APPENDIX A

A.1 CIRCLE EQUATIONS: BILINEAR TRANSFORMATION

The distance between two complex points z and z_o is $|z - z_o|$. Therefore, it follows that the equation of a circle of radius r with center at z_o is given by

$$|z - z_o| = r \tag{A.1}$$

Letting $z = x + jy$ and $z_o = x_o + jy_o$, (A.1) can be expressed in the form

$$|(x - x_o) + j(y - y_o)| = r$$

or

$$(x - x_o)^2 + (y - y_o)^2 = r^2 \tag{A.2}$$

which is the well-known Cartesian representation of a circle centered at (x_o, y_o) with radius r.

Another representation of the circle equation is obtained by squaring (A.1)—namely,

$$|z - z_o|^2 = r^2$$

$$(z - z_o)(z - z_o)^* = r^2$$

$$(z - z_o)(z^* - z_o^*) = r^2$$

Multiplying the left-hand side gives

$$|z|^2 - zz_o^* - z^*z_o + |z_o|^2 = r^2 \tag{A.3}$$

The terms $zz_o^* + z^*z_o$ can also be written as $2\text{Re}[zz_o^*]$. Hence, another form of (A.3) is

$$|z|^2 - 2\mathrm{Re}[zz_o^*] + |z_o|^2 = r^2$$

Several relations in this book involve algebraic manipulations that lead to circle equations. In fact, several relations are of the form

$$w = \frac{az + b}{cz + d} \qquad (ad - bc \neq 0) \tag{A.4}$$

which is recognized as the bilinear transformation. The bilinear transformation transforms (or maps) circles in the w plane into circles in the z plane, and vice versa.

For example, consider the transformation of the circle $|w| = \alpha$ into the z plane. From (A.4),

$$|w| = \alpha = \left|\frac{az + b}{cz + d}\right| \tag{A.5}$$

Then

$$\alpha^2 = \left|\frac{az + b}{cz + d}\right|^2 = \left(\frac{az + b}{cz + d}\right)\left(\frac{az + b}{cz + d}\right)^*$$

which can be expanded as

$$\alpha^2[|cz|^2 + |d|^2 + czd^* + c^*z^*d] = |az|^2 + |b|^2 + azb^* + a^*z^*b$$

or

$$|z|^2[|a|^2 - \alpha^2|c|^2] - z[\alpha^2cd^* - ab^*] - z^*[\alpha^2c^*d - a^*b] + |b|^2 - \alpha^2|d|^2 = 0$$

Finally, we write the preceding expression in the form

$$|z|^2 - z\left[\frac{\alpha^2cd^* - ab^*}{|a|^2 - \alpha^2|c|^2}\right] - z^*\left[\frac{\alpha^2c^*d - a^*b}{|a|^2 - \alpha^2|c|^2}\right] + \frac{|b|^2 - \alpha^2|d|^2}{|a|^2 - \alpha^2|c|^2} = 0 \tag{A.6}$$

Comparing (A.6) with (A.3), it follows that (A.6) is the equation of a circle centered at

$$z_o = \frac{\alpha^2c^*d - a^*b}{|a|^2 - \alpha^2|c|^2} \tag{A.7}$$

and the radius of the circle is obtained from the relation

$$r^2 = |z_o|^2 - \frac{|b|^2 - \alpha^2|d|^2}{|a|^2 - \alpha^2|c|^2} \tag{A.8}$$

Substituting (A.7) into (A.8) gives

$$r^2 = \left|\frac{\alpha^2c^*d - a^*b}{|a|^2 - \alpha^2|c|^2}\right|^2 - \frac{|b|^2 - \alpha^2|d|^2}{|a|^2 - \alpha^2|c|^2} = \frac{\alpha^2|ad - bc|^2}{||a|^2 - \alpha^2|c|^2|^2}$$

or

$$r = \frac{\alpha|ad - bc|}{||a|^2 - \alpha^2|c|^2|} \tag{A.9}$$

Next we consider the transformation of a circle in the w-plane centered at w_o with radius α. That is, for $|w - w_o| = \alpha$ we write, from (A.4),

$$w - w_o = \frac{az + b}{cz + d} - w_o = \frac{(a - cw_o)z + (b - dw_o)}{cz + d}$$

