Lecture 6

Dynamic games of complete information: Applications

Applications with perfect information

Stackelberg model of Duopoly

- 2 firms, 1 and 2 (Leader and Follower)
- Firms choose quantities (as in Cournot) q_1 and q_2.
- Leader (firm 1) moves first and chooses a quantity q_1
- Followers (Firm 2) moves second, observes q_1 and then chooses a quantity q_2
- Each firm faces constant marginal cost c and no fixed cost.

The payoff of firm 1 is:

$$\pi_1(q_1, q_2) = q_1 (P(Q) - c)$$

The payoff of firm 2 is:

$$\pi_2(q_1, q_2) = q_2 (P(Q) - c)$$

where

$P(Q) = a - Q$ is the inverse demand function and

$Q = q_1 + q_2$

Solution by backwards-induction

- We can solve this problem by backwards-induction:

1. We solve the problem that Firm 2 faces for a generic observed quantity q_1

2. The solution gives us the optimal quantity q_2^* as function of the observed quantity q_1:

$q_2^* = R_2(q_1)$ where R_2 () is the reaction function.

3. We solve the problem of Firm 1 assuming that Firm 1 knows $R_2(q_1)$, i.e.

For every quantity (q_1) Firm 1 decides to produces, Firm 1 correctly anticipate the quantity (q_2) Firm 2 will decide to produce.

Firm 2's problem

$$\pi_2(q_1, q_2) = q_2 (P(Q) - c) = q_2 (a - q_1 - q_2 - c)$$

$$\text{Max}_{\{q2\}}\, q_2 (a - q_1 - q_2 - c)$$

Using the FOCs

$$R_2 (q_1) = (a - q_1 - c) / 2$$

Note, this is the same reaction function to that we found in Cournot Oligopoly

Firm 1's problem

$$\pi_2(q_1, q_2) = q_1\,(P(Q) - c) = q_1\,(a - q_1 - q_2 - c)$$

$$\text{Max}_{\{q1\}}\, q_1\,(a - q_1 - q_2 - c)$$

Given that Firm 1 knows $R_2(q_1)$, its problem is

$$\text{Max}_{\{q1\}}\, q_1\,(a - q_1 - R_2(q_1) - c)$$

replacing $R_2(q_1)$ we get:

$$\text{Max}_{\{q1\}}\, q_1\,(a - q_1 - c) / 2$$

Using the FOCs

$$q_1^* = (a - c) / 2$$

Replacing in $R_2(q_1)$ we get:

$$R_2(q_1^*) = (a - c) / 4$$

The backward induction outcome is

$$q_1 = (a - c) / 2$$

$$q_2 = (a - c) / 4$$

The Subgame Perfect Nash Equilibrium is

$$q_1 = (a - c) / 2$$

$$q_2 = (a - q_1 - c) / 2$$

Wage and employment

- Relation between an Union and a Firm
- Union has exclusive control on the wages
- Firm has exclusive control over employment
- Union utility function is U(w, L) where w is the wage the union demands and L is the employment
- U(w, L) is increasing in w and L
- Firm's profit function is:

$$\pi(w, L) = R(L) - wL$$

where R(L) is the revenue of the firm when employment is L.

