Lecture 4

Dynamic games of complete information

Dynamic games

- The strategic form of a game does not represent the timing of moves
- Hence plans of actions are fixed and cannot be changed
- In contrast, **dynamic games** capture the sequential structure of a game

Dynamic games of complete information

- For now, we consider extensive form games with **complete information,** i.e. the utility function (or the preferences) of each player is common knowledge

Extensive – Form Representation

An extensive form representation of a game specifies:

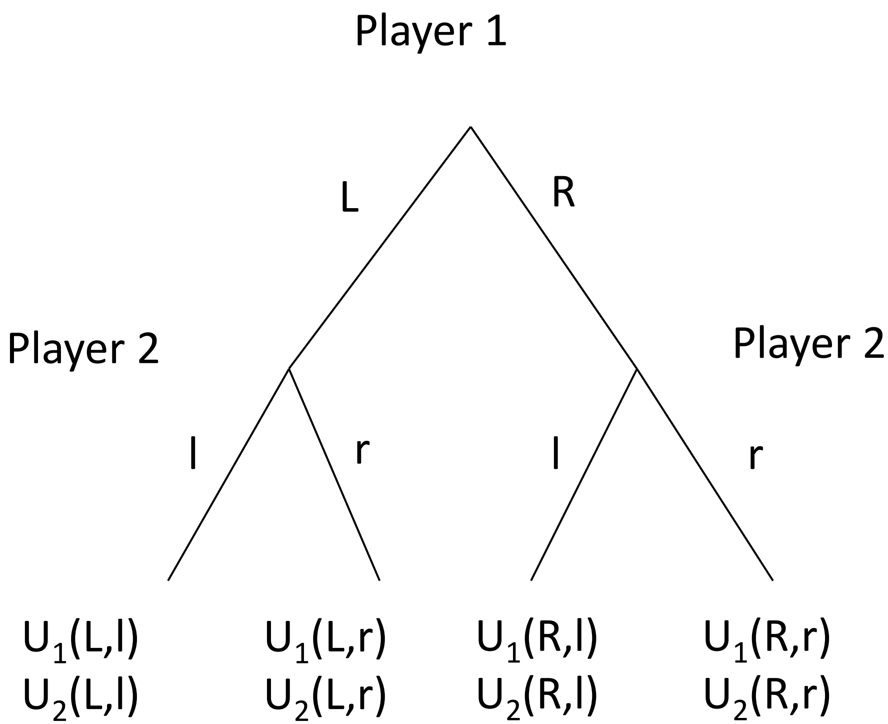
- Players
- When each player has to move
- The actions a player can use at each of his opportunities to move
- What a player knows at each of his opportunities to move
- Payoffs received by each player for each possible outcome

Game Trees

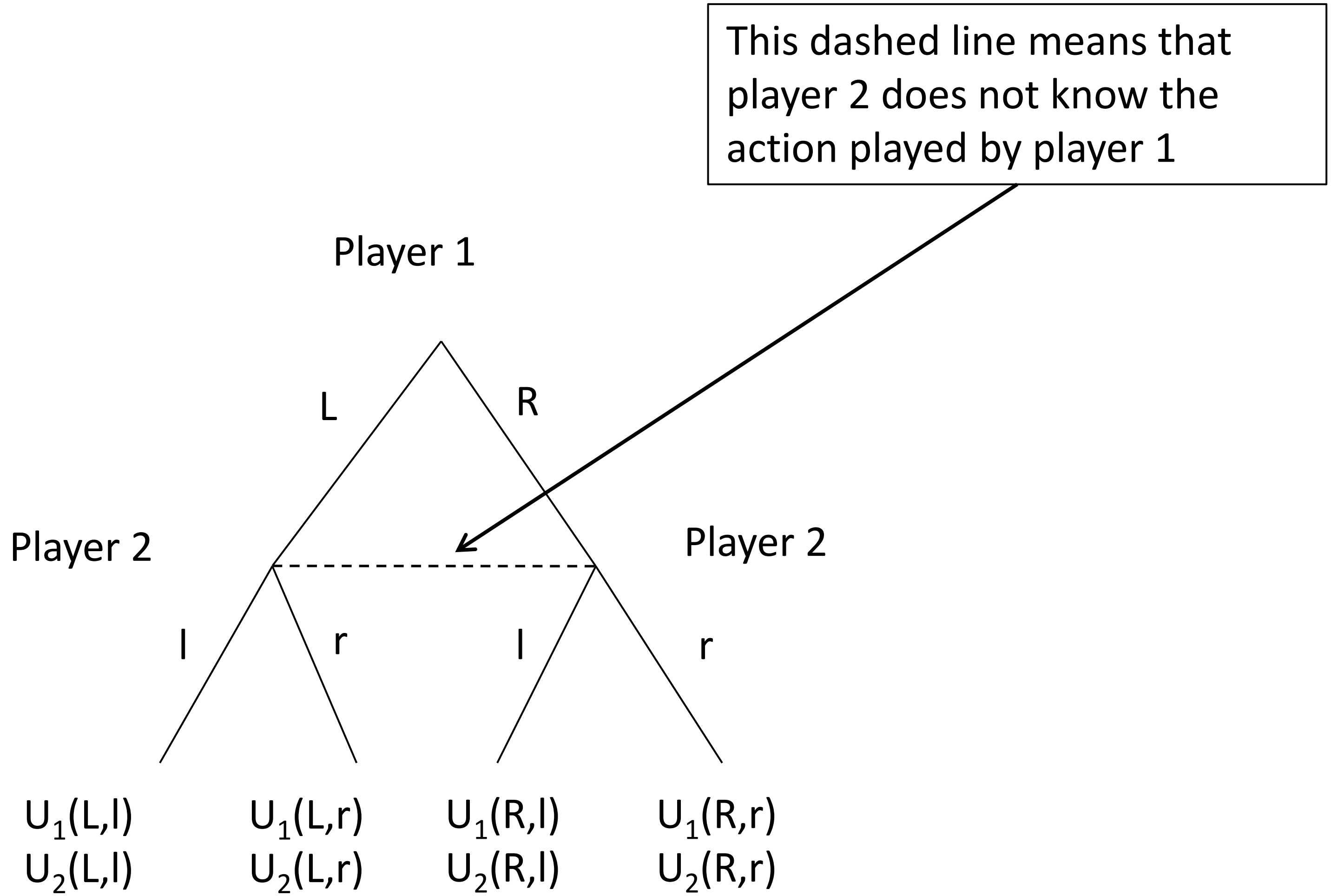
- An extensive form game can be represented in a **game tree**
- This shows
  - who moves when (at the ***nodes***)
  - their available actions (the ***branches***)
  - Their available information
  - and the payoffs over all possible outcomes (at the ***terminal nodes***)

Example 1

The preferences can be represented by a payoff function over the outcomes



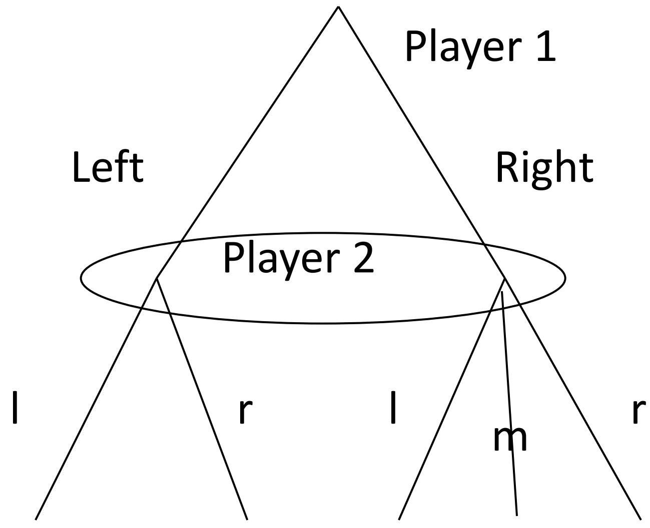
Example 2


This dashed line means that player 2 does not know the action played by player 1
Player 1
L
R
Player 2
Player 2
l
r
l
r
U1(L,l)
U2(L,l)
U1(L,r)
U2(L,r)
U1(R,l)
U2(R,l)
U1(R,r)
U2(R,r)

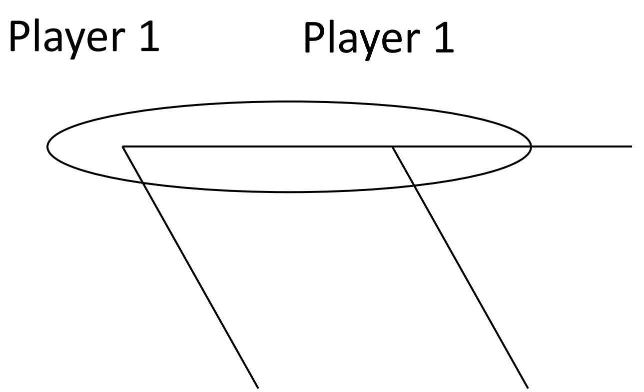
Information set

- It is a collection of decision nodes where:
  - The player has to move at every node in the information set
  - When a player has to move, he cannot distinguish the nodes belonging to the same information set
  - Example 1: player 1 has one info set, player 2 has two info sets
  - Example 2: player 1 has one info set, player 2 has one info set

