $\beta < 1$.

The maximization problem that this individual faces is:

$$\begin{cases} max_{\forall c_t} c_1 + \beta \sum_{t=1}^{\infty} \delta^t \log c_t \\ \text{such that:} \\ R\ (y_t + z_t) = z_{t+1} + y_{t+1} + c_t \\ 0 \leq c_t \leq Ry_t \\ z_t, y_t \geq 0 \\ y_0, z_0 \ \text{ given} \end{cases}$$

Note that this individual can consume only asset y. Moreover this problem is equal to the previous example if asset z does not exist.

We assume that this individual knows that he has time inconsistent preferences. That is, he plans to consume a quantity *c'* in a future period *t'* but knows that when period *t'* will arrive he will change the level of consumption choosing, for example, c''>c'. We see this fact in the previous example.

Suppose at time $t = \bar{t}$ this individual solves his maximization problem and decides the optimal current and future consumption levels $\bar{c}_{\bar{t}}, \bar{c}_{\bar{t}+1}, \bar{c}_{\bar{t}+2}$, ….. Therefore he knows that, at time $t = \bar{t} + 1$, he will not confirm $\bar{c}_{\bar{t}+1}$ but he will switch to higher level of consumption $\hat{c}_{\bar{t}+1}$. Then, he realizes that this deviation causes a lower expected utility, but he cannot manipulate the preferences of his future selves.

To avoid these no desirable deviations of future selves, this individual can use an investment strategy dividing the current endowment in liquid and illiquid asset.

At time $t = \bar{t}$ he plans to consume $\bar{c}_{\bar{t}+1}$ but knows that he will switches to $\hat{c}_{\bar{t}+1} > \bar{c}_{\bar{t}+1}$. Therefore at time $t = \bar{t}$, to avoid this deviation, he invests a quantity $\bar{c}_{\bar{t}+1}/R$ of the endowment in the liquid asset $y_{\bar{t}+1}$ and all the rest in illiquid asset $z_{\bar{t}+1}$. So in period $t = \bar{t} + 1$ this individual want consume $\hat{c}_{\bar{t}+1} > \bar{c}_{\bar{t}+1}$, but he cannot do because $y_{t+1} = \bar{c}_{\bar{t}+1}/R$ and $c_{\bar{t}+1} \leq Ry_{\bar{t}+1}$. So the best that he can do is to consume all the possible that is $c_{\bar{t}+1} = \bar{c}_{\bar{t}+1}$.

So using the result from previous example we have:

In period $t = \bar{t}$ he plans to consume $\bar{c}_{\bar{t}+1} = R(1-\delta)x_{\bar{t}+1}$ but he knows that at period $t = \bar{t} + 1$ he will desire to consume more, that is $\hat{c}_{\bar{t}+1} = R(1-\delta)x_{\bar{t}+1}/\beta$. To avoid this deviation, at time $t = \bar{t}$ he invests $(1-\delta)x_{\bar{t}+1}$ in the liquid asset y and $\delta x_{\bar{t}+1}$ in liquid asset z.

So $y_{\bar{t}+1} = (1-\delta)x_{\bar{t}+1}$ and $z_{\bar{t}+1} = \delta x_{\bar{t}+1}$ is the best investment strategy.

In period $t = \bar{t} + 1$ the best that he can do is to consume all the possible $c_{\bar{t}+1} = Ry_{\bar{t}+1} = \bar{c}_{\bar{t}+1}$= $R(1-\delta)x_{\bar{t}+1}$.

So in every period :

1. At time $t = \bar{t} + 1$, this individual want to consume $R(1-\delta)x_{\bar{t}+1}/\beta$ but the best that he can do is consume $\bar{c}_{\bar{t}+1} = R(1-\delta)x_{\bar{t}+1}$.
2. Therefore, after consumption, his endowment at time $t = \bar{t} + 2$ is $R\delta x_{\bar{t}+1}$.
3. The best investment strategy is $y_{\bar{t}+2} = (1-\delta)R\delta x_{\bar{t}+1}$ and $z_{\bar{t}+2} = \delta R\delta x_{\bar{t}+1}$.

Hint: try to solve similar problems using different utility functions.