Problem set 2

1) Suppose there are n firms in the Cournot oligopoly model. The inverse demand function is $P(Q) = 100 - Q$ where Q is the aggregate quantity on the market. All firms are equal and face the following cost function: $c(q_i) = 2q_i$. Firms choose their quantities simultaneously.

a. Find the Nash equilibrium

b. Find the strategy profile where the aggregate quantity is equal to the monopoly quantity and firms produce the same quantity.

c. Show that the strategy profile at point b) is not an equilibrium (use best responses)

d. Show that firms prefer the strategy profile at point b) respect to the Nash equilibrium (compare profits)

e. Let $n = 2$ and suppose firms can choose to produce the Nash quantity or the quantity you find in point b. No other quantities are feasible. Represent this situation as a normal form game using a payoff table.

f. Let $n = 2$ and $c(q_1) = 2q_1$ $c(q_2) = 3q_2$ (firms have different cost functions). Find the Nash equilibrium.

a. Each firm faces the following problem:

$Max_{q_i}\,(100 - Q - 2)q_i$ where $Q = \sum_{i=1}^{n} q_i$

The FOC are $98 - \sum_{j\neq i} q_j - 2q_i = 0$ then the best response of firm i is:

$$q_i = \frac{98 - \sum_{j\neq i} q_j}{2}$$

Then in equilibrium this condition must be satisfied for every firm, i.e.

$$q_i^* = \frac{98 - \sum_{j\neq i} q_j^*}{2} \text{ for every } i$$

Note that you can rewrite these conditions as:

$q_i^* = 98 - Q^*$ where $Q^* = \sum_{i=1}^{n} q_i^*$. This is enough to state that every firm produces the same quantity, i.e. $q_i^* = q_j^*$ for every i, j.

Then the FOC can be written as:

$q_i^* = \frac{98 - \sum_{j\neq i} q_i^*}{2}$ and $q_i^* = \frac{98}{n+1}$.

Therefore in the Nash equilibrium each firm produces a quantity of $\frac{98}{n+1}$

b. The quantity that maximizes the profit of a monopolist is given by the solution of the following problem: $Max_Q\,(100 - Q - 2)Q$

The FOC is $98 - 2Q = 0$. Then the aggregate quantity that maximize the aggregate profits is $Q = 49$. Then each firm has to produce $\frac{49}{n}$.

c. Take the best response function

$$q_i = \frac{98 - \sum_{j \neq i} q_j}{2}$$

If every firm produces $\frac{49}{n}$ the level of production that maximizes the profits of firm i is:

$$q_i = \frac{98 - (n-1)\frac{49}{n}}{2}$$

Note that this quantity is bigger than $\frac{49}{n}$, indeed

$$\frac{98 - (n-1)\frac{49}{n}}{2} > \frac{49}{n}$$

$$98 - (n-1)\frac{49}{n} > 2\ \frac{49}{n}$$

$$98 > (n+1)\frac{49}{n}$$

That is true for all $n \geq 2$.

Then if all other firms produce $\frac{49}{n}$ each, the best response of firm i is to produce a greater quantity. Then the strategy profile where all firms produce $\frac{49}{n}$ is not a Nash equilibrium.

d. The profit of firm i in the Nash equilibrium are:

$$\pi_i = \left(100 - \frac{98}{n+1}\ n - 2\right)\frac{98}{n+1} = \left(98 - \frac{98}{n+1}\ n\right)\frac{98}{n+1} = \left(\frac{98}{n+1}\right)^2$$

The profit of firm i, when each firm produces $\frac{49}{n}$, are:

$$\hat{\pi}_i = \left(100 - \frac{49}{n}\ n - 2\right)\frac{49}{n} = \frac{49^2}{n}$$

Note that

$$\frac{49^2}{n} > \left(\frac{98}{n+1}\right)^2$$

$$\frac{1}{n} > \left(\frac{2}{n+1}\right)^2$$

That is true for all $n \geq 2$

Then each firm gets more profits if each one produces $\frac{49}{n}$

e. Players: Firm 1 and Firm 2

Strategies: $s_1 \in \left\{\frac{49}{2}, \frac{98}{3}\right\}$ and $s_2 \in \left\{\frac{49}{2}, \frac{98}{3}\right\}$

Payoff:

		Firm 2	
		$\frac{49}{2}$	$\frac{98}{3}$
Firm 1	$\frac{49}{2}$	1200.5, 1200.5	1000.4, 1333.9
	$\frac{98}{3}$	1333.9, 1000.4	1067.1, 1067.1

g.

