Economic Applications

- Traveler’s dilemma Game
- Beauty Contest
- Cournot Model of Duopoly
- Bertrand Model of Duopoly
- Final – Offer Arbitration
- The problem of the Commons

Traveler's Dilemma Game

Game: two players independently and simultaneously choose integer numbers between 180 and 300. Let be n_1 and n_2 the numbers chosen, respectively, by player 1 and player 2

Payoff: both players are paid the lower of the two numbers, and an amount $R > 1$ is transferred from the player with the higher number to the player with the lower number.

For instance:

player 1 chooses 210

player 2 chooses 250,

player 1 receives payoffs of 210 + R

player 2 receives 210 - R

Normal form game representation

Players: Player 1 and Player 2

Strategies: $S_1=\{180, 181, ..., 300\}$ i.e. $s_1 = n_1$

$S_2=\{180, 181, ..., 300\}$ i.e. $s_2 = n_2$

Payoff:

$$\pi_1 = n_1 + R \quad if\ n_1 < n_2$$
$$\pi_1 = n_1 \quad if\ n_1 = n_2$$
$$\pi_1 = n_2 - R \quad if\ n_1 > n_2$$

$$\pi_2 = n_2 + R \quad if\ n_2 < n_1$$
$$\pi_2 = n_2 \quad if\ n_2 = n_1$$
$$\pi_2 = n_1 - R \quad if\ n_2 > n_1$$

Solution

Best responses of player 1

If $n_1 > n_2$ $\pi_1 = n_2 - R$

If $n_1 = n_2$ $\pi_1 = n_2$

If $n_1 < n_2$ $\pi_1 = n_1 + R$

If $n_1 = n_2 - x$ $\pi_1 = n_2 - x + R$

To play $n_1 = n_2$ is strictly better than $n_1 > n_2$ because $n_2 > n_2 - R$

To play $n_1 < n_2$ is strictly better that $n_1 = n_2$ if $x < R$ because in this case $n_2 - x + R > n_2$

Given that payoff is decreasing in x the best response is

$$n_1 = n_2 - 1$$

Repeating this reasoning for player 2 we find that player 2's best response is:

$$n_2 = n_1 - 1$$

There is an unique Nash equilibrium:

$$n_1{}^* = n_2{}^* = 180$$

The payoff is $\pi_1{}^* = \pi_2{}^* = 180$

Suppose that a player plays an higher number, his payoff reduces to $180 - R$. Then there are not profitable deviations.

So the strategy profile $n_1{}^* = n_2{}^* = 180$ is Nash equilibrium

There are other Nash equilibria?

No because the best response is to pay one unit less than the opponent

Suppose player 1 plays $n_1 = 300$

The best response of player 2 to $n_1 = 300$ is $n_2 = 299$

The best response of player 1 to $n_2 = 299$ is $n_1 = 298$

The best response of player 2 to $n_1 = 298$ is $n_2 = 297$

……. repeating we get:

The best response of player 2 to $n_1 = 182$ is $n_2 = 181$

The best response of player 1 to $n_2 = 181$ is $n_1 = 180$

The best response of player 2 to $n_1 = 180$ is $n_2 = 180$

The best response of player 1 to $n_2 = 180$ is $n_1 = 180$

Starting with Player 2 we get the same result

Beauty contest

In the *p*-beauty contest game *n* participants are asked to simultaneously submit a number between 0 and 100.

The winner of the contest is the person(s) whose number is closest to p times the average of all numbers submitted

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Normal form game representation

Players: n individuals denoted by $i \in \{1,\ 2, \dots, n\}$

Strategies: $s_i \in S_i = \{0, 1, \dots, 100)\ \forall i \in \{1,\ 2, \dots, n\}$

Payoff:

$\pi_i = k > 0$ if s_i is the closest to $p\frac{\sum_i s_i}{n}$ otherwise $\pi_i = 0$.