Defining $a' = a - cw_o$ and $b' = b - dw_o$, we have

$$w - w_o = \frac{a'z + b'}{cz + d}$$

Then

$$|w - w_o| = \alpha = \left|\frac{a'z + b'}{cz + d}\right| \tag{A.10}$$

and it follows that the center and radius of the circle in the z plane are given by (A.7) and (A.9), with a replaced by a' and b replaced by b'—namely,

$$z_o = \frac{\alpha^2 c^* d - (a')^* b'}{|a'|^2 - \alpha^2|c|^2} \tag{A.11}$$

and

$$r = \frac{\alpha^2|a'd - b'c|}{||a'|^2 - \alpha^2|c|^2|} \tag{A.12}$$

A.2 DERIVATION OF THE INPUT AND OUTPUT STABILITY CIRCLES [EQUATIONS (3.3.5) AND (3.3.6)]

From (3.3.3), the values of Γ_L that produce $|\Gamma_{\mathrm{IN}}| = 1$ are

$$|\Gamma_{\mathrm{IN}}| = 1 = \left|S_{11} - \frac{S_{12}S_{21}\Gamma_L}{1 - S_{22}\Gamma_L}\right| = \left|\frac{S_{11} - \Delta\Gamma_L}{1 - S_{22}\Gamma_L}\right| \tag{A.13}$$

which is recognized to be a bilinear transformation. Comparing (A.13) with (A.4) (with $\Gamma_L = z$ and $\Gamma_{\mathrm{IN}} = w$), it follows that $a = -\Delta$, $b = S_{11}$, $c = -S_{22}$, and $d = 1$. The transformation (A.13) maps the circle $|\Gamma_{\mathrm{IN}}| = 1$ into a circle in the Γ_L plane whose center and radius are given by (A.7) and (A.9), respectively. From (A.7), denoting the center by C_L, we obtain

$$C_L = \frac{c^*d - a^*b}{|a|^2 - |c|^2} = \frac{-S_{22}^* + \Delta^* S_{11}}{|\Delta|^2 - |S_{22}|^2} = \frac{(S_{22} - \Delta S_{11}^*)^*}{|S_{22}|^2 - |\Delta|^2}$$

and, from (A.9), denoting the radius by r_L, we obtain

$$r_L = \frac{|ad - bc|^2}{||a|^2 - |c|^2|^2} = \left|\frac{-\Delta + S_{11}S_{22}}{|\Delta|^2 - |S_{22}|^2}\right| = \left|\frac{S_{12}S_{21}}{|S_{22}|^2 - |\Delta|^2}\right|$$

The circle in the Γ_L plane is given by

$$|\Gamma_L - C_L| = r_L$$

which is recognized as (3.3.5). The derivation of (3.3.6) is similar.

APPENDIX B

STABILITY CONDITIONS

The necessary and sufficient conditions for a two-port network to be unconditionally stable can be derived from (3.3.1) to (3.3.4) (Kurokawa [B.1] and Ha [B.2]). An alternate derivation of the stability conditions begins with (3.3.11) and (3.3.12). Both derivations are now presented.

FIRST DERIVATION

The conditions for a two-port network to be unconditionally stable are given in (3.3.1) to (3.3.4)—namely,

$$|\Gamma_s| < 1 \tag{B.1}$$

$$|\Gamma_L| < 1 \tag{B.2}$$

$$|\Gamma_{IN}| = \left| S_{11} + \frac{S_{12}S_{21}\Gamma_L}{1 - S_{22}\Gamma_L} \right| < 1 \tag{B.3}$$

and

$$|\Gamma_{OUT}| = \left| S_{22} + \frac{S_{12}S_{21}\Gamma_s}{1 - S_{11}\Gamma_s} \right| < 1 \tag{B.4}$$

Equations (B.2) and (B.3) state that for all passive load impedances, the real part of the input impedance must be positive; while (B.1) and (B.4) state that for all passive source impedances, the real part of the output impedance must be positive.