Note: What can an info set NOT look like



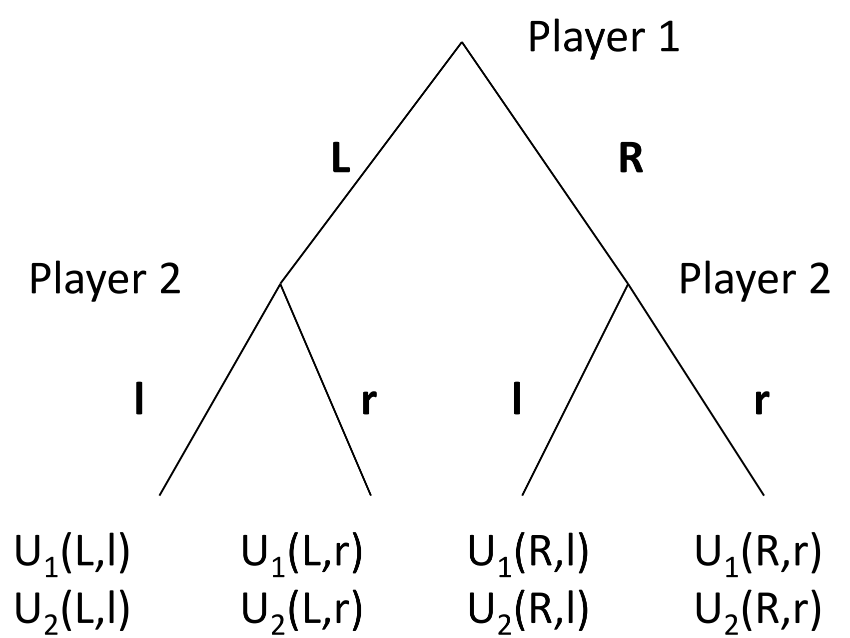
The two nodes in the information set have different number of available actions, then player 2 can distinguish the node



This could be true only assuming that player 1 does not remember his move in the first node

Strategies

- A **strategy** is a complete description of a player's actions at all the information sets when it's his turn to move, e.g.
  - for player 2 to choose **r** after **L** and **l** after **R**, i.e. (r, l) .
  - Player 2 has 4 strategies: {(l,l),(l,r),(r,l),(r,r)}



Definition of subgame:

A subgame starts at an information set with a single node ***n***

- it contains all decision and terminal nodes following ***n***
- an information set cannot belong to two different subgames

Note: someone considers the whole game a subgame, others do not consider the whole game a subgame.

In the following we use the first approach

Dynamic games of perfect and imperfect information

- **perfect information,** i.e. when choosing an action a player knows the actions chosen by players moving before her

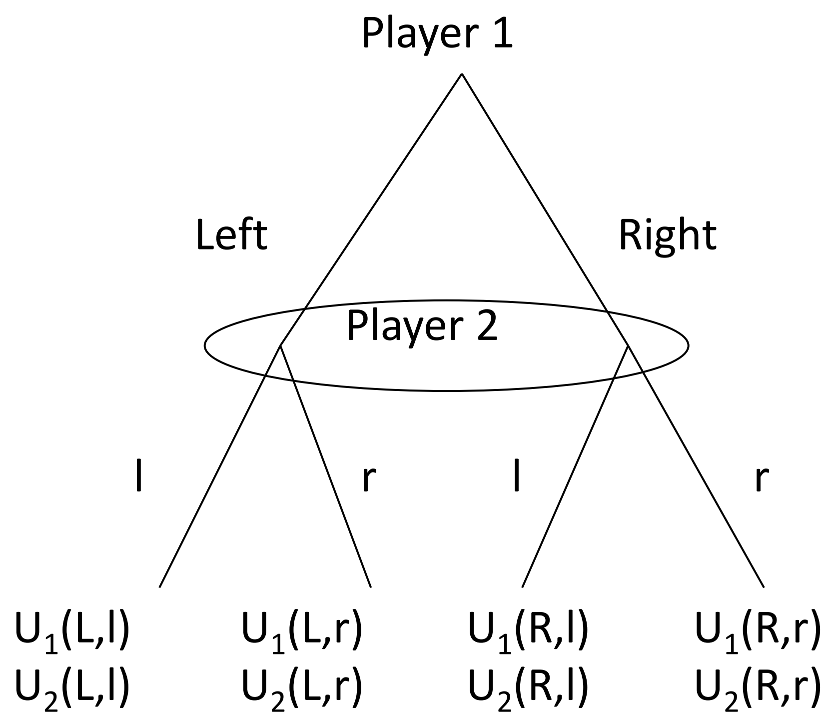
  i.e. all previous moves are observed before the next move is chosen

- **Imperfect information** when at least one player does not know the history by the time he chooses.

  At least one player does not know all the actions chosen by players moving before her

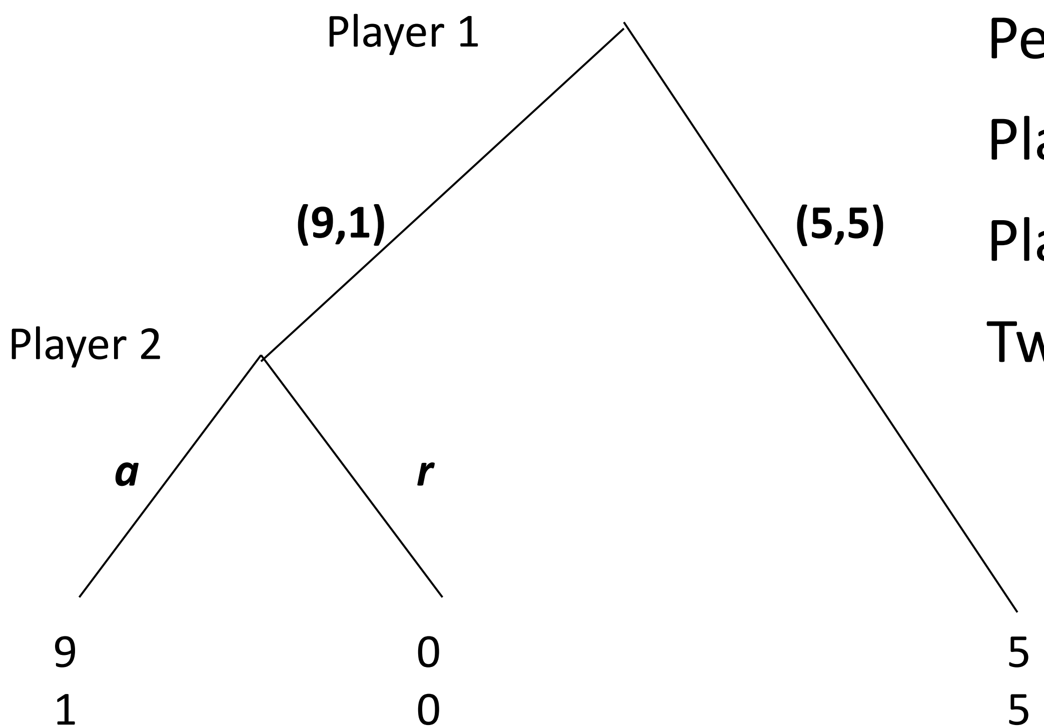
In other words, when a game is of imperfect info, there exists at least an information set with more than one decision node

Player 2 knows that he is in the information set, but not in which specific node



Example 1: Mini Ultimatum Game

- Proposer (Player 1) can suggest one of two splits of £10: (5,5) and (9,1).
- Responder (Player 2) can decide whether to accept or reject (9,1), but has to accept (5,5). Reject leads to 0 for both



