Solutions

1) Find all mixed strategy Nash equilibrium of the following game (you have to use the property of the Nash equilibrium in mixed strategies)

		Player 2		
		L	M	R
Player 1	T	2, 2	0, 3	1, 3
	B	3, 2	1, 1	0, 2

Notation

t is the probability to play T

l is the probability to play L

m is the probability to play M

Player 1's strategy *(t, 1-t)*

Player 2's strategy *(l, m, 1 – l - m)*

E(X) expected value from playing the pure strategy X

We start considering all possible cases for player 1: (1, 0), (0, 1), (t, 1-t) for 0 < t < 1

For each one we explore if it can be played in a NE

Case (1, 0)

In this case for player1 must be: E(T)≥ E(B)

Now we look at the expected payoff of player 2 when player 1's strategy is (1, 0)

E(M) = E(R) = 3 > E(L) = 2

Then a mixed strategy for player 2 must be of the type (0, m, 1 – m)

Given this possible player 2's strategy we compute the expected payoff of player 1

E(T) = 1 – m , E(B) = m

Given that in equilibrium E(T)≥ E(B) must be satisfied this is true only if m ≤ 0.5

Therefore all strategy profiles (1, 0) (0, m, 1 –m) with m ≤ 0.5 are Nash equilibria

Note there are infinite equilibria and, among them, the NE in pure strategy {(1, 0) (0, 0, 1)}
(Player 1 plays T and Player 2 plays R)

Case (0, 1)

In this case for player1 must be: E(B) ≥ E(T)

Now we look at the expected payoff of player 2 when player 1's strategy is (0, 1)

E(L) = E(R) =23 > E(M) =1

Then a mixed strategy for player 2 must be of the type (l, 0, 1 – l)

Given this possible player 2's strategy we compute the expected payoff of player 1

E(T) = 1 + l , E(B) = 3l

Given that in equilibrium E(B)≥ E(L) must be satisfied this is true only if l ≥ 0.5

Therefore all strategy profiles (0, 1) (l, 0, 1 – l) with l ≥ 0.5 are Nash equilibria

Note there are infinite equilibria and, among them, the NE in pure strategy {(0, 1) (1, 0, 0)} (Player 1 plays T and Player 2 plays R)

Case (t, 1 – t) for t $\in$ (0, 1)

In this case for player1 must be: E(B) = E(T)
Now we look at the expected payoff of player 2 when player 1's strategy is (t, 1 – t) for t $\in$ (0, 1)

E(L) = 2
E(M) =3 t + (1 – t) = 1 + 2t
E(R) =3t + 2(1 – t) = 2 + t

We have to explore all the possible classes of player 2's strategy, i.e. for each case we have to verify the equilibrium conditions given that player 1 strategy is of the type (t, 1 – t) for t $\in$ (0, 1).
All cases are:

1) (1, 0, 0)
2) (0, 1, 0)
3) (0, 0, 1)
4) (l, 1 – l , 0)
5) (l, 0, 1 – l)
6) (0, m, 1 – m)
7) (l, m 1 – l – m)

Note that we can reduce the number of cases. Indeed for every t $\in$ (0, 1) strategy L is dominated by R (2 + t > 2 for t > 0) and strategy M is dominated by R (2 + t > 1 + 2 for t < 1) Then we can eliminate all cases where either L or M or both are played by strictly positive probability.

Then it remains to explore only strategy (0, 0, 1).
In this case the player 1's expected payoffs are
E(T) = 1 > E(B) = 0
Then condition E(T) = E(B) does not hold.

Final results:
There are two sets of Nash equilibria:

1) (1, 0) (0, m, 1 –m) with m ≤ 0.5
2) (0, 1) (l, 0, 1 – l) with l ≥ 0.5

Each set contains an equilibrium in pure strategies, respectively, (1, 0) (0, 0, 1) and (0, 1) (1, 0, 0).

2) Consider the following game

		Player 2	
		L	R
Player 1	T	1, 2	1, 3
	M	4, 1	0, 1
	B	0, 3	3, 2

Find all mixed strategies that dominate strategy T

Notation
l is the probability that Player 2 plays L
t, m, 1 – t – m are, respectively, the probabilities that player 1 plays actions T,M, B.
σ1 = (t, m, 1 – t – m) denotes a player 1's mixed strategy; when the strategy is a pure strategy we call it with the name of the action, i.e. σ1 = (1, 0, 0) = T.
σ2 = (l, 1 – l) denotes a player 2's mixed strategy; when the strategy is a pure strategy we call it with the name of the action, i.e. σ2 = (1, 0) = L.