R(L) is increasing and concave

Timing of the game

1. The union makes a wage demand w

2. The firm observes w and then chooses employment L

3. Firms and Union receive their payoffs, π(w, L) and U(w, L)

Solution by backwards-induction

1. We analyze (and solve) the Firm problem for a generic observed wage w.

2. The solution gives us the optimal level of employment for any salary level.

3. Then we solve the problem of the Union assuming that Union knows the reaction of the firm to any wage demand w.

Firm's problem

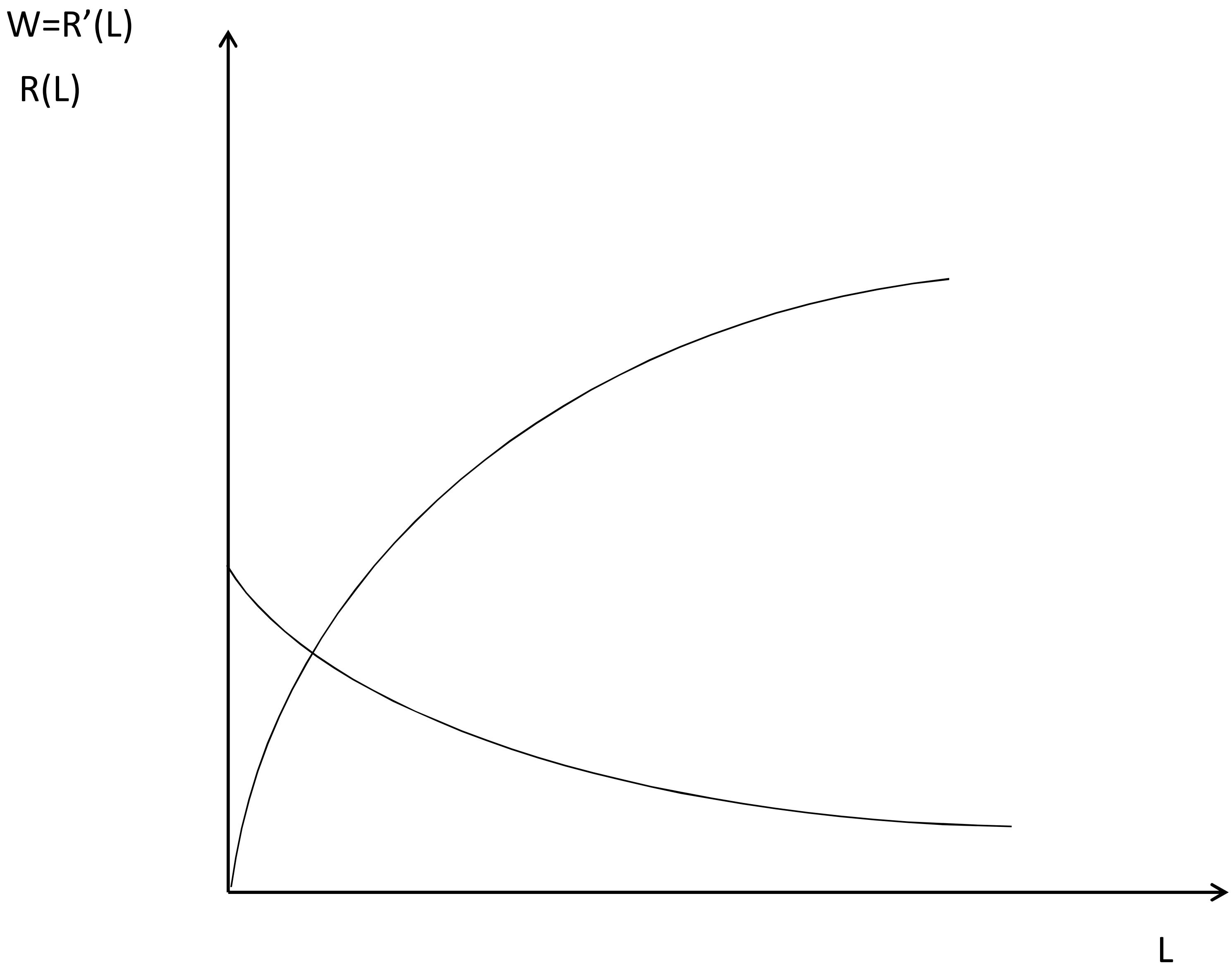
$$\text{Max}_{\{L\}} \; R(L) - w\,L$$

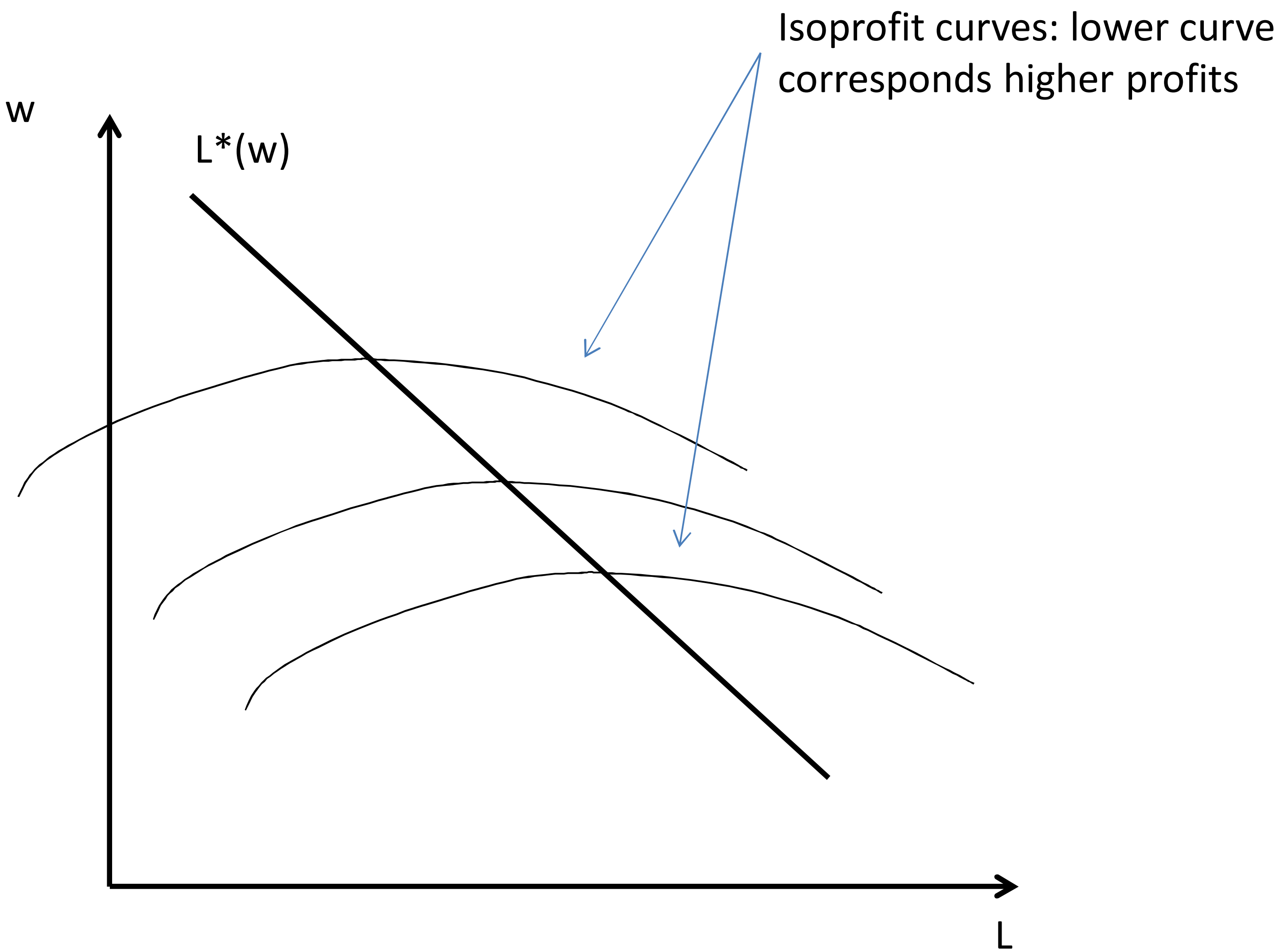
FOCs are:

$$R'(L) - w = 0 \;\rightarrow\; w = R'(L)$$

The solution will give us the reaction (best response) of the firm to a salary demand w, i.e. $L^*(w)$

Given that R(L) is increasing and concave, it follows that R'(L) is decreasing respect to L , $L^*(w)$ will be decreasing respect to w