Perfect information

Player 1 has one information set

Player 2 has one information set

Two subgames

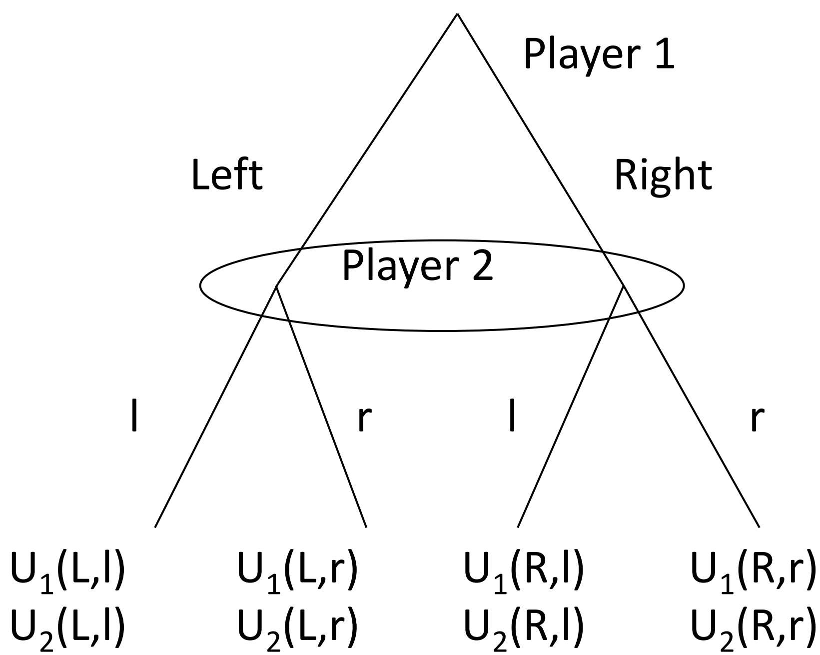
Example 2

Imperfect information

Player 1 has one information set

Player 2 has one information set

One subgame



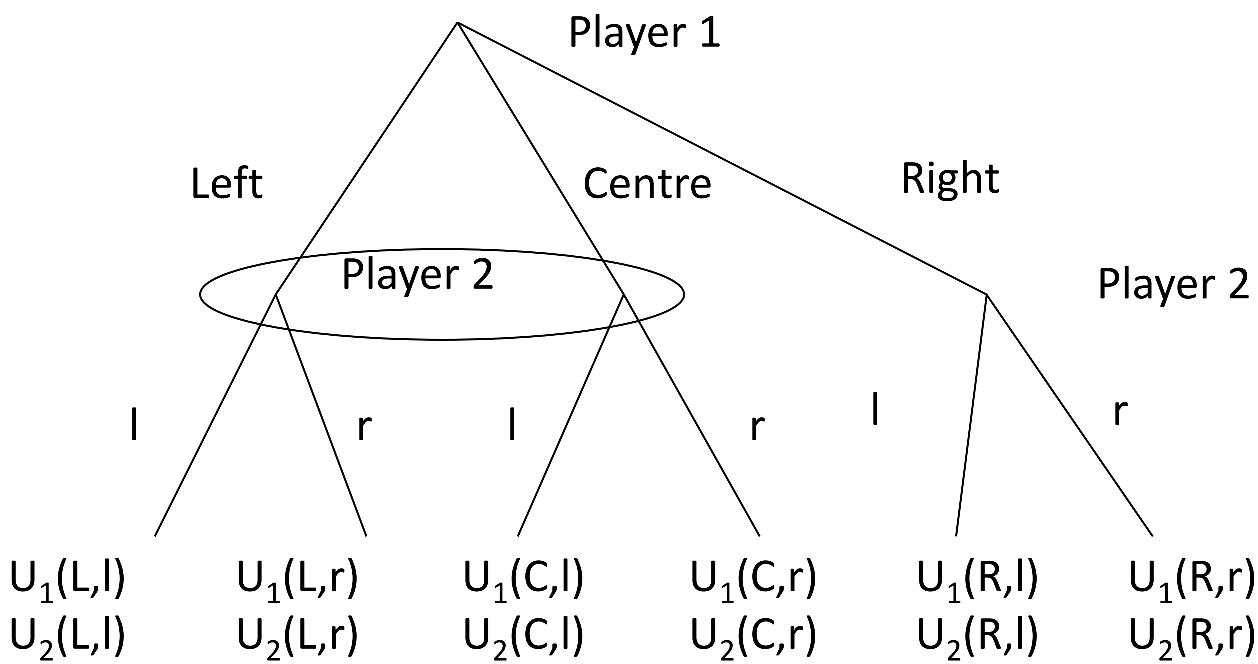
Example 3

Imperfect information

Player 1 has one information set

Player 2 has two information sets

Two subgames



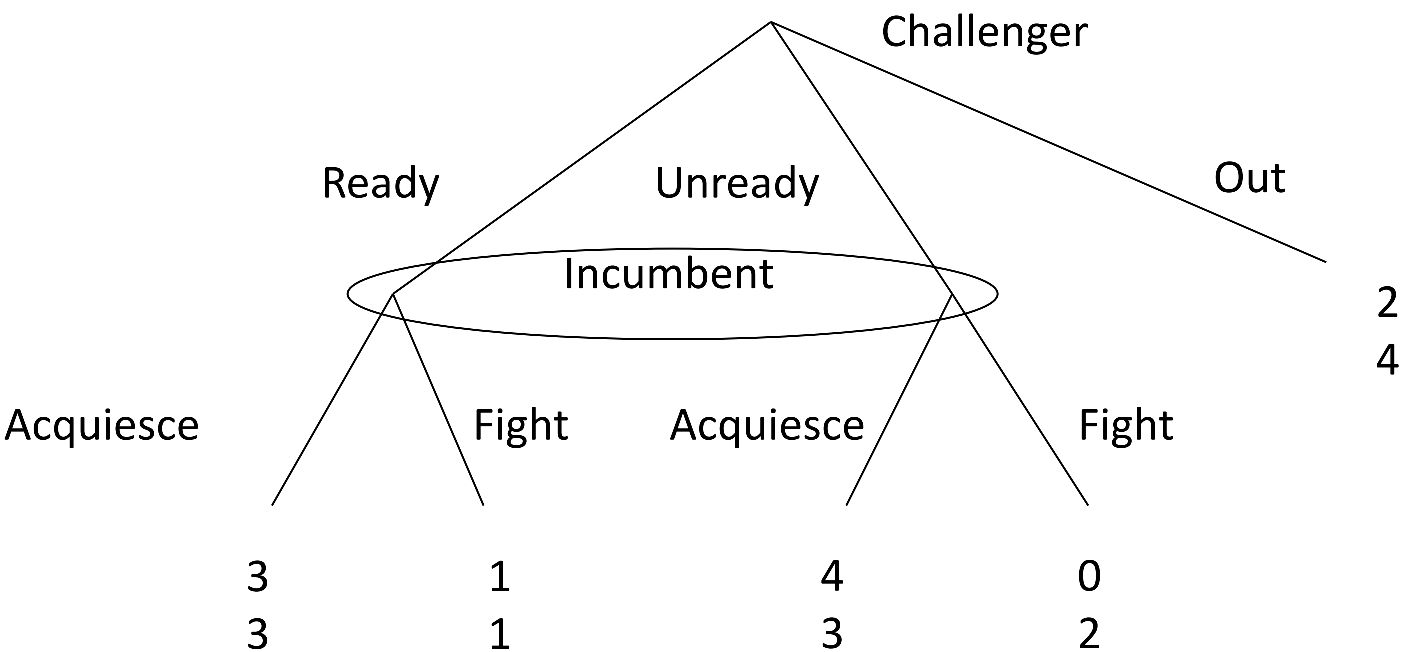
Example 4

Imperfect information

Challenger: one information set

Incumbent: one information set

One subgame



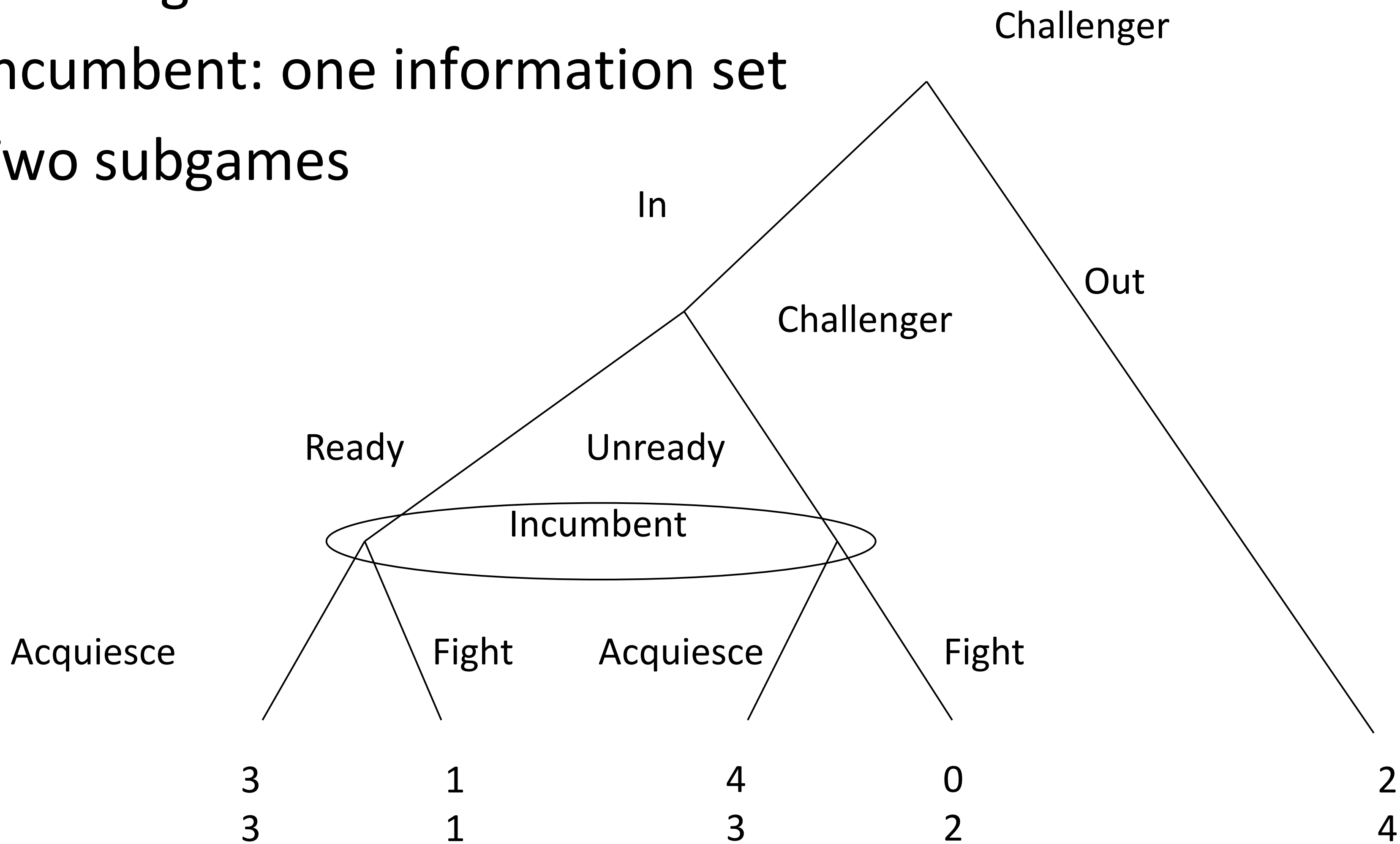
Example 5

Imperfect information

Challenger: two information sets

Incumbent: one information set

Two subgames



Representation of a sequential game using the normal form

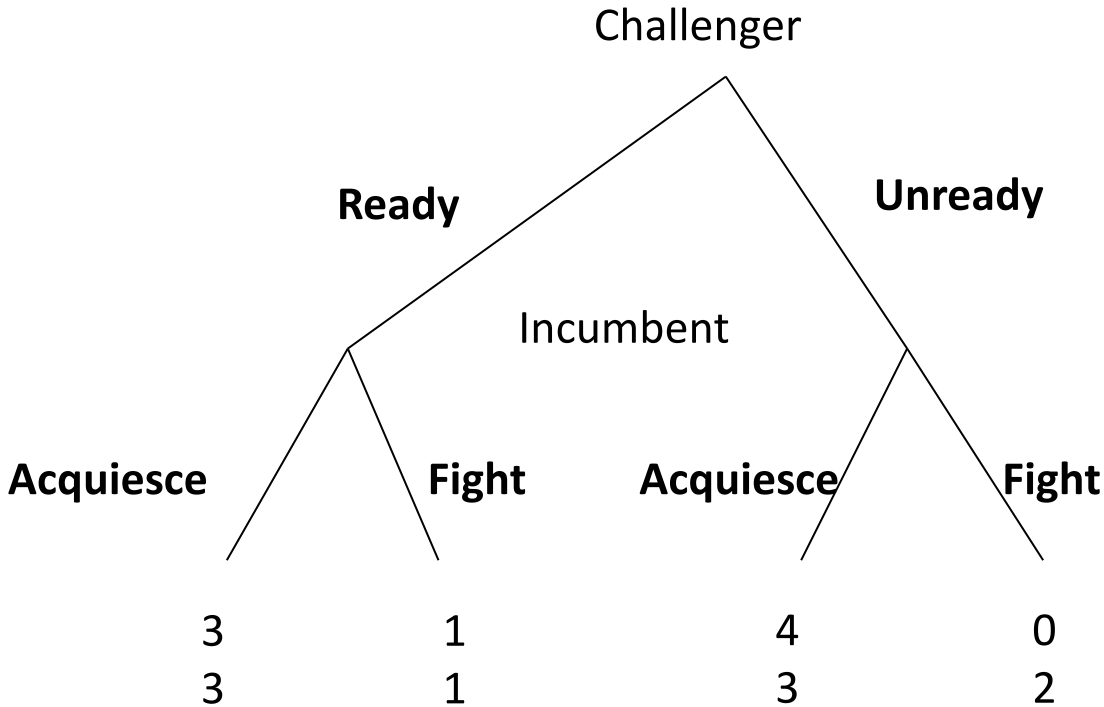
Case of two players: 1 and 2

Label the rows of the normal form with the player 1's strategies