E(σ1|σ2) is the expected payoff from strategy σ1 given that player 2's strategy σ2.

Note, strategy T is dominated by a strategy σ1 only if E(σ1) > E(T) for all l ∈ [0, 1]
Expected payoffs are:
E(T|σ2) = 1
E(σ1|σ2) = t + 4 m l + 3 (1 – t – m) (1 – l)

$$\frac{dE(\sigma1)}{dl} = 4\,m - 3(1 - t - m)$$

E(σ1|σ2) is increasing if l if

$$4\,m - 3(1 - t - m) > 0$$
$$m > \frac{3(1-t)}{7}$$

It is decreasing in l if:

$$m < \frac{3(1-t)}{7}$$

It is constant and equal to $\frac{1}{7}(12 - 5t)$ if:

$$m = \frac{3(1-t)}{7}$$

By this last point we see that $\frac{1}{7}(12 - 5t) > 1$ for all t < 1. Then for every value of t' < 1 it exists a value of m' such that strategy $\sigma1 = (t', m', 1 - t' - m')$ dominates T. But these are not the only mixed strategies dominating T.

When $m > \frac{3(1-t)}{7}$ E(σ1|σ2) is increasing in l, then its minimum value is at l = 0, i.e
E(σ1|R) = t + 3 (1 – t – m)
In order to dominate T we need E(σ1|R) = t + 3 (1 – t – m) > 1

That is satisfied when $m < \frac{2\,(1-t)}{3}$

When $< \frac{3(1-t)}{7}$, E(σ1|σ2) is decreasing in l, then its minimum value is at l = 1, i.e
E(σ1|L) = t + 4 m
In order to dominate T we need E(σ1|L) = t + 4 m > 1
That is satisfied when $m > \frac{(1-t)}{4}$

Then all mixed strategies σ1 = (t, m, 1 – t – m) **with t <1** and $\frac{(1-t)}{4} < m < \frac{2\,(1-t)}{3}$ are dominating strategy T.

3) Is the following statement true?
"A mixed strategy that assigns positive probability to a strictly dominated action is strictly dominated"

It is true.

Notation

By s_{-i} we denote the strategies of the opponent and by S_{-i} the set of all possible combination of the opponents' strategies

Suppose that action X is dominated by strategy $\sigma 1$, it means that:

$$E(\sigma 1 | s_{-i}) > E(X | s_{-i}) \text{ for all } s_{-i} \in S_{-i}$$

Now suppose a strategy $\sigma 2$ that prescribes to play by probability p (> 0) action X and by probability 1 – p (> 0) a strategy $\sigma 3$.

Replacing X with strategy $\sigma 1$ we produce a mixed strategy, $\sigma 4$, that dominates $\sigma 2$.

Proof

$\sigma 2$: X by probability p and $\sigma 3$ by probability 1 - p
$\sigma 4$: $\sigma 1$ by probability p and $\sigma 3$ by probability 1 - p

$E(\sigma 2 | s_{-i}) = p\, E(X | s_{-i}) + (1- p)\, E(\sigma 3 | s_{-i})$
$E(\sigma 4 | s_{-i}) = p\, E(\sigma 1 | s_{-i}) + (1- p)\, E(\sigma 3 | s_{-i})$

$\sigma 4$ dominates $\sigma 2$ if:

$$E(\sigma 2 | s_{-i}) < E(\sigma 4 | s_{-i}) \text{ for all } s_{-i} \in S_{-i}$$

Replacing the expected values we have:

$$p\, E(X | s_{-i}) + (1- p)\, E(\sigma 3 | s_{-i}) < p\, E(\sigma 1 | s_{-i}) + (1- p)\, E(\sigma 3 | s_{-i}) \text{ for all } s_{-i} \in S_{-i}$$

that is:

$$E(X | s_{-i}) < E(\sigma 1 | s_{-i}) \text{ for all } s_{-i} \in S_{-i}$$

That is true by assumption.