Solution

Best responses of player i is to submit a number that is equal to p times the average of all submitted numbers, i.e.:

$$s_i = \frac{\sum_{j=1}^{n} s_j}{n} p$$

Then her best response is

$$s_i = \frac{p \sum_{j \neq i} s_j}{n - p}$$

best response is $s_i = \frac{p \sum_{j \neq i} s_j}{n-p}$

Suppose $p < 1$

The best response is to play a number that is smaller than p times the average of the others' numbers

1. A strategy profile where there are at least two players playing different numbers is not a Nash equilibrium. Indeed the player with the number above p times the average has an incentive to play a smaller number.

2. A strategy profile where all players play the same number s and $s > 0$ is not a NE. Any player has an incentive to play a small number

3. A strategy profile where all players play the same number s=0 is the unique NE

Cournot Model of Duopoly

- q_1 and q_2 are the quantities of an homogeneous product produced by firms 1 and 2
- linear inverse demand $P(Q) = a - Q$ for $Q < a$

  and $P(Q) = 0$ if $Q \geq a$
- total quantity $Q = q_1 + q_2$
- The cost to produce q_i is $C_i(q_i) = c\, q_i$
- Firms choose their quantities simultaneously.

Normal form game representation

Players: Firm 1 and Firm 2

Strategies: $S_1=[0, \infty)$ i.e. $s_1 = q_1$

$S_2=[0, \infty)$ i.e. $s_2 = q_2$

Payoff: $\pi_1 = q_1 P(Q) - C_1(q_1)$

$\pi_2 = q_2 P(Q) - C_2(q_2)$

replacing inverse demand and cost functions, we have:

$$\pi_1 = q_1(a - q_1 - q_2) - c\, q_1$$

$$\pi_2 = q_2(a - q_1 - q_2) - c\, q_2$$

Solution: Nash Equilibrium

- Let be q_1^* *and* q_2^* the quantities produced in a NE
- *In a Nash equilibrium each player strategy is a best response to the other players' strategies*
- We look for the best response function of firm 1 to q_2^* that is given by the solution of the following problem:

$$\max_{q_1 \geq 0} q_1 \left(a - c - q_1 - q_2^* \right)$$

The FOC are $q_1 = \dfrac{a - c - q_2^*}{2}$

- In similar way we find the best response function of firm 2 to q_1^*:

$$q_2 = \frac{a - c - q_1^*}{2}$$

- $q_1^* \; and \; q_2^*$ are Nash equilibrium if

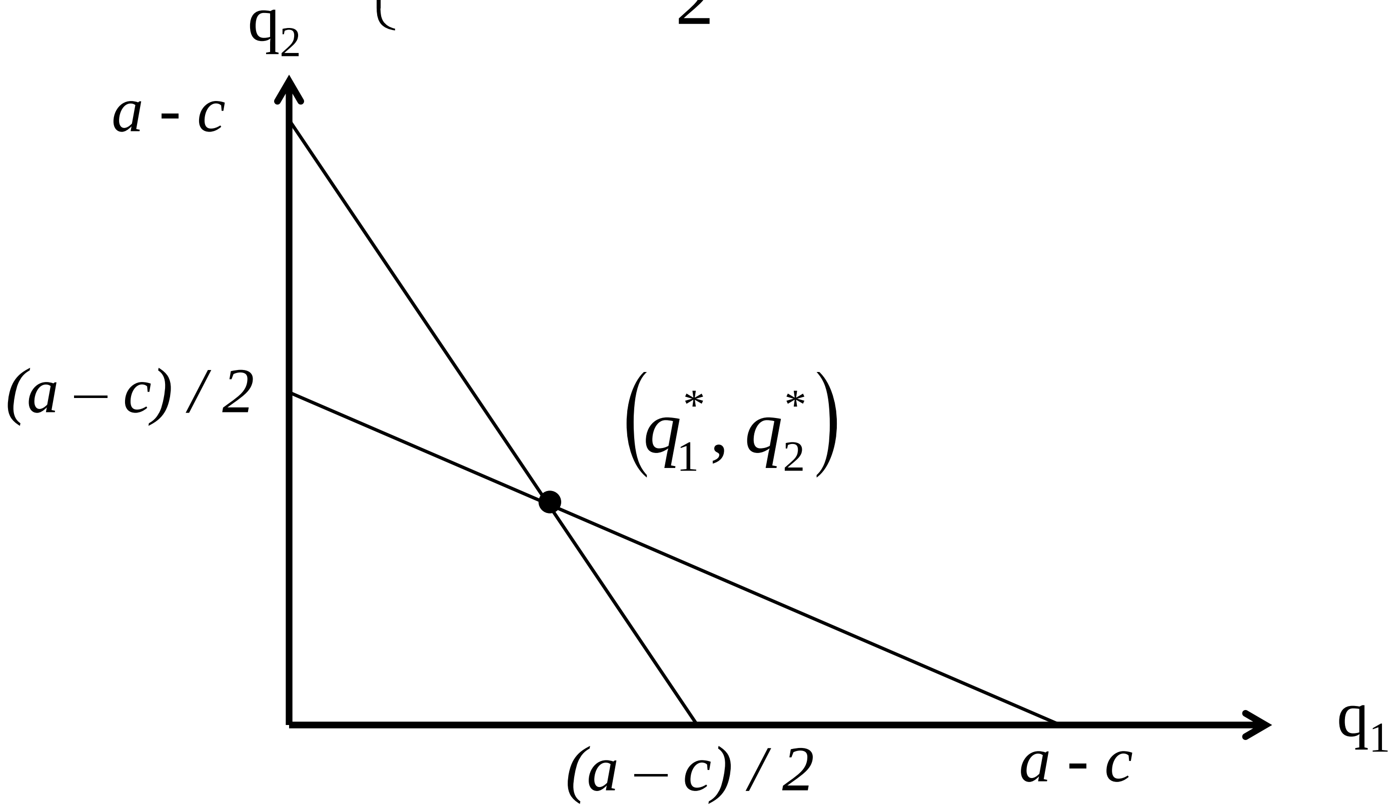
$$\begin{cases} q_1^* = \dfrac{a - c - q_2^*}{2} \\ q_2^* = \dfrac{a - c - q_1^*}{2} \end{cases}$$