W=R'(L)
R(L)
L


Isoprofit curves: lower curve
corresponds higher profits
w
L*(w)
L

Union's Problem

$$\max_{\{w\}} U(w, L)$$

Note that Union can anticipate the firm reaction to a wage demand w

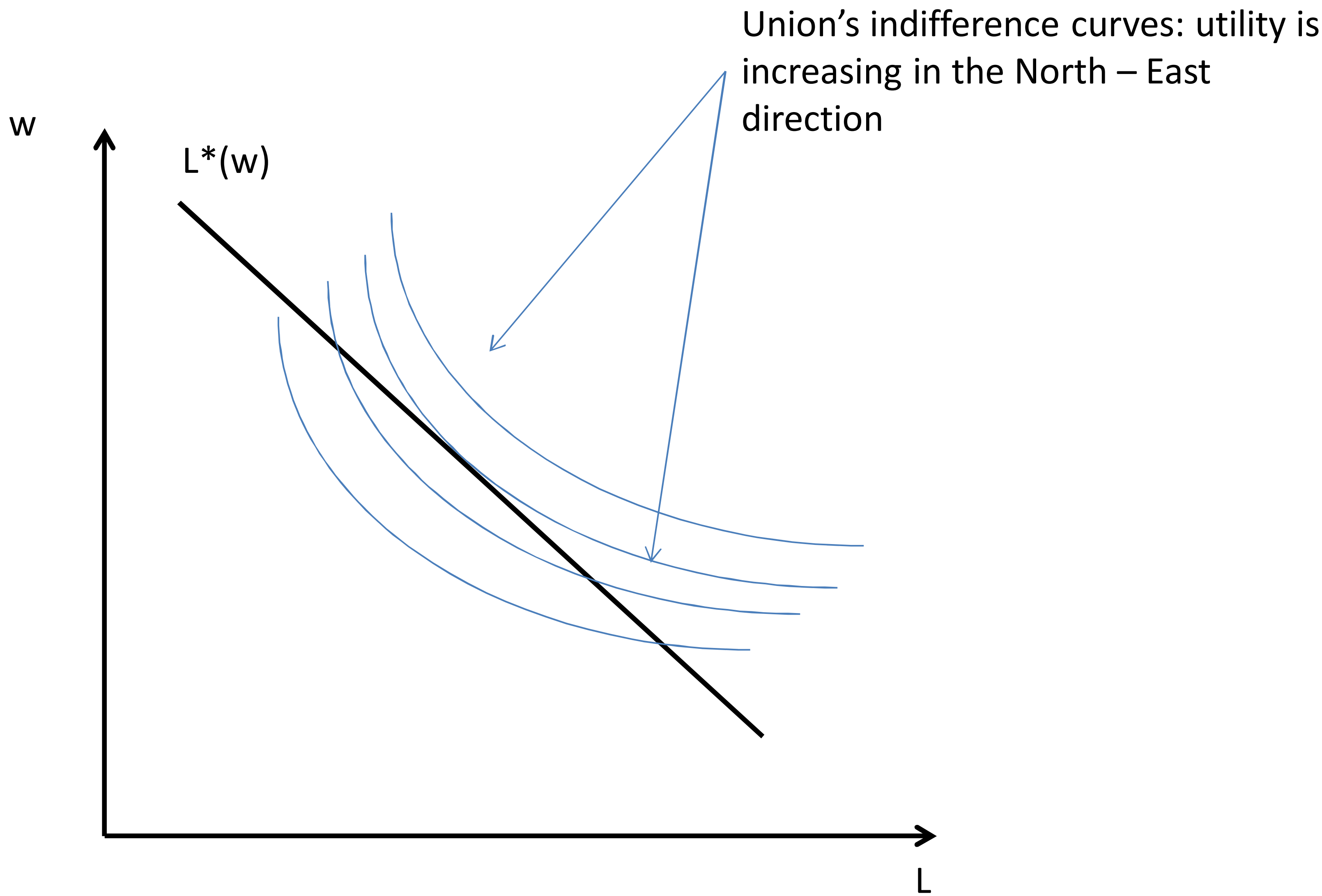
(Union can solve the firm's problem as well as the firm can solve it)

Then its problem is:

$$\max_{\{w\}} U(w, L^*(w))$$

FOCs are:

$$\frac{d\mathrm{U}(\mathrm{w}, L^*(\mathrm{w}))}{dw} = U'_1 + U'_2 L^{*\prime} = 0$$