Starting with (B.3), we can write the inequality in the form

$$\left|S_{11} + \frac{S_{12}S_{21}\Gamma_L}{1 - S_{22}\Gamma_L}\right| = \left|\frac{S_{11} - \Delta\Gamma_L}{1 - S_{22}\Gamma_L}\right| = \left|\frac{\Delta - \Delta\Gamma_L S_{22} + S_{21}S_{12}}{S_{22}(1 - S_{22}\Gamma_L)}\right|$$

$$= \left|\frac{1}{S_{22}}\left(\Delta + \frac{S_{12}S_{21}}{1 - S_{22}\Gamma_L}\right)\right| < 1 \tag{B.5}$$

where

$$\Delta = S_{11}S_{22} - S_{12}S_{21}$$

Hence, we need to determine the conditions that the S parameters must satisfy so that (B.5) holds for all values of Γ_L such that $|\Gamma_L| < 1$ [i.e., so that (B.2) is satisfied]. To this end, we write (B.5) as

$$|z| < 1 \tag{B.6}$$

where

$$z = \frac{1}{S_{22}}\left(\Delta + \frac{S_{12}S_{21}}{1 - S_{22}\Gamma_L}\right) \tag{B.7}$$

Next, we analyze the bilinear transformation in (B.7) to determine the mapping of the unit circle $|\Gamma_L| = 1$ on the z plane. The mapping of $|\Gamma_L| = 1$ on the z plane according to (B.7) can be viewed as a series of successive mappings. That is, let

$$z = \frac{1}{S_{22}}(\Delta + q) \tag{B.8}$$

where

$$q = \frac{S_{12}S_{21}}{1 - S_{22}\Gamma_L} = S_{12}S_{21}t$$

$$t = \frac{1}{1 - S_{22}\Gamma_L} = \frac{1}{w}$$

and

$$w = 1 - S_{22}\Gamma_L$$

The transformation $w = 1 - S_{22}\Gamma_L$ is a translation that maps the unit circle $|\Gamma_L| = 1$ onto the w plane as a circle, centered at 1 with radius $|S_{22}|$ (see Fig. B.1a). The inverse transformation

$$t = \frac{1}{w} = \frac{1}{1 - S_{22}\Gamma_L}$$

produces the mapping shown in Fig. B.1b. The center C_1 is located on the real axis at the midpoint between $1/(1 - |S_{22}|)$ and $1/(1 + |S_{22}|)$. That is,