- Solving the system we get:

$$q_1^* = q_2^* = \frac{a-c}{3}$$

- Alternatively we could consider the best response of firm 1 (firm 2) to an arbitrary strategy of firm 2 (firm 1)

$$\begin{cases} q_1 = \dfrac{a-c-q_2}{2} & \text{Best response of firm 1} \\[2ex] q_2 = \dfrac{a-c-q_1}{2} & \text{Best response of firm 2} \end{cases}$$



Bertrand Model of Duopoly

- We consider the case of differentiated products
- p_1 and p_2 are the prices of two slight differentiated goods produced respectively by firms 1 and 2 (goods are substitutes)
- Simultaneously each firm chooses a price and satisfies all the demand at that price
- The demands are
  - for firm 1: $q_1(p_1, p_2) = a - p_1 + b\, p_2$
  - for firm 2: $q_2(p_1, p_2) = a - p_2 + b\, p_1$
  - b (< 2)reflects the level of substitutability between the two goods
- No fixed cost, constant marginal cost c ($< a$)

Normal form game representation

Players: Firm 1 and Firm 2

Strategies: $S_1=[0, \infty)$ i.e. $s_1 = p_1$

$S_2=[0, \infty)$ i.e. $s_2 = p_2$

Payoff: $\pi_1 = q_1(p_1, p_2)\ [p_1 - c]$

$\pi_2 = q_2(p_1, p_2)\ [p_2 - c]$

replacing demand function, we have:

$\pi_1 = (a - p_1 + b\, p_2)\ [p_1 - c]$

$\pi_2 = (a - p_2 + b\, p_1)\ [p_2 - c]$

Solution: Nash Equilibrium

- Let be p_1^* *and* p_2^* the prices in a NE
- *In a Nash equilibrium each player strategy is a best response to the other players' strategies*
- We look for the best response of firm 1 to p_2^* that is given by the solution of the following problem:

$$\max_{p_1 \geq 0} \left(a - p_1 + bp_2^*\right) \cdot \left(p_1 - c\right)$$

The FOC are $p_1 = \dfrac{a + c + bp_2^*}{2}$

- In similar way we find the best function of firm 2 to p_1^*:

$$p_2 = \frac{a + c + bp_1^*}{2}$$

- p_1^* *and* p_2^* are Nash equilibrium if

$$\begin{cases} p_1^* = \dfrac{a+c+bp_2^*}{2} \\ p_2^* = \dfrac{a+c+bp_1^*}{2} \end{cases}$$

- Solving the system we get:

$$p_1^* = p_2^* = \frac{a+c}{2-b}$$

Final – Offer Arbitration

- Two types of arbitration: Final - Offer and Conventional
  - Final - offer: the two sides make offers and then the arbitrator picks one as settlement
  - Conventional : the arbitrator is free to impose any settlement.