The union's problem is to choose a w such that (w, L*(w)) is on the highest possible indifference curve

The solution will be:

(w*, L*(w*))

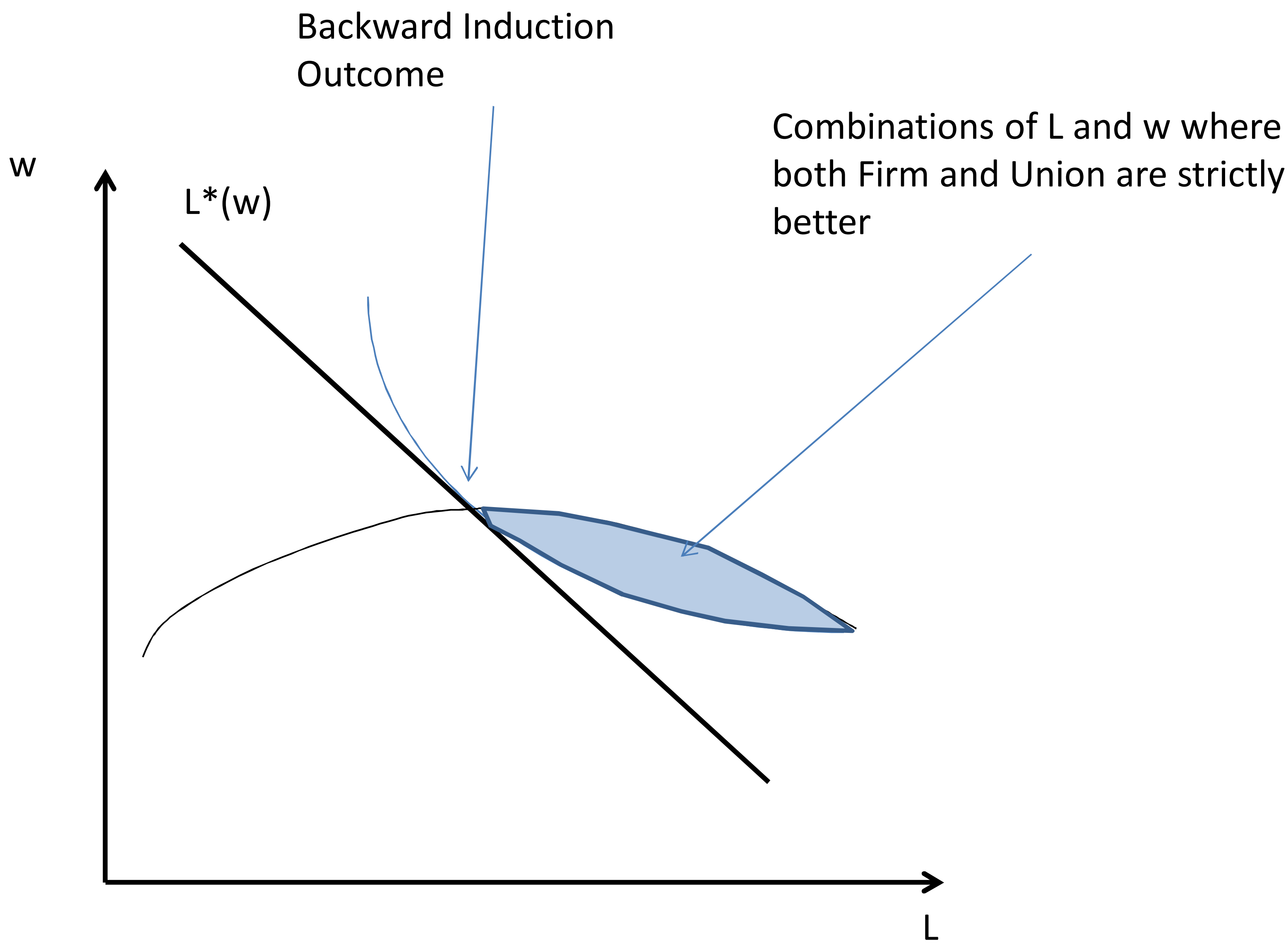
where the union indifference curve through the point (w*, L*(w*)) will be tangent to L*(w) at that point

This solution is not efficient, in the sense that exist other combinations of L and w where Union and Firm are strictly better.