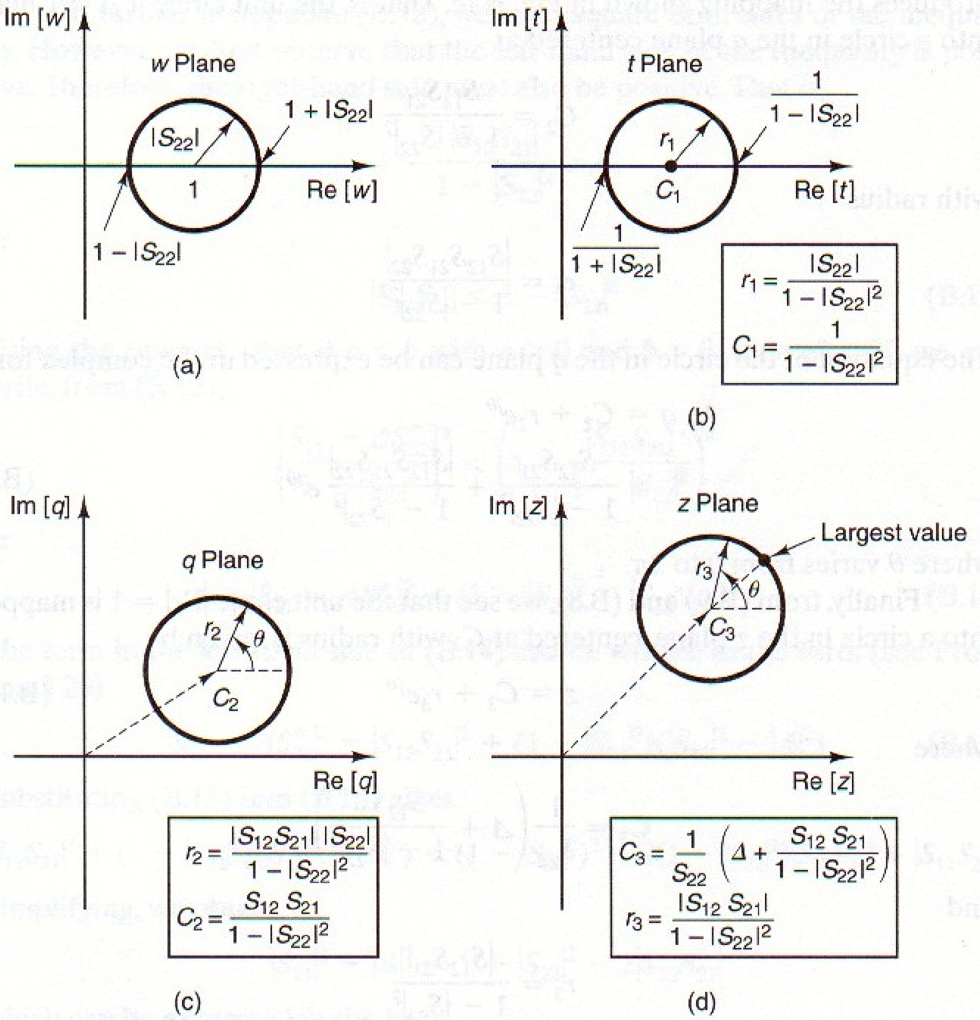

Figure B.1 (a) Mapping of $|\Gamma_L| = 1$ on the w plane, where $w = 1 - S_{22}\Gamma_L$; (b) mapping of $|\Gamma_L| = 1$ on the t plane, where $t = 1/w$; (c) mapping of the $|\Gamma_L| = 1$ plane on the q plane, where $q = S_{12}S_{21}t$; (d) mapping of the $|\Gamma_L| = 1$ plane on the z plane, where $z = 1/S_{22}(\Delta + q)$.

$$C_1 = \frac{1}{2}\left[\frac{1}{1 - |S_{22}|} + \frac{1}{1 + |S_{22}|}\right] = \frac{1}{1 - |S_{22}|^2}$$

The radius r_1 is given by

$$r_1 = \frac{1}{2}\left[\frac{1}{1 - |S_{22}|} - \frac{1}{1 + |S_{22}|}\right] = \frac{|S_{22}|}{1 - |S_{22}|^2}$$

Then, the transformation

$$q = S_{12}S_{21}t = \frac{S_{12}S_{21}}{1 - S_{22}\Gamma_L}$$

produces the mapping shown in Fig. B.1c. That is, the unit circle $|\Gamma_L| = 1$ maps into a circle in the q plane centered at

$$C_2 = \frac{S_{12}S_{21}}{1 - |S_{22}|^2}$$

with radius

$$r_2 = \frac{|S_{12}S_{21}S_{22}|}{1 - |S_{22}|^2}$$

The equation of the circle in the q plane can be expressed in the complex form

$$\begin{aligned} q &= C_2 + r_2 e^{j\theta} \\ &= \frac{S_{12}S_{21}}{1 - |S_{22}|^2} + \frac{|S_{12}S_{21}S_{22}|}{1 - |S_{22}|^2} e^{j\theta} \end{aligned} \tag{B.9}$$

where θ varies from 0 to 2π.

Finally, from (B.9) and (B.8), we see that the unit circle $|\Gamma_L| = 1$ is mapped into a circle in the z plane, centered at C_3 with radius r_3, given by

$$z = C_3 + r_3 e^{j\theta} \tag{B.10}$$

where

$$C_3 = \frac{1}{S_{22}}\left(\Delta + \frac{S_{12}S_{21}}{1 - |S_{22}|^2}\right)$$

and

$$r_3 = \frac{|S_{12}S_{21}|}{1 - |S_{22}|^2}$$

The mapping is shown in Fig. B.1d.

From (B.10), we observe that $|z|$ has its largest value when θ is equal to the phase of the C_3 term. It follows that regardless of the value of θ, the inequality $|z| < 1$ [i.e., (B.6)] is satisfied when

$$\left|\frac{1}{S_{22}}\left(\Delta + \frac{S_{12}S_{21}}{1 - |S_{22}|^2}\right)\right| + \frac{|S_{12}S_{21}|}{1 - |S_{22}|^2} < 1 \tag{B.11}$$