- Suppose the following case of final – offer arbitration:
  - A firm and a union dispute about wages
  - Firm likes low wages as possible
  - Union likes high wages as possible
  - Firm and union simultaneously make offers, w_f and w_u.

– Arbitrator has an ideal settlement, denoted by *x*, and she/he chooses the offer that is closer to *x* (as settlement):

Arbitrator chooses: $\min\{w_f, w_u\}$ if $x < (w_f + w_u) / 2$

$\max\{w_f, w_u\}$ if $x > (w_f + w_u) / 2$

– Arbitrator knows *x*

– Firm and union don't know *x*, they know that *x* is randomly distributed according a cumulative probability distribution $F(x)$.

Normal form game representation

Players: Firm and Union

Strategies: $S_{\mathrm{f}}=[0, \infty)$ i.e. $s_{\mathrm{f}} = w_{\mathrm{f}}$

$S_{\mathrm{u}}=[0, \infty)$ i.e. $s_{\mathrm{u}} = w_{\mathrm{u}}$

Payoff: $\pi_u = w$

$\pi_f = K - w$ where K is a positive number

Solution: Nash Equilibrium

We look for Firm and Union best responses

For the firm all offers $w_f > w_u$ never are a best response

For the union all offers $w_u < w_f$ never are a best response

Proof

Consider the firm and an offer $w_f > w_u$

The expected payoff is $- w_f\, p - w_u\, (1 - p)$, where p is some probability depending on the offers and $F(x)$

Note that $- w_f\, p - w_u\, (1 - p) < - w_u$

Note that $w_f > w_u$ cannot be a best response to w_u because by $w_f < w_u$ $\quad - w_f\, p' - w_u\, (1 - p') > - w_u$

For the union the proof follows similar steps. ■

It follows that:

for the firm, the best response to w_u has to be $w_f \leq w_u$

for the union, the best response to w_f has to be $w_u \geq w_f$

Therefore we concentrate our attention on the case $w_f \leq w_u$

Arbitrator chooses: w_f if $x < \frac{w_f + w_u}{2}$

w_u if $x > \frac{w_f + w_u}{2}$

Then:

$$\Pr(w_f) = \Pr\left(x < \frac{w_f + w_u}{2}\right) = F\left(\frac{w_f + w_u}{2}\right)$$

$$\Pr(w_u) = \Pr\left(x > \frac{w_f + w_u}{2}\right) = 1 - F\left(\frac{w_f + w_u}{2}\right)$$

The expected wage settlement is:

$$E(w) = w_f \Pr(w_f) + w_u \Pr(w_u)$$

$$= w_f F\left(\frac{w_f + w_u}{2}\right) + w_u\left(1 - F\left(\frac{w_f + w_u}{2}\right)\right)$$

The Firm problem is:

$$\min_{w_f} w_f F\left(\frac{w_f + w_u}{2}\right) + w_u\left(1 - F\left(\frac{w_f + w_u}{2}\right)\right)$$

The FOC are

$$F\left(\frac{w_f + w_u}{2}\right) + w_f\, f\left(\frac{w_f + w_u}{2}\right)\frac{1}{2} - w_u\, f\left(\frac{w_f + w_u}{2}\right)\frac{1}{2} = 0$$

$$F\left(\frac{w_f + w_u}{2}\right) = \left(w_u - w_f\right) f\left(\frac{w_f + w_u}{2}\right)\frac{1}{2}$$

The Union problem is:

$$\max_{w_u} w_f F\left(\frac{w_f + w_u}{2}\right) + w_u\left(1 - F\left(\frac{w_f + w_u}{2}\right)\right)$$