Label the columns of the normal form with the player 2's strategies

Compute the payoffs to the players for each possible combination of strategies

<u>Using The normal form representation is possible to find all Nash equilibrium</u>



Player 1'strategies: {Ready, Unready}

Player 2'strategies: {(A, A), (A, F), (F, A), (F, F)}

Player 1’strategies: {Ready, Unready}

Player 2’strategies: {(A, A), (A, F), (F, A), (F, F)}

		Incumbent			
		(A, A)	(A, F)	(F, A)	(F, F)
Challenger	Ready	3, 3	3, 3	1, 1	1, 1
	Unready	4, 3	0, 2	4.3	0, 2

Three Nash equilibria

1. Ready, (A, F)
2. Unready, (A, A)
3. Unready, (F, A)

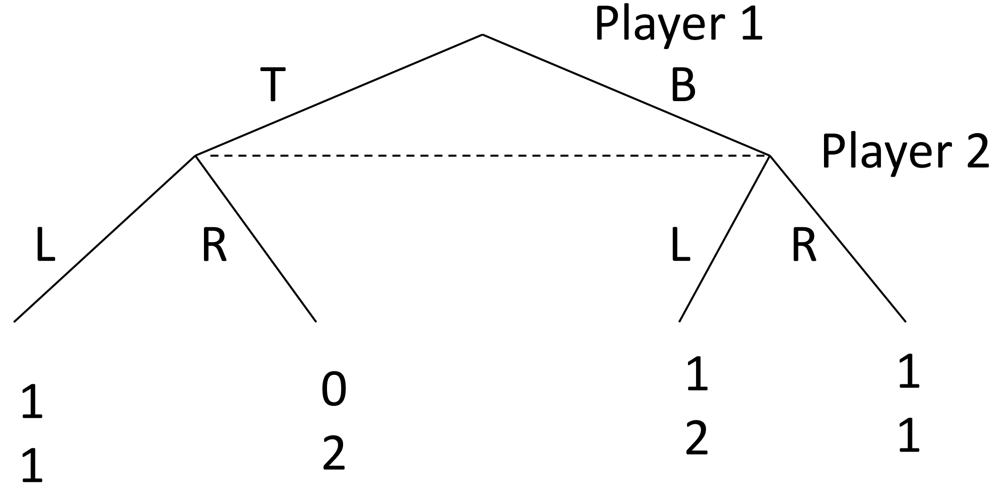
Representation of a static game using the extensive form

A simultaneously game is equivalent to a sequential game where the second player cannot observe the first player's move.

Consider the game:

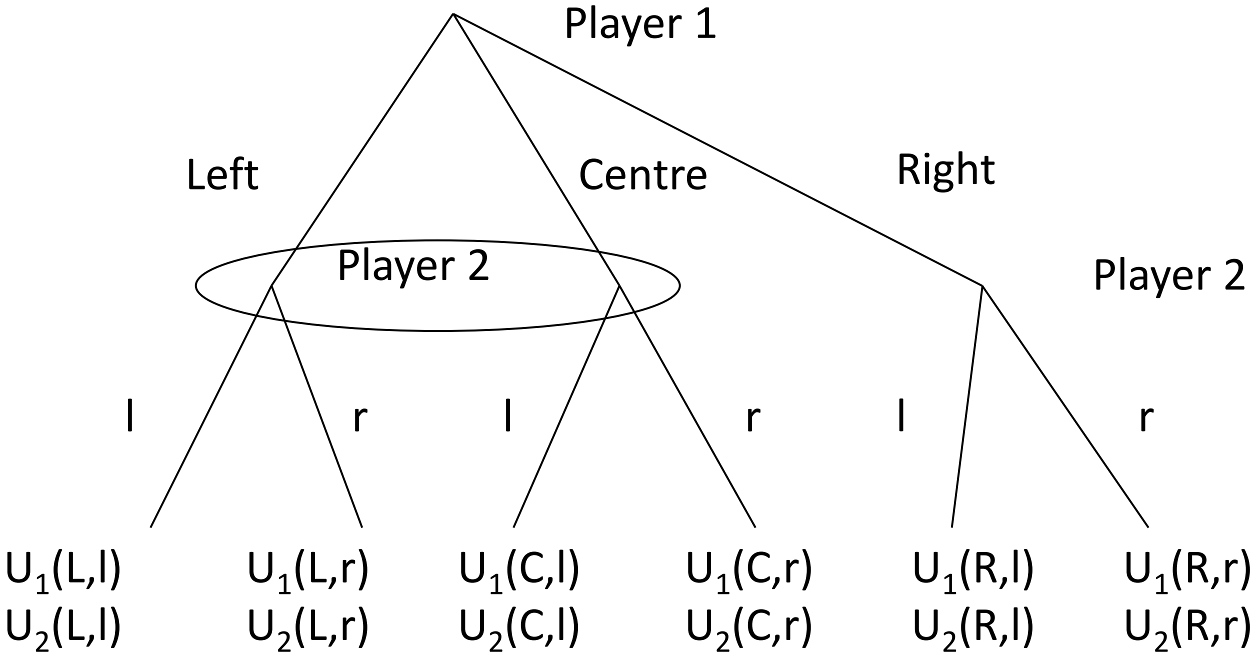
		Player 2	
		L	R
Player 1	T	1,1	0, 2
	B	1,2	1,1