Since

$$\frac{1}{S_{22}}\left(\Delta + \frac{S_{12}S_{21}}{1 - |S_{22}|^2}\right) = \frac{S_{11} - \Delta S_{22}^*}{1 - |S_{22}|^2}$$

we can write (B.11) in the form

$$\left|\frac{S_{11} - \Delta S_{22}^*}{1 - |S_{22}|^2}\right| < 1 - \frac{|S_{12}S_{21}|}{1 - |S_{22}|^2} \tag{B.12}$$

To further manipulate (B.12), we must square both sides of the inequality. However, we first observe that the left-hand side of the inequality is positive. Therefore, the right-hand side must also be positive. That is,

$$1 - \frac{|S_{12}S_{21}|}{1 - |S_{22}|^2} > 0$$

or

$$|S_{12}S_{21}| < 1 - |S_{22}|^2 \tag{B.13}$$

Using the property that if $a < b$ with $a > 0$ and $b > 0$, then $a^2 < b^2$, we can write, from (B.12),

$$\left|\frac{S_{11} - \Delta S_{22}^*}{1 - |S_{22}|^2}\right|^2 < \left(1 - \frac{|S_{12}S_{21}|}{1 - |S_{22}|^2}\right)^2$$

or

$$|S_{11} - \Delta S_{22}^*|^2 < (1 - |S_{22}|^2 - |S_{12}S_{21}|)^2 \tag{B.14}$$

The term in the left-hand side of (B.14) can be written in the form (see Problem 3.20)

$$|S_{11} - \Delta S_{22}^*|^2 = |S_{12}S_{21}|^2 + (1 - |S_{22}|^2)(|S_{11}|^2 - |\Delta|^2) \tag{B.15}$$

Substituting (B.15) into (B.14) gives

$$|S_{12}S_{21}|^2 + (1 - |S_{22}|^2)(|S_{11}|^2 - |\Delta|^2) < (1 - |S_{22}|^2)^2 - 2(1 - |S_{22}|^2)|S_{12}S_{21}| + |S_{12}S_{21}|^2$$

Simplifying, we obtain

$$|S_{11}|^2 - |\Delta|^2 < 1 - |S_{22}|^2 - 2|S_{12}S_{21}|$$

which can be expressed in the form

$$K > 1$$

where K is

$$K = \frac{1 - |S_{11}|^2 - |S_{22}|^2 + |\Delta|^2}{2|S_{12}S_{21}|} \tag{B.16}$$

Thus far, we have shown that (B.2) and (B.3) are satisfied when the S parameters satisfy (B.13) and (B.16). Starting with (B.1) and (B.4), the derivation is similar. In fact, it follows that the conditions for unconditional stability at the output port are simply obtained by interchanging S_{11} by S_{22} and S_{22} by S_{11} in (B.13) and (B.16)—that is, when

$$|S_{12}S_{21}| < 1 - |S_{11}|^2 \tag{B.17}$$

and

$$K > 1$$

In conclusion, from (B.13), (B.16), and (B.17), the two-port network is unconditionally stable when

$$K = \frac{1 - |S_{11}|^2 - |S_{22}|^2 + |\Delta|^2}{2|S_{12}S_{21}|} > 1$$

$$|S_{12}S_{21}| < 1 - |S_{11}|^2$$

and

$$|S_{12}S_{21}| < 1 - |S_{22}|^2$$

SECOND DERIVATION

The conditions for a two-port network to be unconditionally stable can be expressed in the form given in (3.3.11) and (3.3.12)—namely,

$$|S_{11}| < 1 \tag{B.18}$$

with

$$\begin{cases} |C_L| - r_L > 1 & \text{(B.19)} \\ \quad\text{or} \\ r_L - |C_L| > 1 & \text{(B.20)} \end{cases}$$

and

$$|S_{22}| < 1 \tag{B.21}$$

with

$$\begin{cases} |C_s| - r_s > 1 & \text{(B.22)} \\ \quad\text{or} \\ r_s - |C_s| > 1 & \text{(B.23)} \end{cases}$$

where C_L, r_L, C_s, and r_s are given by (3.3.7) to (3.3.10).

Equation (B.19) states that the output stability circle is completely outside the Smith chart (see Fig. B.2a), and (B.20) states that the output stability circle completely encloses the Smith chart (see Fig. B.2b). Equation (B.18) makes the inside of the Smith chart the stable region (see Figs. B.2a and B.2b). Equations (B.21), (B.22), and (B.23) place similar conditions on the input stability circle.