The FOC are:

$$w_f f\left(\frac{w_f + w_u}{2}\right)\frac{1}{2} + 1 - F\left(\frac{w_f + w_u}{2}\right) - w_u f\left(\frac{w_f + w_u}{2}\right)\frac{1}{2} =$$

$$1 - F\left(\frac{w_f + w_u}{2}\right) = \left(w_u - w_f\right) f\left(\frac{w_f + w_u}{2}\right)\frac{1}{2}$$

Let $\left(w_f^* \; w_u^*\right)$ be a Nash equilibrium, then both FOCs must be satisfied, then:

$$F\left(\frac{w_f^*+w_u^*}{2}\right)=\left(w_u^*-w_f^*\right)f\left(\frac{w_f^*+w_u^*}{2}\right)\frac{1}{2}$$

$$1-F\left(\frac{w_f^*+w_u^*}{2}\right)=\left(w_u^*-w_f^*\right)f\left(\frac{w_f^*+w_u^*}{2}\right)\frac{1}{2}$$

Note, the RHSs are equal, then:

$$1-F\left(\frac{w_f^*+w_u^*}{2}\right)=F\left(\frac{w_f^*+w_u^*}{2}\right)$$

It implies that

$$F\left(\frac{w_f^*+w_u^*}{2}\right)=0.5$$

Finally $F\left(\frac{w_f^* + w_u^*}{2}\right) = 0.5$ implies that

$\frac{w_f^* + w_u^*}{2} = m$ where m is the median of x (the arbitrator ideal settlement)

Replacing in the FOCs we get:

$$\frac{1}{2} = \left(w_u^* - w_f^*\right) f(m) \frac{1}{2}$$

$$w_u^* - w_f^* = \frac{1}{f(m)}$$

Finally all Nash equilibria must satisfy:

$$\frac{w_f^* + w_u^*}{2} = m \quad w_u^* - w_f^* = \frac{1}{f(m)}$$

The problem of the Commons

n farmer in a village graze their goats on the village green.

g_i is the number of goats of the i^{th} farmer

The total number of goats is denote by $G = g_1 + \ldots + g_n$

c is the cost of a goat

Value of a goat is $v(G)$ where $v' < 0$, $v'' < 0$ and

$v(G) > 0$ if $G < G_{max}$.

During the spring farmers simultaneously choose how many goats to own.

Normal form game representation

Players: *n* farmers

Strategies:

i^{th} player's set of strategy is $S_{\mathrm{i}}=[0, \infty)$ i.e. $s_{\mathrm{i}} = g_{\mathrm{i}}$

Payoff:

i^{th} player's payoff is $\pi_i = g_{\mathrm{i}}\, V(G) - c\, g_{\mathrm{i}}$

Solution: Nash Equilibrium

$\left(g_1^*,....g_n^*\right)$ is a Nash equilibrium if every g_i^* is the solution to the following farmer's problem:

$$\max_{\{g_i\}} g_i \cdot v\left(g_1^* + ... + g_i + ... + g_n^*\right) - g_i \cdot c$$

The FOC are:

$$v\left(g_1^* + ... + g_i + ... + g_n^*\right) + g_i \cdot v'\left(g_1^* + ... + g_i + ... + g_n^*\right) - c = 0$$

Then in a Nash equilibrium must be:

$$v\left(g_1^* + ... + g_i^* + ... + g_n^*\right) + g_i \cdot v'\left(g_1^* + ... + g_i^* + ... + g_n^*\right) - c = 0$$

for all i.

Denoting by G* the total number of goats in equilibrium, for every *i* the FOC is written as:

$$v\left(G^*\right)+g_i\cdot v'\left(G^*\right)-c=0$$

Summing up all *n* FOCs we have

$$n\cdot v\left(G^*\right)+G^*\cdot v'\left(G^*\right)-n\cdot c=0$$

$$v\left(G^*\right)+\frac{G^*}{n}\cdot v'\left(G^*\right)-c=0$$

The social optimum G^{**} is given by the solution of the following problem:

$$\max_{\{G\}} G \cdot v(G) - G \cdot c$$

The FOC is:

$$v(G^{**}) + G \cdot v'(G^{**}) - c = 0$$

Then in The Nash equilibrium farmers choose to buy more goats that the social optimum.