It is equivalent to



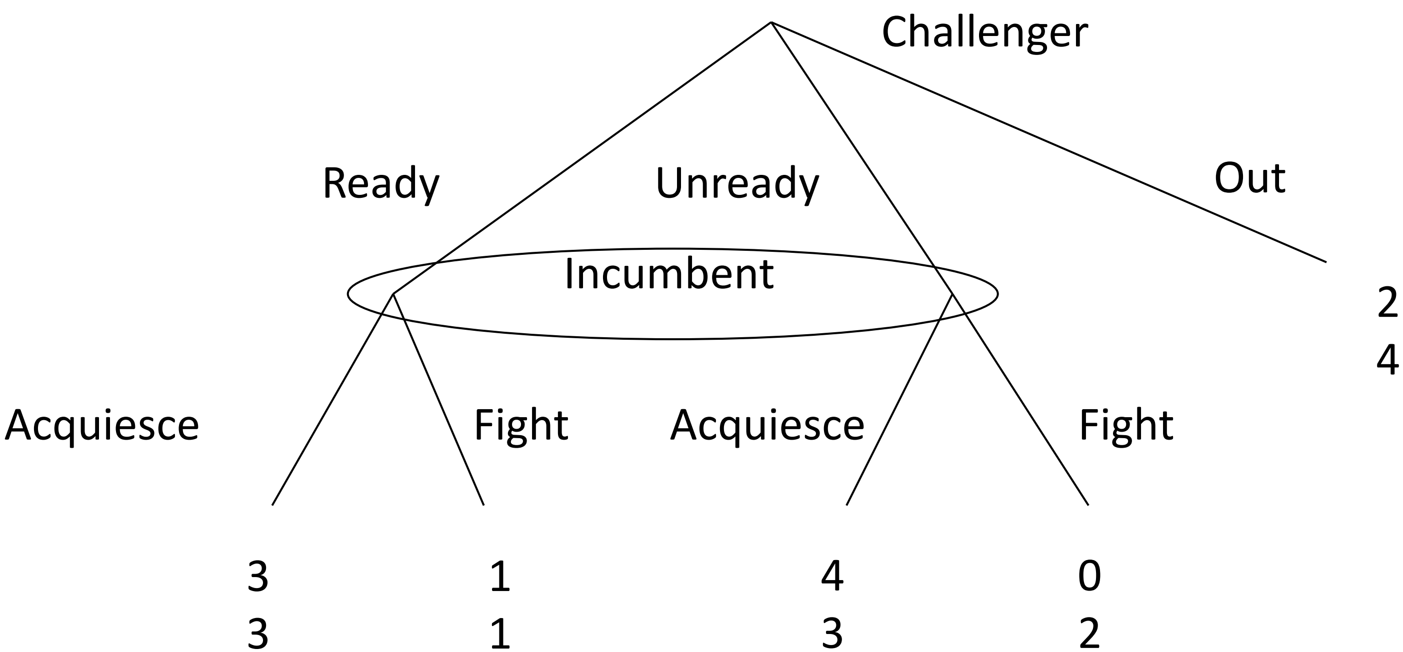
Example 3

		Player 2			
		l, l	*l, r*	*r, l*	*r, r*
Player 1	Left	$U_1(L,l)$, $U_2(L,l)$	$U_1(L,l)$, $U_2(L,l)$	$U_1(L,r)$, $U_2(L,r)$	$U_1(L,r)$, $U_2(L,r)$
	Centre	$U_1(C,l)$, $U_2(C,l)$	$U_1(C,l)$, $U_2(C,l)$	$U_1(C,r)$, $U_2(C,r)$	$U_1(C,r)$, $U_2(C,r)$
	Right	$U_1(R,l)$, $U_2(R,l)$	$U_1(R,r)$, $U_2(R,r)$	$U_1(R,l)$, $U_2(R,l)$	$U_1(R,r)$, $U_2(R,r)$



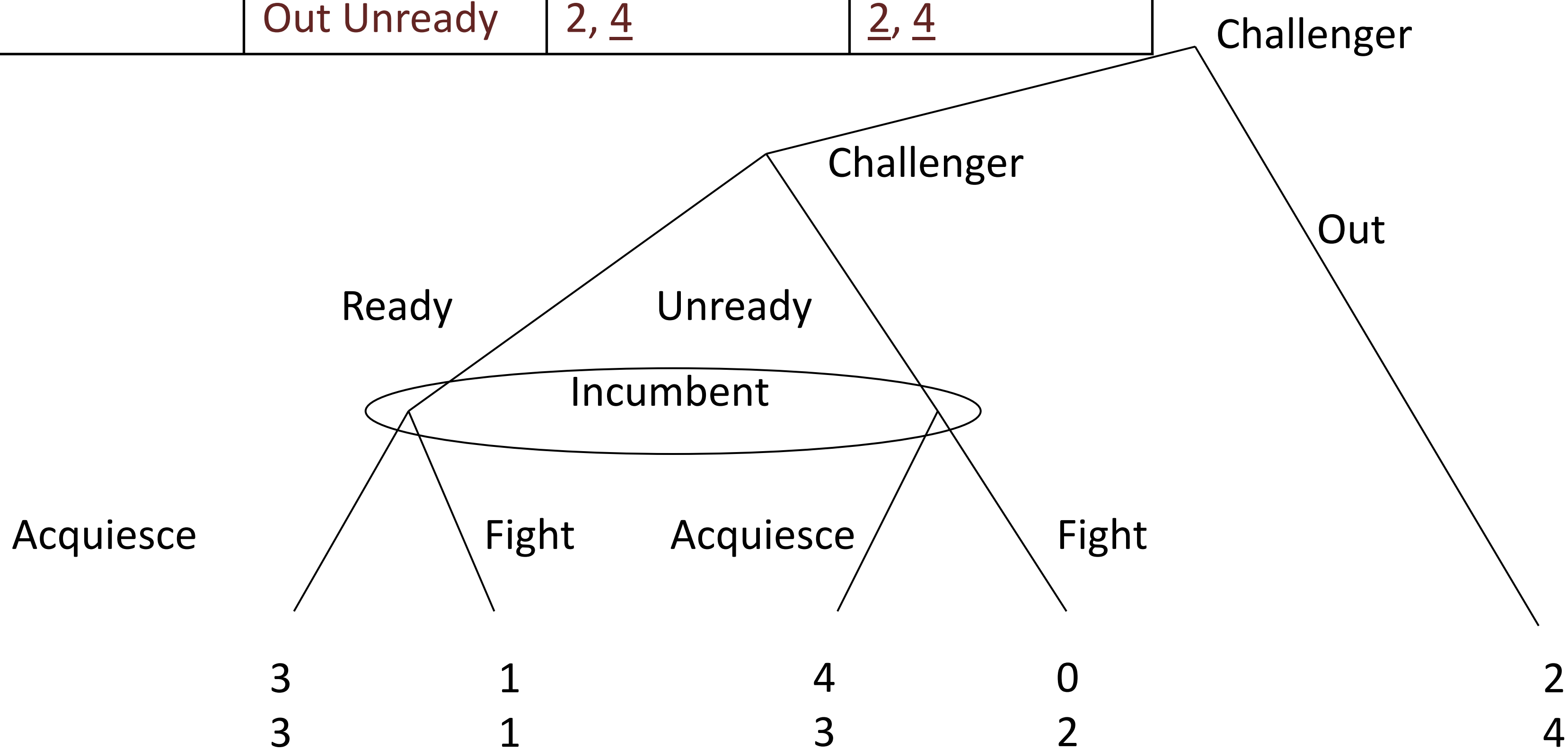
Example 4

		Incumbent	
		Acquiesce	Fight
Challenger	Ready	3, 3	1,1
	Unready	4, 3	0, 2
	Out	2, 4	2, 4



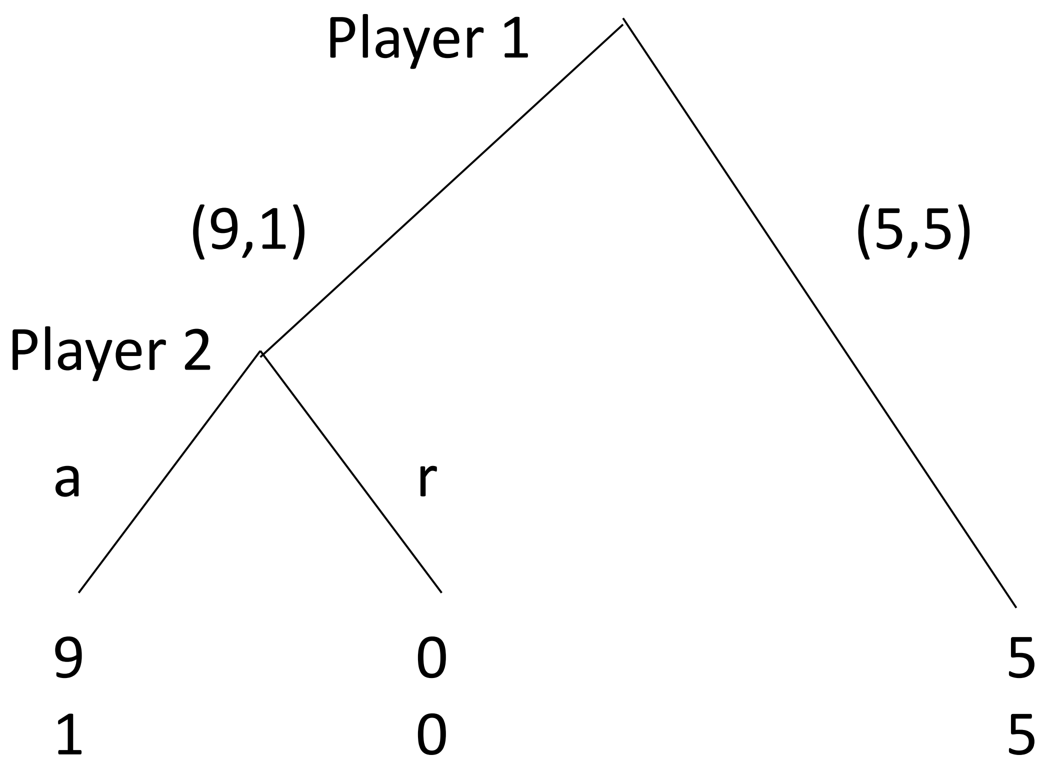
Example 5

		Incumbent	
		Acquiesce	Fight
Challenger	In Ready	3, 3	1,1
	In Unready	4, 3	0, 2
	Out Ready	2, 4	2, 4
	Out Unready	2, 4	2, 4