Substituting (3.3.7) and (3.3.8) into (B.19) gives

$$\left| \frac{S_{22} - \Delta S_{11}^*}{|S_{22}|^2 - |\Delta|^2} \right| > 1 + \frac{|S_{12}S_{21}|}{|S_{22}|^2 - |\Delta|^2} \tag{B.24}$$

In writing (B.24), we used the fact that when the origin is not enclosed by the output stability circle, it follows that $|S_{22}| > |\Delta|$ (see Problem 3.11). Therefore,

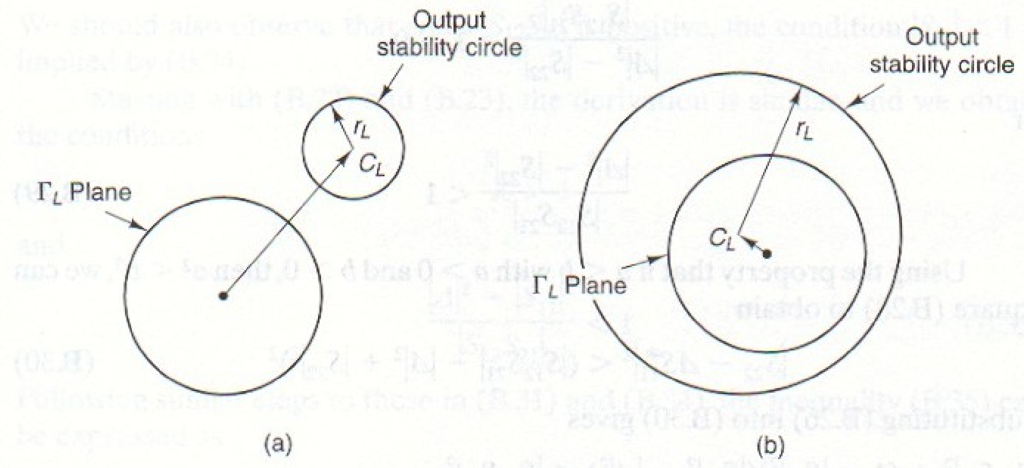

Figure B.2 (a) A typical output stability circle outside the Smith chart for unconditional stability; (b) a typical output stability circle enclosing the Smith chart for unconditional stability.

the term $|S_{12}S_{21}|/(|S_{22}|^2 - |\Delta|^2)$ (which is the expression for r_L) is positive. We can then square both sides of (B.24) to obtain

$$|S_{22} - \Delta S_{11}^*|^2 > (|S_{22}|^2 - |\Delta|^2 + |S_{12}S_{21}|)^2 \tag{B.25}$$

Using the identity (see Problem 3.20)

$$|S_{22} - \Delta S_{11}^*|^2 = |S_{12}S_{21}|^2 + (1 - |S_{11}|^2)(|S_{22}|^2 - |\Delta|^2) \tag{B.26}$$

we can write (B.25) in the form

$$|S_{12}S_{21}|^2 + (1 - |S_{11}|^2)(|S_{22}|^2 - |\Delta|^2) > (|S_{22}|^2 - |\Delta|^2)^2 + 2(|S_{22}|^2 - |\Delta|^2)|S_{12}S_{21}| + |S_{12}S_{21}|^2$$

Simplifying, we obtain

$$1 - |S_{11}|^2 > |S_{22}|^2 - |\Delta|^2 + 2|S_{12}S_{21}|$$

or simply

$$K = \frac{1 - |S_{11}|^2 - |S_{22}|^2 + |\Delta|^2}{2|S_{12}S_{21}|} > 1 \tag{B.27}$$

Next, substituting (3.3.7) and (3.3.8) into (B.20) gives

$$\left|\frac{S_{22} - \Delta S_{11}^*}{|S_{22}|^2 - |\Delta|^2}\right| < \frac{|S_{12}S_{21}|}{|\Delta|^2 - |S_{22}|^2} - 1 \tag{B.28}$$

