- An individual has to choose the optimal levels of consumption in future periods.
- Individual starts with a given endowment x_0 that evolves by a given (exogenous) rate of interest:
  - In each period t, she consumes c_t and saves x_t;
  - at the beginning of period $t+1$, the saving of previous period, x_t, either produce interests (if $R>1$) or is subjected to a decay ($R<1$).
  - The budget for period t+1 is given by $R\ x_t$.
- We assume that this individual is characterized by exponential discounting.

Finite time

Consider the following problem (from the perspective of $t = 1$)

$$
(1) \qquad \begin{cases} \max_{\forall c_t} \sum_{t=1}^{n} u(c_t)\, \delta^{t-1} \\ \text{such that:} \\ R\, x_t = x_{t+1} + c_{t+1} \\ c_t \geq 0 \\ x_t \geq 0 \\ x_0 \ \text{ given} \end{cases}
$$

Where:

1. x_t is a state variable (for example the capital).
2. R > 0 is the growth rate of the state variable; it could be greater than 1 (growth) or smaller (decay);
3. δ^t is the discount function where $0 < \delta < 1$.
4. x_0 is the endowment at time 0
5. maximization is with respect to variables c_t for each $t \in \{ 1,2,\dots,n\}$

The problem (1) can be rewritten as:

(2)
$$\begin{cases} \max_{\forall x_t} \sum_{t=0}^{n-1} u(R\, x_t - x_{t+1})\, \delta^t \\ \text{such that:} \\ 0 \le x_t \le R x_{t-1} \\ x_0 \text{ given} \end{cases}$$

where maximization is with respect to state variables x_t .

If $u(.)$ is increasing and concave, the first order conditions are necessary and sufficient for a maximum.

<u>first order conditions</u>:

$$-u'(R\, x_{t-1} - x_t)\delta^{t-1} + u'(R\, x_t - x_{t+1})R\, \delta^t = 0\ \forall t \in \{1,2,..n\}$$

These conditions are called Euler equations.

(3)
$$-u'(R\, x_{t-1} - x_t) + u'(R\, x_t - x_{t+1})R\delta = 0$$

and using the relation $R\, x_{t-1} = x_t + c_t$ we have:

(4) $$-u'(c_t) + u'(c_{t+1}) R\, \delta = 0$$

If $R\, \delta \geq 1$ (≤ 1) the marginal utility is decreasing (increasing) on the time; it follows that consumption is increasing (decreasing) on the time.

Solution:

(5) $$c_t = f\,(c_{t-1}).$$

In an optimal consumption plan, the individual has to consume the endowment as possible: $x_{n+1} = 0$

An optimum consumption plan has to satisfy the following condition:

(6) $$\sum_{t=1}^{n} \frac{c_t}{R^t} = x_0$$

Using the relation (5) we can solve by c_1.

Example 1 (logarithmic utility in finite time):

Assume $u(c_t) = \ln c_t$.

The first order condition is:

$$-\frac{1}{c_{t-1}} + R\,\delta\frac{1}{c_t} = 0\,,$$

that rewritten is:

$$c_t = R\,\delta c_{t-1}.$$

We can explicit each c_t as a function of c_1:

$$c_t = (R\,\delta)^{t-1}c_1.$$

Using relation (5) we can find c_1 as follows:

$$\sum_{t=1}^{n}\frac{(R\,\delta)^{t-1}c_1}{R^t} = x_0 \qquad \frac{c_1}{R}\sum_{t=1}^{n}\delta^{t-1} = x_0;$$

$$c_1 = \frac{1-\delta}{1-\delta^n}x_0\,.$$

Infinite time ($n = \infty$)

The first order conditions are the same

$$-u'(R\,x_{t-1} - x_t)\delta^{t-1} + u'(R\,x_t - x_{t+1})R\,\delta^t = 0\ \forall t \in \{1,2,..n\}$$

These conditions are called Euler equations.

(1) $$-u'(R\,x_{t-1} - x_t) + u'(R\,x_t - x_{t+1})R\delta = 0$$

and using the relation $R\,x_{t-1} = x_t + c_t$ we have:

(2) $$-u'(c_t) + u'(c_{t+1})R\,\delta = 0$$

To find the initial level of consumption, we have to solve:

$$\sum_{t=1}^{\infty} \frac{c_t}{R^t} = x_0$$

The solution has to satisfy the following condition (we check ex post):

$$\lim_{t\to\infty} P_t \delta^t x_t = 0 \quad \text{where } P_t = \frac{\partial u(R\, x_t - x_{t+1})}{\partial x_t}$$