Example: Mini Ultimatum Game

- Proposer (Player 1) can suggest one of two splits of £10: (5,5) and (9,1).
- Responder (Player 2) can decide whether to accept or reject (9,1), but has to accept (5,5). Reject leads to 0 for both



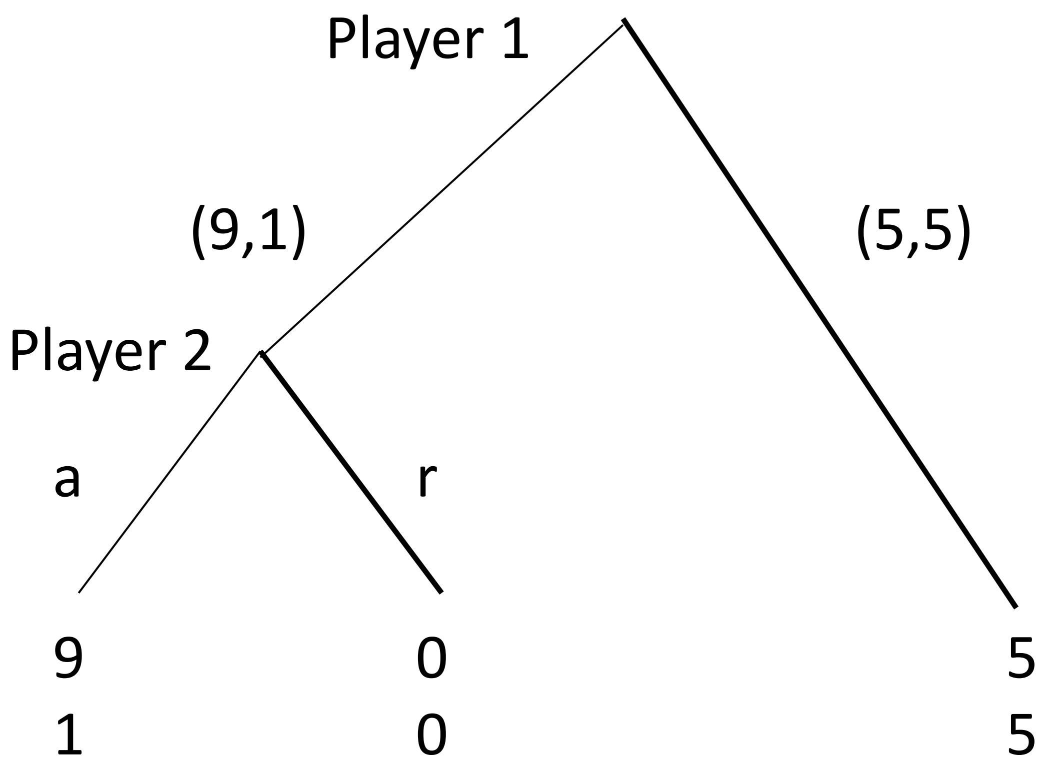
Mini Ultimatum Game in Strategic Form

		Player 2	
		accept (9,1)	reject (9,1)
Player 1	propose (5,5)	5,5	5,5
	propose (9,1)	9,1	0,0

- There are **two** equilibria:
  1. (propose (9,1), accept (9,1))
  2. (propose (5,5), reject (9,1)).
- Equilibrium 2 is in **weakly dominated** strategies (reject (9,1) is weakly dominated)

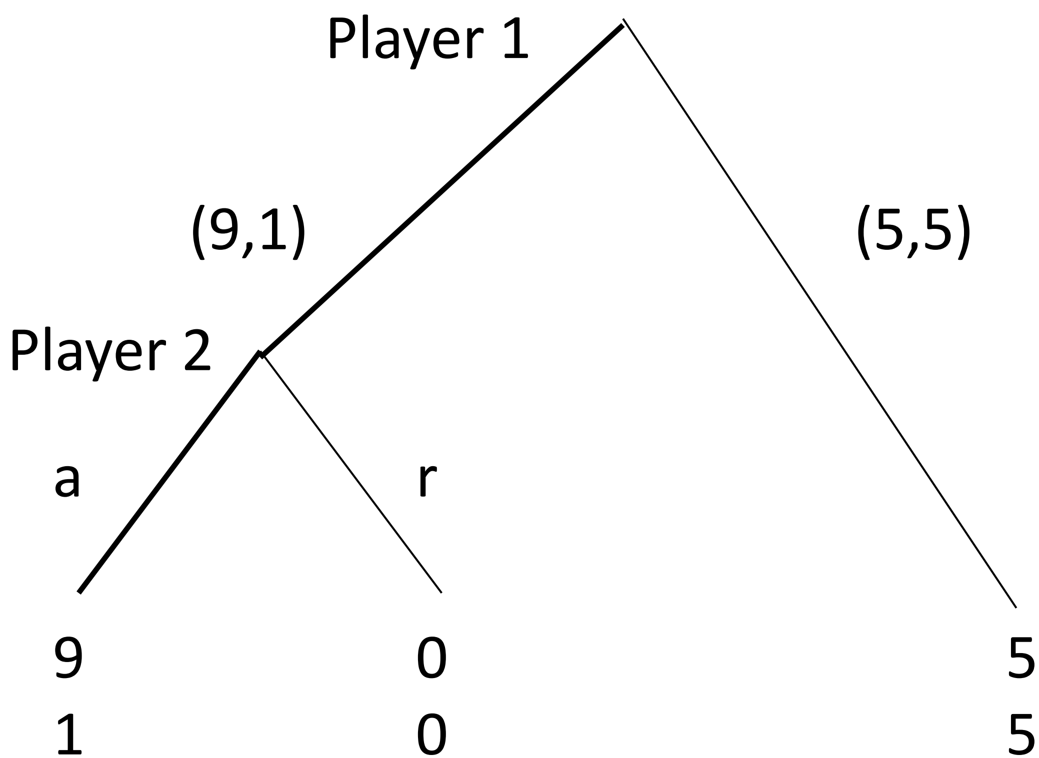
Note that equilibrium 2 (propose (5,5), reject (9,1)) is not convincing because it relies on a **non-credible** threat: if the 1 proposes (9,1) player 2 has an incentive to deviate (i.e. to accept).

Formally, the decision taken in the subgame starting at player 2's decision node is not optimal



equilibrium 1 (propose (9,1), accept) is more convincing because the decision taken in the subgame starting at player 2's decision node is optimal

We say that equilibrium 1 is "*subgame perfect*" because in every subgame the decisions are optimal



Definition:
Subgame perfect Nash equilibrium

A Nash equilibrium is subgame perfect (Nash equilibrium) if the players' strategies constitute a Nash equilibrium in every subgame.

(Selten 1965)

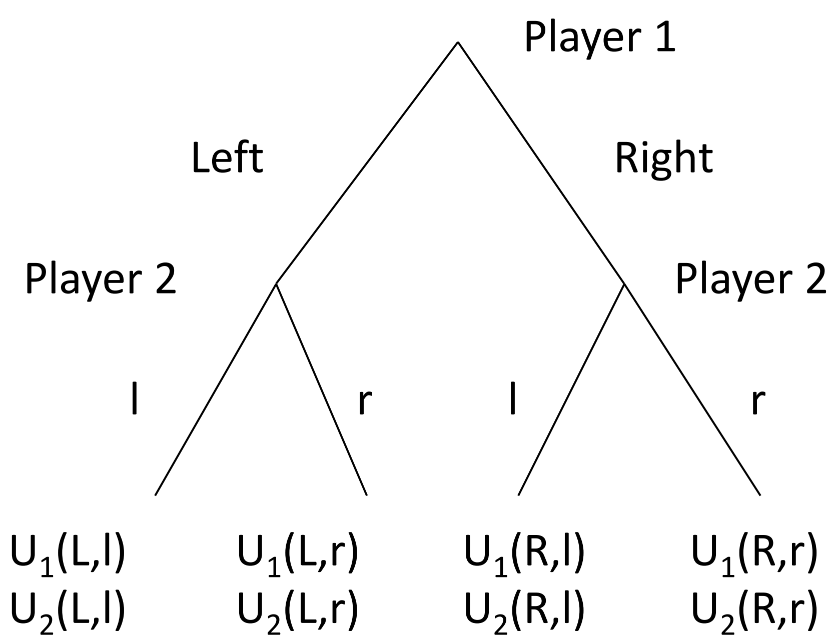
Note that every finite sequential game of complete information has at least one subgame perfect Nash equilibrium

We can find all subgame perfect NE using backward induction

Backward Induction in dynamic games of perfect information

- **Procedure:**
  - We start at the end of the trees
  - first find the optimal actions of the last player to move
  - then taking these actions as given, find the optimal actions of the second last player to move
  - continue working backwards
- If in each decision node there is only one optimal action, this procedure leads to a **Unique Subgame Perfect Nash equilibrium**