4) Each of two firms has a job opening. The firms offer different wages: firm *i* offers wage w_i where $0.5 \cdot w_1 < w_2 < 2 \cdot w_1$.
There are two workers that want to apply for a job. Each of whom can apply to only one firm. The workers simultaneously decide whether apply to firm 1 or to firm 2.
If only one worker applies to a given firm, that worker gets the job. If both workers apply to one firm, the firm hires one worker at random and the other worker remains unemployed.
   a) Rapresent this game using the normal form
   b) Solve for the Nash equilibria (pure a mixed strategies)

a)

		Worker 2	
		Firm 1	Firm 2
Worker 1	Firm 1	$\frac{w1}{2}, \frac{w1}{2}$	$w1, w2$
	Firm 2	$w2, w1$	$\frac{w2}{2}, \frac{w2}{2}$

b) Notation
p is the probability that Player 1 play Firm 1
q is the probability that Player 2 play Firm 1
the player 1's mixed strategy is σ1 = (p, 1 – p)
the player 2's mixed strategy is σ2 = (q, 1 – q)

We compute the expected values for each single action given a strategy of the opponent
Player 1

$$E_1(Firm1|\sigma 2) = q\frac{w1}{2} + (1-q)w1 = w1\,(1-\frac{q}{2})$$
$$E_1(Firm2|\sigma 2) = qw2 + (1-q)\frac{w2}{2} = w2\,(\frac{1+q}{2})$$

Player 2

$$E_2(Firm1|\sigma 1) = p\frac{w1}{2} + (1-p)w1 = w1\,(1-\frac{p}{2})$$
$$E_2(Firm2|\sigma 1) = pw2 + (1-p)\frac{w2}{2} = w2\,(\frac{1+p}{2})$$

<u>Suppose σ1 = (1, 0)</u>

Player 2' s expected profits are:

$$E_2(Firm1|\sigma 1) = \frac{w1}{2}$$
$$E_2(Firm2|\sigma 1) = w2$$

Given that $w2 > \frac{w1}{2}$ the best response for player 2 is σ2 = (0, 1)
Given σ2 = (0, 1) the expected payoff of player 1 are:

$$E_1(Firm1|\sigma 2) = w1$$

$$E_1(Firm2|\sigma 2) = \frac{w2}{2}$$

Given that $w1 > \frac{w2}{2}$ the best response for player 1 is σ1 = (1, 0)
Then σ1 = (1, 0) σ2 = (0, 1) is a Nash equilibrium

<u>Suppose σ1 = (0, 1)</u>
Player 2' s expected profits are:

$$E_1(Firm1|\sigma 2) = w1$$
$$E_1(Firm2|\sigma 2) = \frac{w2}{2}$$

Given that $w1 > \frac{w2}{2}$ the best response for player 2 is σ2 = (1, 0)
Given σ2 = (0, 1) the expected payoff of player 1 are:

$$E_1(Firm1|\sigma 1) = \frac{w1}{2}$$
$$E_1(Firm2|\sigma 1) = w2$$

Given that $w2 > \frac{w1}{2}$ the best response for player 1 is σ1 = (0, 1)
Then σ1 = (0, 1) σ2 = (1, 0) is a Nash equilibrium

<u>Suppose σ1 = (p, 1 – p) for p ∈ (0, 1)</u>
In this case must be that $E_1(Firm1|\sigma 2) = E_1(Firm2|\sigma 2)$ i.e.

$$w1\left(1 - \frac{q}{2}\right) = w2\left(\frac{1+q}{2}\right)$$

This is true only if $= \frac{2w1-w2}{w1+w2}$, i.e. $\sigma_2 = \left(\frac{2w1-w2}{w1+w2}, \frac{2w2-w1}{w1+w2}\right)$
But for player 2, in order to play such a strategy, the condition

$$E_2(Firm1|\sigma 1) = E_2(Firm2|\sigma 1)$$

must be satisfied, i.e.:

$$w1\left(1 - \frac{p}{2}\right) = w2\left(\frac{1+p}{2}\right)$$

This is true only if $= \frac{2w1-w2}{w1+w2}$, i.e. $\sigma_1 = \left(\frac{2w1-w2}{w1+w2}, \frac{2w2-w1}{w1+w2}\right)$

Then the strategy profile $\sigma_1 = \left(\frac{2w1-w2}{w1+w2}, \frac{2w2-w1}{w1+w2}\right)$ $\sigma_2 = \left(\frac{2w1-w2}{w1+w2}, \frac{2w2-w1}{w1+w2}\right)$ is a Nah equilibrium.

Final results:
There are 3 Nash equilibria:

1) σ1 = (1, 0) σ2 = (0, 1)
2) σ1 = (0, 1) σ2 = (1, 0)
3) $\sigma_1 = \left(\frac{2w1-w2}{w1+w2}, \frac{2w2-w1}{w1+w2}\right)$ $\sigma_2 = \left(\frac{2w1-w2}{w1+w2}, \frac{2w2-w1}{w1+w2}\right)$