Note:

(w*, L*(w*)) is the backward induction outcome

(w*, L*(w)) is the Subgame Perfect Nash Equilibrium



Backward Induction
Outcome
Combinations of L and w where both Firm and Union are strictly better
w
L*(w)
L

Example

$$\pi(\text{w, L}) = 10\,L - L^2 - wL$$

$$U(w, L) = \text{L} + \ln w$$

Applications with imperfect information

Bank Runs

- Two investors, one bank
- Each investor has deposited D with a bank
- The bank has invested $2D$ in a long term project
- If the bank liquidates the investment before the end, it will get back $2r$, where $D/2 < r < D$
- Otherwise the bank will get R, where $R > D$

- Investors can make withdrawals from the bank at:
  - Date 1, before the end of the investment
  - Date 2, after the end of the investment
- It is enough that one investor makes withdrawal at date 1 to force the bank to liquidate the investment

Payoffs:

- Both investors make withdrawals at date 1:
  - each one receives *r*.
- Only one investor makes withdrawal at date 1:
  - he receives *D*, the other receives *2r – D*.
- Neither investor makes withdrawal at date 1:
  - Both investors will take a withdrawal decision at date 2
- Both investors make withdrawals at date 2:
  - each receive *R*
- Only one investor makes withdrawal at date 2:
  - he receives *2R - D*, the other receives *D*.
- Neither investor makes withdrawal at date 2:
  - The Bank returns *R* to each investor

Date 1			
		Investor 2	
		Withdraw	No withdraw
Investor 1	Withdraw	r, r	D, 2r - D
	No withdraw	2r – D, D	Next stage

Date 2			
		Investor 2	
		Withdraw	No withdraw
Investor 1	Withdraw	R, R	2R – D, D
	No withdraw	D, 2R – D	R, R

We solve the game in date 2

Date 2			
		Investor 2	
		Withdraw	No withdraw
Investor 1	Withdraw	*R, R*	*2R – D, D*
	No withdraw	*D, 2R – D*	*R, R*

In date 2's game there is only one Nash equilibrium:

{(Withdraw), (Withdraw)}

where each Investor gets *R*

In date 1 the two investors anticipate that in the case neither investor makes withdrawal at date 1, the game goes in the second stage (date 2) and that in date two the outcome will be (the NE):

{(Withdraw), (Withdraw)}

where each Investor gets *R.*

Then the game in date 1 can be written as:

Date 1			
		Investor 2	
		Withdraw	No withdraw
Investor 1	Withdraw	*r, r*	*D, 2r - D*
	No withdraw	*2r – D, D*	*R, R*

We solve the game in date 1:

		Date 1	
		Investor 2	
		Withdraw	No withdraw
Investor 1	Withdraw	$\underline{r}, \underline{r}$	$D, 2r - D$
	No withdraw	$2r – D, D$	$\underline{R}, \underline{R}$

There are two Nash equilibria in date 1 game:

{(Withdraw), (Withdraw)}

{(No withdraw), (No Withdraw)}

Date 2 game, one NE: {(Withdraw), (Withdraw)}

(reduced) Date 1 game, two NE:

1. {(Withdraw), (Withdraw)}
2. {(No withdraw), (No Withdraw)}

Whole game:

Two Backward Induction Outcomes (BIO):