The best response for firm 1 is

$$q_1 = \frac{100 - q_2 - 2}{2}$$

The best response for firm 2 is

$$q_2 = \frac{100 - q_1 - 3}{2}$$

In a Nash equilibrium both must be satisfied, i.e.

$$\begin{cases} q_1^* = \dfrac{100 - q_2^* - 2}{2} \\ q_2^* = \dfrac{100 - q_1^* - 3}{2} \end{cases}$$

Solving the system we have:

$q_1^* = 33$ and $q_2^* = 32$ that is the strategy profile in The Nash equilibrium

2) Consider the Bertrand duopoly model with homogeneous product. The demand function of firm 1 is $q_1 = \begin{cases} 100 - p_1 \; if \; p_1 < p_2 \\ \frac{100-p_1}{2} \; if \; p_1 = p_2 \\ 0 \; if \; p_1 > p_2 \end{cases}$; that of firm 2 is $q_2 = \begin{cases} 100 - p_2 \; if \; p_2 < p_1 \\ \frac{100-p_2}{2} \; if \; p_2 = p_1 \\ 0 \; if \; p_2 > p_1 \end{cases}$.

The two firms are equal and face the following cost function: $c(q_i) = c \cdot q_i$

Show that the unique Nash equilibrium is $p_2 = p_1 = c$.

1) p_1 > c and p_2 > c is not an equilibrium because the firm with the higher price (zero profits) has a positive incentive to set a price a bit lower than the other (so it gets strictly positive profits).
2) p_1 < c and p_2 <c is not an equilibrium because the firm with the lower price (strictly negative profit) has a positive incentive to set a price a bit higher than the other (so it gets zero profits).
3) p_1 < c and p_2 >c (or p_1 > c and p_2<c) is not an equilibrium because the firm with the lower price (strictly negative profits) has a positive incentive to set a price a bit lower than the other but above c (so it gets strictly positive profits) .

Therefore the only pair of prices that is a best response of each other is $p_2 = p_1 = c$.

Otherwise we look for best responses.

Consider the best response of Firm 1 to:

p_2 > c. It is to set $p_1 = p_2 - \varepsilon$ where ε is infinitely small.

p_2 = c. It is to set $p_1 \geq p_2$ where ε is infinitely small.

p_2 < c. It is to set $p_1 > p_2$

Consider the best response of Firm 2 to:

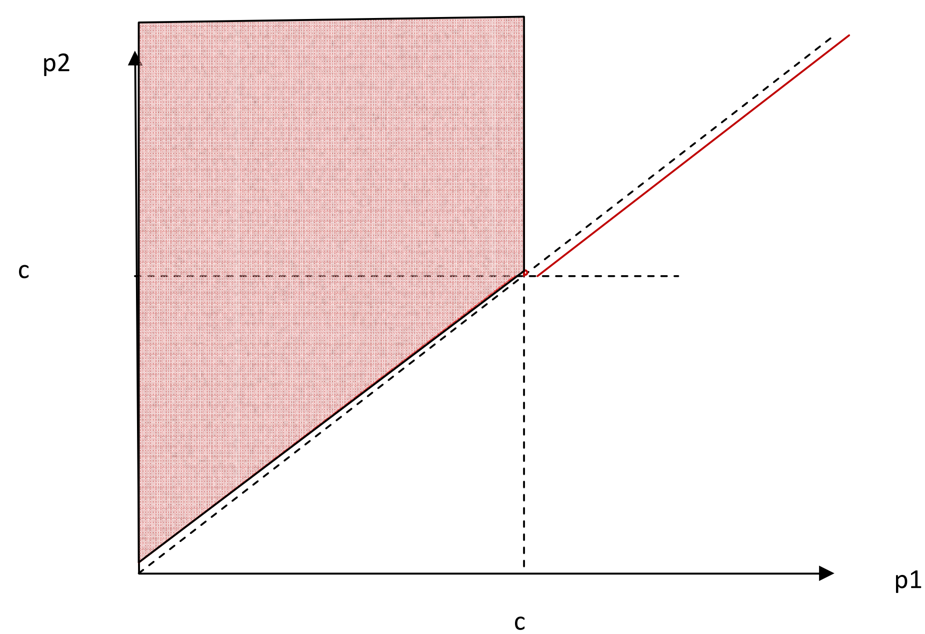
p_1 > c. It is to set $p_2 = p_1 - \varepsilon$ where ε is infinitely small.

p_1 = c. It is to set $p_2 \geq p_1$ where ε is infinitely small.

p_1 < c. It is to set $p_2 > p_1$

Therefore the only pair of prices that is a best response of each other is $p_2 = p_1 = c$.