- Player 1 choose action a_1 from the set A_1={Left, Right}
- Player 2 observes a_1 and choose an action a_2 from the set A_2={l, r}
- Payoffs are $u_1(a_1, a_2)$ and $u_2(a_1, a_2)$



- When Player 2 gets the move, she observes player's 1 action a_1 and faces the following problem

$$\underset{a_2 \in \{A_2\}}{Max}\ u_2(a_1, a_2)$$

- Solving this problem, for each possible a_1 □ A_1, we get the best response of Player 2 to Player 1's action.
- We denote it by $R_2(a_1)$, the reaction function of Player 2.
- Player 1 can anticipate Player 2's reaction, then Player 1's problem is:

$$\underset{a_1 \in \{A_1\}}{Max}\ u_1(a_1, R_2(a_1))$$

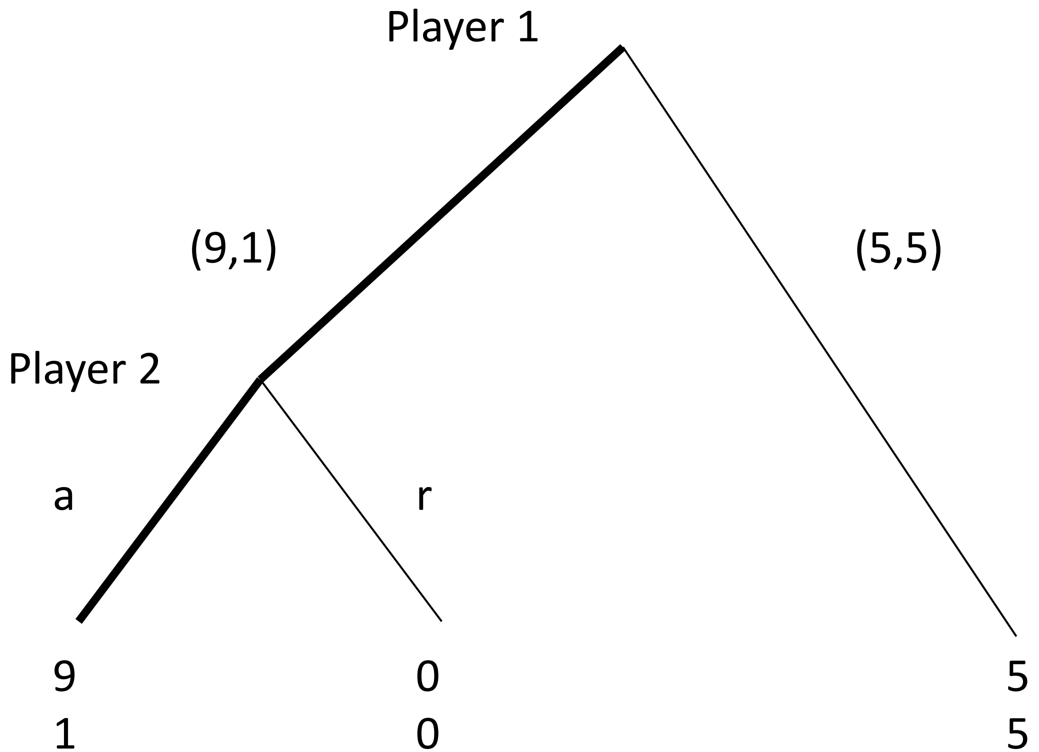
- The ***backwards induction outcome*** is denoted by

$$(a_1^*, R_2(a_1^*))$$

- It is different from the description of the equilibrium
- To describe the Nash equilibrium we need to describe the equilibrium strategies:
  - Action of Player 1 (Left or Right)
  - Action of Player 2 after Left, action of player 2 after Right

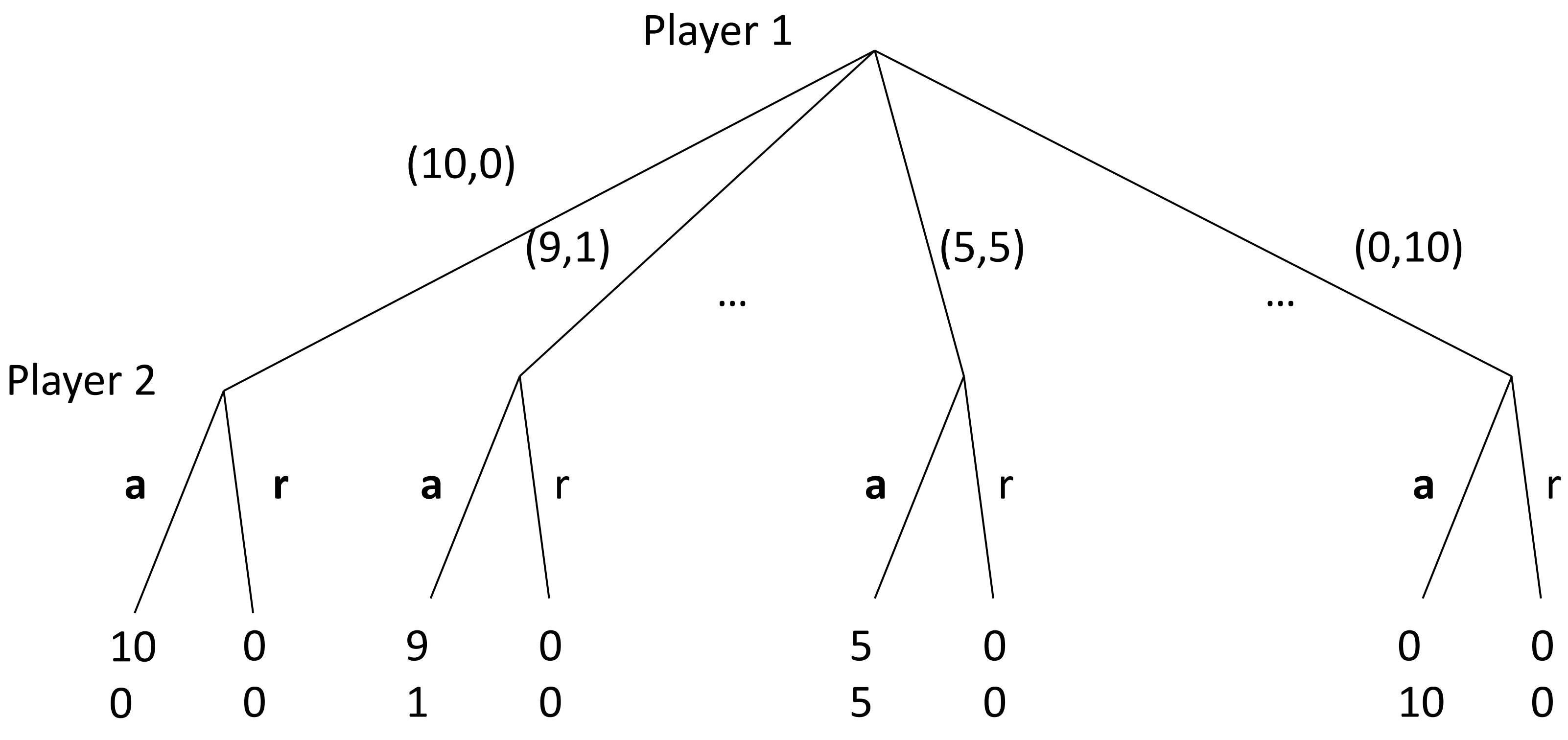
Example: Mini Ultimatum Game

- Consider Player 2, the optimal action is accept
- Taking "accept" as given, we see that (9,1) is the optimal action for player 1