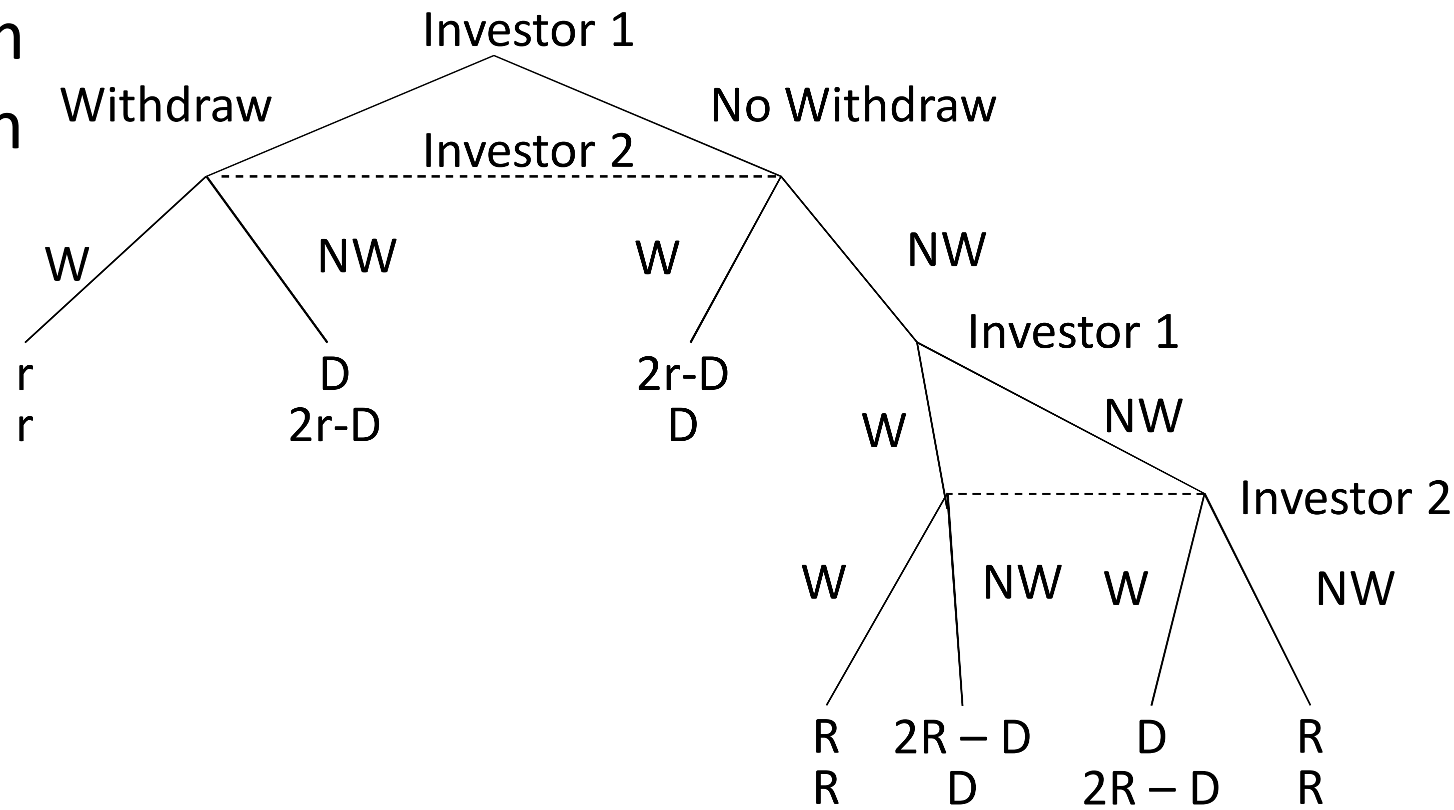
1) {(Withdraw), (Withdraw)} in date 1
2) {(No withdraw), (No Withdraw)} in date 1, {(Withdraw), (Withdraw)} in date 2

Two subgame perfect NE (SPNE):

1) {(Withdraw, Withdraw), (Withdraw, Withdraw)}
2) {(No withdraw, Withdraw), (No Withdraw, Withdraw)}

Note SPNE 1) supports BIO 1), SPNE 2) supports BIO 2)