In the figure the best response of firm 2 (shaded area and red lines)



3) Consider the model of final offer arbitration. Find the Nash equilibrium when

a. $F(x) = \frac{x^2}{10000}$ for $0 \leq x \leq 100$ and $F(x) = 1$ for $x > 100$

b. $F(x) = 0.01 \cdot x$ with $0 \leq x \leq 100$

a. To find the median you have to solve $\frac{x^2}{10000} = 0.5$ by x.

$x^2 = 5000$

$x = 70.7 = m$

Then one condition for a NE is $\frac{w_f + w_u}{2} = 70.7$

Then we need to find $f(x) = \frac{dF(x)}{dx}$

$$f(x) = \frac{x}{5000}$$

Then the second condition is:

$$w_u - w_f = \frac{1}{\frac{m}{5000}} = \frac{5000}{70.7} = 70.7$$

We have to solve the system

$$\begin{cases} w_u - w_f = 70.7 \\ \frac{w_f + w_u}{2} = 70.7 \end{cases}$$

The solution is

$$w_u \cong 106$$

$$wf \cong 35.3$$

But if w_u>0 which is the solution?

We have to use Kuhn Tucker condition for constrained maximization.

b. We have to solve the system

$$\begin{cases} w_u - w_f = \frac{1}{0.01} \\ \frac{w_f + w_u}{2} = 50 \end{cases}$$

$$w_u = 100$$

$$wf = 0$$

4. Consider the Problem of the Commons. Assume that $n = 3$and that $v(x) = 120 - x$. Compute the Nash equilibrium, the total number of goats in the Nash equilibrium and the number of goats that maximize the social welfare.

The problem of farmer 1 is

$$max_{g1} g_1(120 - c - g_1 - g_2 - g_3)$$

Compute the FOC to find its best response, that is:

$$g_1 = \frac{(120 - c - g_2 - g_3)}{2}$$

As in Cournot model is possible to show that $g_1 = g_2 = g_3$, then we have that:

$$g_1 = g_2 = g_3 = \frac{(120 - c)}{4}$$

That is the a Nash equilibrium

The total number of goats in equilibrium is :

$$G = \frac{3(120 - c)}{4}$$

The number of goats that maximizes the social welfare is the number that maximizes the aggregate profits

The problem is:

$$max_G G(120 - c - G)$$

Using the first order condition we find that:

$$G = \frac{(120 - c)}{2}$$

That is smaller than in the Nash equilibrium

5. Represent by a table a traveler's dilemma game with two players. They can choose integer numbers between 1 and 4 and R=2. Find the Nash equilibrium

		Player 2			
		1	2	3	4
Player 1	1	1, 1	3, -1	3, -1	3, -1
	2	-1, 3	2, 2	4, 0	4, 0
	3	-1, 3	0, 4	3, 3	5, 1
	4	-1, 3	0, 4	1, 5	4, 4

The unique Nash equilibrium is: Player 1 plays 1, Player 2 plays 1

6. Represent a beauty contest game with two players. They can choose integer numbers between 1 and 4 :
   c. when p=0.5
   d. when p=1
   e. when p=2

   In all cases find the Nash equilibria

   Let 100 be the prize. When both players are at same distance from $p * average$, each one receives 50

P=0.5

		Player 2			
		1	2	3	4
Player 1	1	50, 50	100, 0	100, 0	100, 0
	2	0, 100	50, 50	100, 0	100, 0
	3	0, 100	0, 100	50, 50	100, 0
	4	0, 100	0, 100	0, 100	50, 50

Unique Nash Equilibrium: Player 1 plays 1, Player 2 plays 1

p=1

		Player 2			
		1	2	3	4
Player 1	1	50, 50	50, 50	50, 50	50, 50
	2	50, 50	50, 50	50, 50	50, 50
	3	50, 50	50, 50	50, 50	50, 50
	4	50, 50	50, 50	50, 50	50, 50

All strategy combinations are Nash equilibria

p=2

		Player 2			
		1	2	3	4
Player 1	1	50, 50	0, 100	0, 100	0, 100
	2	100, 0	50, 50	0, 100	0, 100
	3	100, 0	100, 0	50, 50	0, 100
	4	100, 0	100, 0	100, 0	50, 50

Unique Nash Equilibrium: Player 1 plays 4, Player 2 plays 4