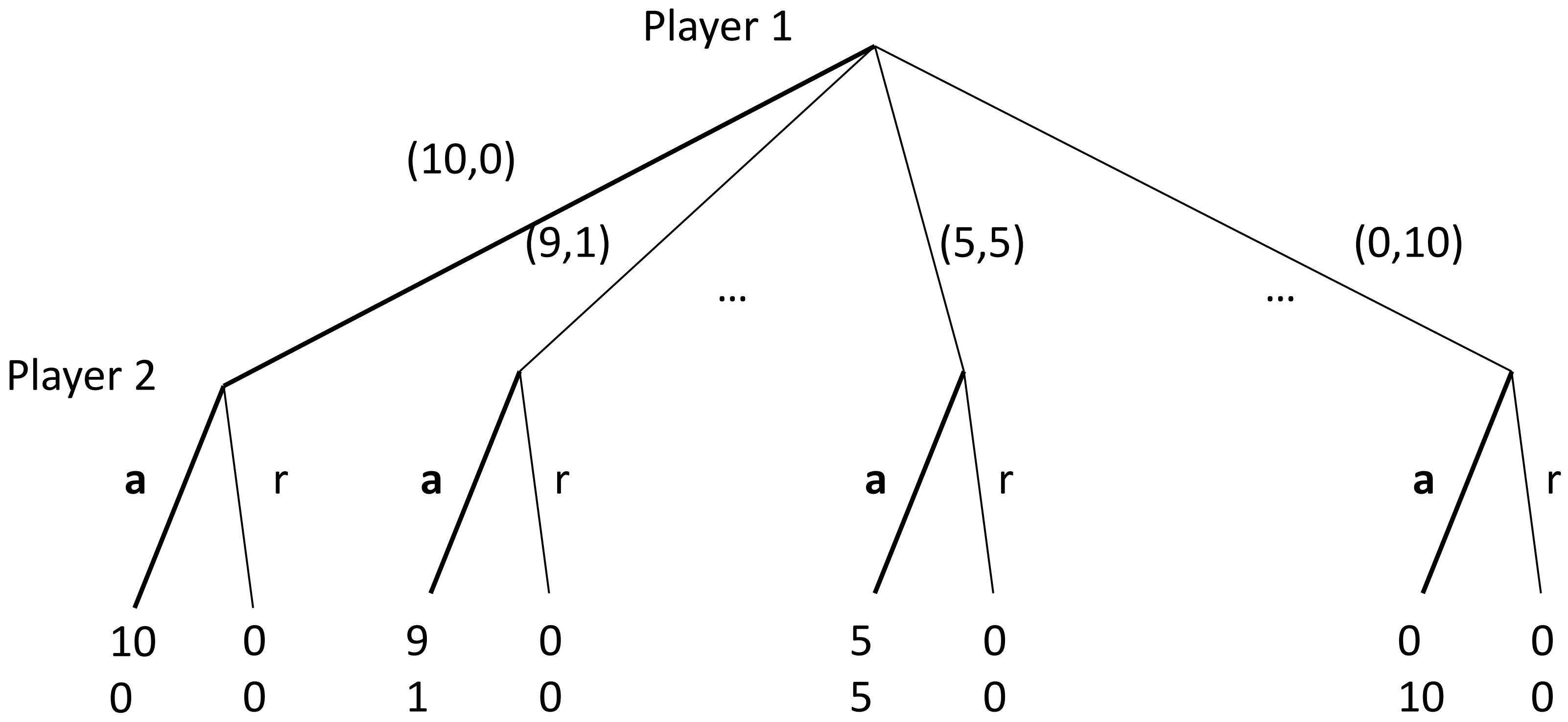
The Ultimatum Game

- Proposer (Player 1) suggest (integer) split of a fixed pie, say £10.
- Responder (Player 2) accepts the proposal or rejects (both receive 0)
- There is no unique best response following (10,0), so we have two SPNE



The Ultimatum Game

- First SPNE:
- Player 1 proposes (10, 0)
- Player 2 accepts in all of his decision nodes (a, a, a, a, a, a, a, a, a, a, a)



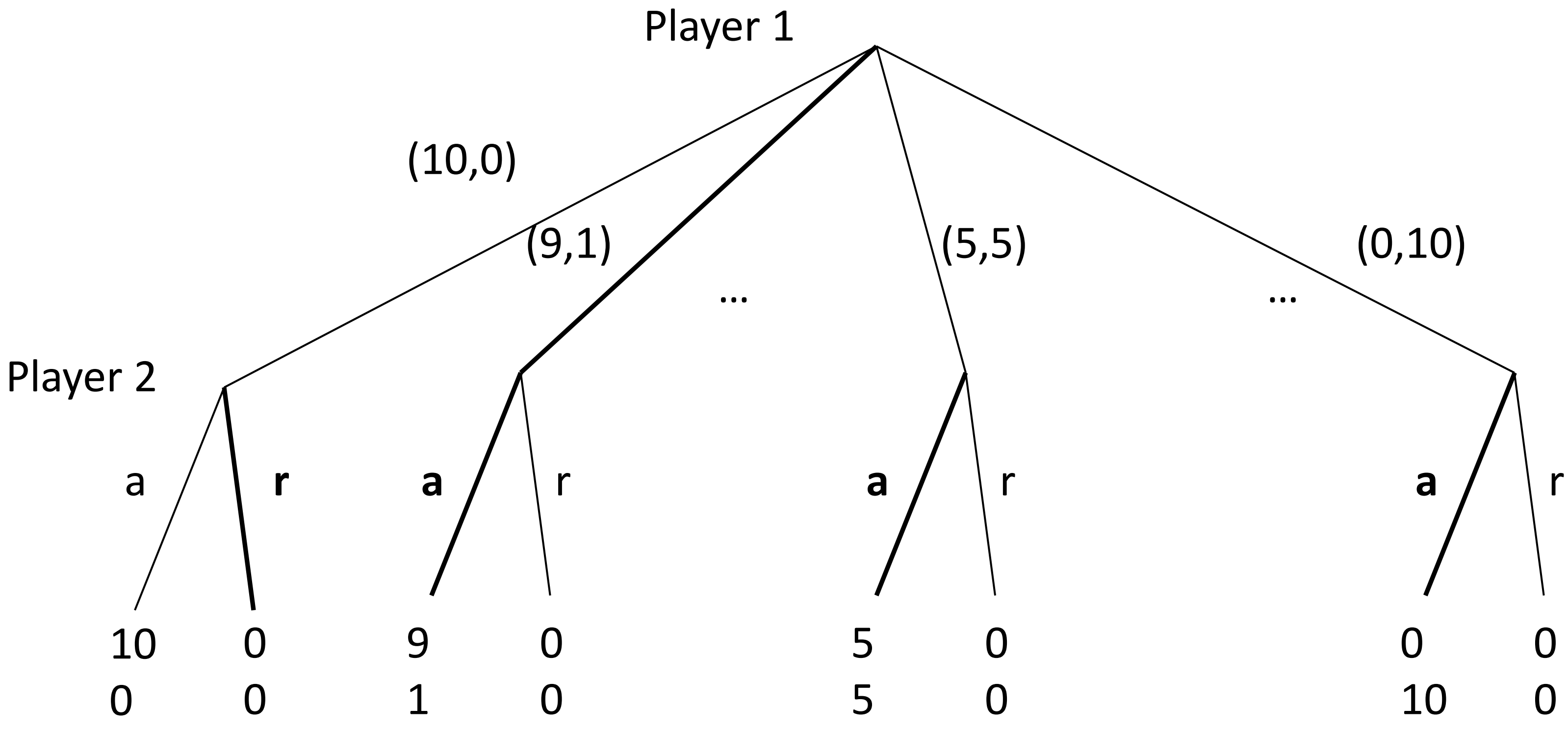
The Ultimatum Game

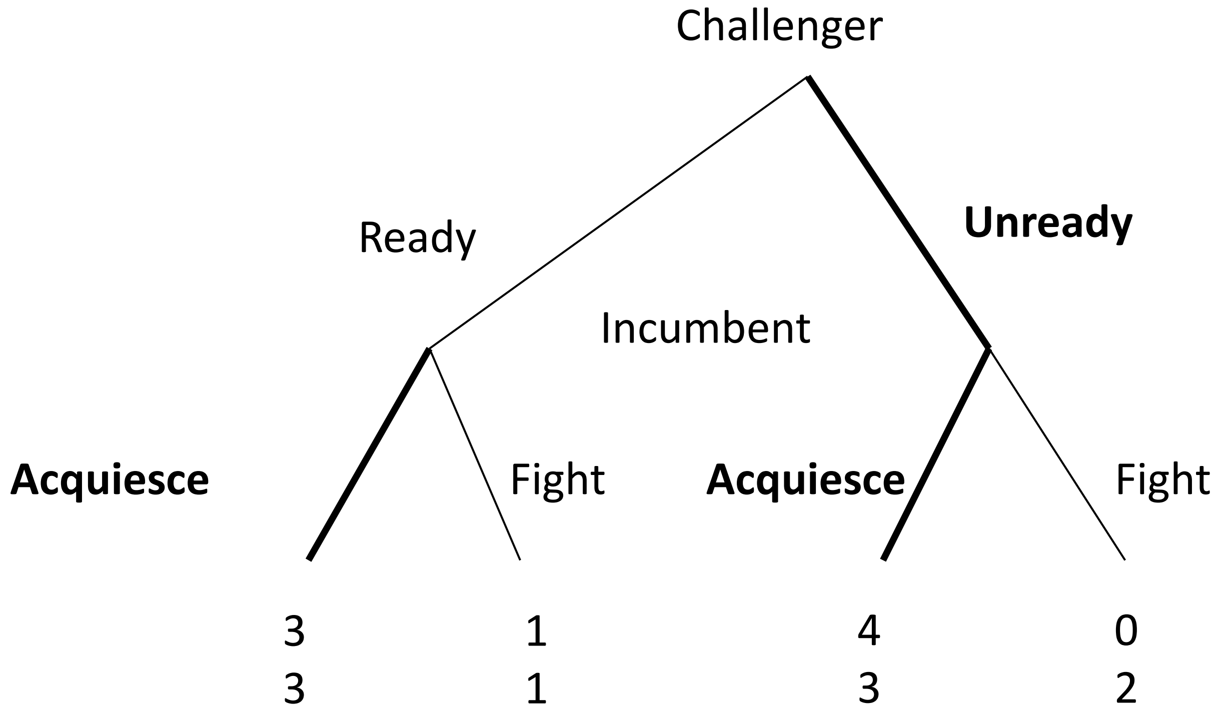
Second SPNE:

Player 1 proposes (9, 1)

Player 2 rejects after (10, 0) and accepts in all other decision nodes

(r, a, a, a, a, a, a, a, a, a, a)





Three Nash equilibria

1. Ready, (A, F)
2. Unready, (A, A)
3. Unready, (F, A)

Only {Unready, (A, A)} is subgame perfect

		Incumbent			
		(A, A)	(A, F)	(F, A)	(F, F)
Challenger	Ready	3, 3	3, 3	1, 1	1, 1
	Unready	4, 3	0, 2	4.3	0, 2