Extensive form representation



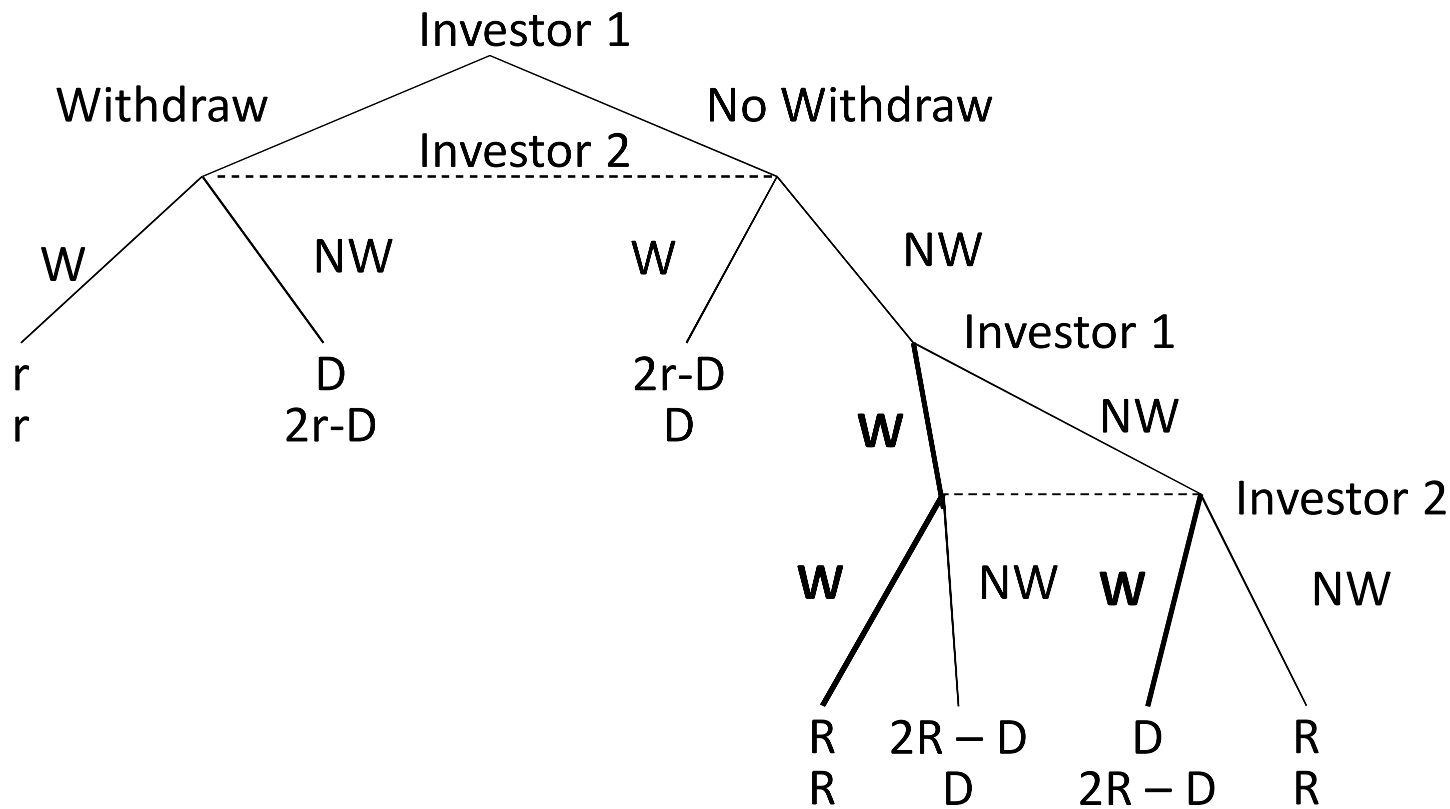
Investor 1: 2 information sets

Investor 2: 2 information sets

Investor 1's strategies: {(W, W), (W, NW), (NW, W), (NW, NW)}

Investor 2's strategies: {(W, W), (W, NW), (NW, W), (NW, NW)}

Backward Induction



Investor 1: 2 information sets
Investor 2: 2 information sets
Investor 1's strategies: {(W, W), (W, NW), (NW, W), (NW, NW)}
Investor 2's strategies: {(W, W), (W, NW), (NW, W), (NW, NW)}

Reduced game