In writing (B.28), we used the fact that when the origin is enclosed by the output stability circle, it follows that $|S_{22}| < |\Delta|$ (see Problem 3.11). Therefore, the term $|S_{12}S_{21}|/(|\Delta|^2 - |S_{22}|^2)$ (which is the expression for r_L) is positive. Since the left-hand side of (B.28) is positive, the right-hand side must also be positive. That is,

$$\frac{|S_{12}S_{21}|}{|\Delta|^2 - |S_{22}|^2} - 1 > 0$$

or

$$\frac{|\Delta|^2 - |S_{22}|^2}{|S_{12}S_{21}|} < 1 \tag{B.29}$$

Using the property that if $a < b$ with $a > 0$ and $b > 0$, then $a^2 < b^2$, we can square (B.28) to obtain

$$|S_{22} - \Delta S_{11}^*|^2 < (|S_{12}S_{21}| - |\Delta|^2 + |S_{22}|^2)^2 \tag{B.30}$$

Substituting (B.26) into (B.30) gives

$$|S_{12}S_{21}|^2 + (1 - |S_{11}|^2)(|S_{22}|^2 - |\Delta|^2) < |S_{12}S_{21}|^2 - 2|S_{12}S_{21}|(|\Delta|^2 - |S_{22}|^2) + (|\Delta|^2 - |S_{22}|^2)^2$$

Simplifying, we obtain

$$-1 + |S_{11}|^2 < -2|S_{12}S_{21}| + |\Delta|^2 - |S_{22}|^2$$

or simply

$$K = \frac{1 - |S_{11}|^2 - |S_{22}|^2 + |\Delta|^2}{2|S_{12}S_{21}|} > 1$$

which is the same condition as (B.27). Thus far, we have shown that a two-port network is unconditionally stable when the conditions $K > 1$ and (B.29) are satisfied.

Next, we will show that the inequality (B.29) can be expressed as the inequality in (B.17). To show this, we write (B.27) in the form

$$2K = \frac{1 - |S_{11}|^2}{|S_{12}S_{21}|} + \frac{|\Delta|^2 - |S_{22}|^2}{|S_{12}S_{21}|} \tag{B.31}$$

Since from (B.29) the last term in (B.31) is less than 1, we can write

$$\frac{|\Delta|^2 - |S_{22}|^2}{|S_{12}S_{21}|} = 1 - \alpha \tag{B.32}$$

where α is a positive number smaller than 1. Substituting (B.32) into (B.31) gives

$$\frac{1 - |S_{11}|^2}{|S_{12}S_{21}|} = 2K - 1 + \alpha \tag{B.33}$$

Since $K > 1$, we conclude from (B.33) that

$$\frac{1 - |S_{11}|^2}{|S_{12}S_{21}|} > 1$$

or

$$|S_{12}S_{21}| < 1 - |S_{11}|^2 \tag{B.34}$$

We should also observe that since $|S_{12}S_{21}|$ is positive, the condition $|S_{11}| < 1$ is implied by (B.34).

Starting with (B.22) and (B.23), the derivation is similar, and we obtain the conditions

$$K > 1$$

and

$$\frac{|\Delta|^2 - |S_{11}|^2}{|S_{12}S_{21}|} < 1 \tag{B.35}$$

Following similar steps to those in (B.31) and (B.34), the inequality (B.35) can be expressed as

$$|S_{12}S_{21}| < 1 - |S_{22}|^2 \tag{B.36}$$

In conclusion, from (B.27), (B.34), and (B.36), the two-port network is unconditionally stable when

$$K > 1$$

$$|S_{12}S_{21}| < 1 - |S_{11}|^2$$

and

$$|S_{12}S_{21}| < 1 - |S_{22}|^2$$

OTHER DERIVATIONS

The stability criterion of active two-port networks has also been analyzed by other researchers working in active network theory. Their results are usually given in terms of *z*, *y*, or *h* parameters. For those who want to delve further into this topic the papers by Ku [B.3, B.4] and the textbook by Mitra [B.5] are recommended.

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APPENDIX C

UNCONDITIONAL STABILITY CONDITIONS: $K > 1$ AND $B_1 > 0$

From (3.3.17), we have

$$|\Delta| = |S_{11}S_{22} - S_{12}S_{21}| \leq |S_{11}S_{22}| + |S_{12}S_{21}|$$

and

$$|\Delta|^2 \leq |S_{11}S_{22}|^2 + 2|S_{11}S_{22}||S_{12}S_{21}| + |S_{12}S_{21}|^2$$

Then

$$\begin{aligned} B_1 &= 1 + |S_{11}|^2 - |S_{22}|^2 - |\Delta|^2 \\ &\geq 1 + |S_{11}|^2 - |S_{22}|^2 - |S_{11}S_{22}|^2 - 2|S_{11}S_{22}||S_{12}S_{21}| - |S_{12}S_{21}|^2 \end{aligned} \tag{C.1}$$

Using (3.3.15)—namely,

$$1 - |S_{22}|^2 > |S_{12}S_{21}|$$

we can express the inequality in (C.1) as

$$\begin{aligned} B_1 &> |S_{11}|^2 + |S_{12}S_{21}| - |S_{11}S_{22}|^2 - 2|S_{11}S_{22}||S_{12}S_{21}| - |S_{12}S_{21}|^2 \\ B_1 &> |S_{12}S_{21}|(1 - 2|S_{11}S_{22}| - |S_{12}S_{21}|) + |S_{11}|^2(1 - |S_{22}|^2) \\ B_1 &> |S_{12}S_{21}|[1 - 2|S_{11}S_{22}| - (1 - |S_{22}|^2)] + |S_{11}|^2|S_{12}S_{21}| \\ B_1 &> |S_{12}S_{21}|(-2|S_{11}S_{22}| + |S_{22}|^2 + |S_{11}|^2) \end{aligned}$$

or

$$B_1 > |S_{12}S_{21}|(|S_{22}|^2 - |S_{11}|^2)^2 \tag{C.4}$$

The right-hand side of (C.4) is greater than zero. Hence,

$$B_1 > 0$$

Similarly, using (3.3.14), it follows that

$$B_2 = 1 + |S_{22}|^2 - |S_{11}|^2 - |\Delta|^2 > 0$$

In Problem 3.13 it is shown that if $B_1 > 0$, then $B_2 > 0$, and vice versa. Hence the unconditional stability conditions in (3.3.13), (3.3.14), and (3.3.15) can be expressed as $K > 1$ and $B_1 > 0$, or as $K > 1$ and $B_2 > 0$.

APPENDIX D

DERIVATION OF THE UNILATERAL CONSTANT-GAIN CIRCLES [EQUATION (3.4.10)]

From (3.4.9),

$$g_i = \frac{1 - |\Gamma_i|^2}{|1 - S_{ii}\Gamma_i|^2}(1 - |S_{ii}|^2)$$

Then

$$g_i(1 + |S_{ii}\Gamma_i|^2 - S_{ii}\Gamma_i - S_{ii}^*\Gamma_i^*) = 1 - |\Gamma_i|^2 - |S_{ii}|^2 + |\Gamma_i|^2|S_{ii}|^2$$

Factoring $|\Gamma_i|^2$, we can write

$$|\Gamma_i|^2(1 - |S_{ii}|^2 + g_i|S_{ii}|^2) - g_i S_{ii}\Gamma_i - g_i S_{ii}^*\Gamma_i^* = 1 - g_i - |S_{ii}|^2$$

or

$$|\Gamma_i|^2 - \frac{g_i S_{ii}\Gamma_i}{1 - |S_{ii}|^2(1 - g_i)} - \frac{g_i S_{ii}^*\Gamma_i^*}{1 - |S_{ii}|^2(1 - g_i)} = \frac{1 - g_i - |S_{ii}|^2}{1 - |S_{ii}|^2(1 - g_i)} \qquad (D.1)$$

Comparing (D.1) with the circle equation (A.3), it follows that the center of the circle, denoted by C_{g_i}, is

$$C_{g_i} = \frac{g_i S_{ii}^*}{1 - |S_{ii}|^2(1 - g_i)} \qquad (D.2)$$

and the radius r_{g_i} follows from

$$r_{g_i}^2 - |C_{g_i}|^2 = \frac{1 - g_i - |S_{ii}|^2}{1 - |S_{ii}|^2(1 - g_i)} \qquad (D.3)$$

Substituting (D.2) into (D.3) and simplifying gives

$$r_{g_i} = \frac{\sqrt{1 - g_i}\,(1 - |S_{ii}|^2)}{1 - |S_{ii}|^2(1 - g_i)}$$

Hence, the constant-gain circles are given by

$$|\Gamma_i - C_{g_i}| = r_{g_i}$$