Lecture 8
Strategic Games with Incomplete Information

Incomplete Information and Bayesian Games

- We have considered games of **complete information:** all players know the preferences of all others
- We now consider situations, where players have **incomplete information:** they do not know some relevant characteristic of other players. This may include the payoffs, the actions, and the beliefs

- In a Bayesian game there are different **types** of the players and players know their own type but not the type of the other players

- Note that in Osborne's book this class of games is denoted as "*imperfect information*". In literature this term is used to denote sequential games (extensive form) with at least one information set containing more than one decision node.

Example Cournot with incomplete and asymmetric information

Normal Form Representation of Bayesian games

A normal form specifies:

1. the *agents* in the game,
2. for each agent $\boldsymbol{i}$ the set of *available actions* $\boldsymbol{A_i}$ where $\boldsymbol{a_i}$ is an element of $\boldsymbol{A_i}$
3. for each agent $\boldsymbol{i}$ *the set of possible types* $\boldsymbol{T_i}$ where $\boldsymbol{t_i}$ is an element of $\boldsymbol{T_i}$
4. for each agent $\boldsymbol{i}$ *the belief* $\boldsymbol{p_i}$, i.e. the probability distribution over all possible realizations of types

5. for each agent i, and for each possible type in T_i, the payoff received for each possible combination of strategies.

$$G = \{A_1, \ldots\ldots A_n;\ T_1, \ldots\ldots T_n;\ p_1 \ldots p_n\ ;\ u_1 \ldots u_n\}$$

Note:

$\boldsymbol{u_i}$ $(\boldsymbol{a_1} \ldots \boldsymbol{a_i} \ldots \boldsymbol{a_n}; \boldsymbol{t_i})$ denotes the payoff function of player $\boldsymbol{i}$ where $(\boldsymbol{a_1} \ldots \boldsymbol{a_i} \ldots \boldsymbol{a_n})$ are the actions chosen by the players and $\boldsymbol{t_i}$ is his/her type realization.

Player type $\boldsymbol{t_i}$ is privately observed by player $\boldsymbol{i}$ (or by a subset of players)

By $\boldsymbol{p_i(t_{-i}|ti)}$ we denote the beliefs of player i on the possible realizations of the other players' types, given her realization of type $\boldsymbol{t_i}$

$$\boldsymbol{p_i(t_{-i}|t_i) = \frac{p_i(t_{-i}, t_i)}{p_i(t_i)}}$$

We can represent this class of games assuming there exists **a player "nature"** drawing the types and the players not being perfectly informed about nature's moves

1) Nature draws a type vector $t = (t_1,\ldots,t_n)$
2) Nature reveals t_i to player i
3) Players simultaneously choose actions
4) Payoffs $u_i(a_1 \ldots a_i \ldots a_\mathrm{n}; t_i)$ are received

By introducing the player "Nature" we have described the game of incomplete information as a game of imperfect information.

Example, 2 players, two types, two actions

Definition of strategy

In a Bayesian game

$G = \{A_1, \ldots\ldots A_n; T_1, \ldots\ldots T_n; p_1 \ldots p_n; u_1 \ldots u_n\}$

a strategy for player *i* specifies an action a_i from the feasible set A_i **for each type t_i in T_i**

The strategy can be represented by a function $s_i(t_i)$

i.e. a strategy is a *contingent plan of actions*

Example:

Definition of Bayesian Nash equilibrium

In the Bayesian game

$G = \{A_1, \ldots.. A_n; T_1, \ldots.. T_n; p_1 \ldots p_n ; u_1 \ldots u_n \}$

The strategies $\boldsymbol{s^* = (s_1^*, \ldots s_n^*)}$ are a *Bayesian Nash equilibrium* if for all i and all types in T_i

$s_i^*(t_i)$ is a best response to the others' strategies in s^*

i.e. $s_i^*(t_i)$ solves

$$\max_{a_i \in A_i} \sum_{t_{-1} \in T_{-1}} u_i\left(s_1^*(t_1), \ldots, s_{i-1}^*(t_{i-1}), a_i, s_{i+1}^*(t_{i+1}), \ldots s_n^*(t_n); t_i\right) \cdot p_i(t_{-i} \mid t_i)$$

(See solutions 1 and 2 of the following example)

Bayesian Nash equilibrium (2)

The Bayesian Nash equilibrium of the game

$$G = \{A_1, \ldots\ldots A_n;\ T_1, \ldots\ldots T_n;\ p_1 \ldots p_n;\ u_1 \ldots un\}$$

is the Nash equilibrium of the game:

$$G' = \{S_1, \ldots\ldots S_n;\ u'1 \ldots u'_n\}$$

where $\boldsymbol{S_i}$ is the strategy space of players *i*

and $\boldsymbol{u'_i}$ is the function that gives the player *i*'s expected payoff for each possible combination of players' strategies

(See solution 3 of the following example)

Example 1: Prisoner's dilemma

Consider the following modification of the prisoner dilemma game:

a) Player 1 herself is not selfish, but a ***conditional cooperator***, i.e. she likes to cooperate as long as others do.

b) Player 2 is ***selfish*** by probability p and ***cooperative*** by probability 1 – p,

c) Assume $0 < p < 1$

Then if Player 2 is selfish, the matrix is

		Player 2	
		C(ooperate)	D(efect)
Player 1	C(ooperate)	3,2	0,3
	D(efect)	2,0	1,1

But if Player 2 is cooperative, the payoff matrix is, e.g.

		Player 2	
		C(ooperate)	D(efect)
Player 1	C(ooperate)	3,3	0,2
	D(efect)	2,1	1,0

It now matters which type of Player 2 Player 1 believes to face

What is a strategy in this context?

For Player 1 is an action

For Player 2 is a contingent plan of actions:

> the action to play when he is cooperative and the action he plays when he is selfish: *(x, y) means to play x when he is selfish and to play y when he is cooperative*

In the example there are 4 strategies for player 2:

(C, C), (C, D), (D, C), (D, D).

Solution 1

		Player 2 $(a_2(t_1), a_2(t_2))$			
		(C,C)	(C,D)	(D,C)	(D,D)
Player 1	C	3	$3p$	$3(1-p)$	0
	D	2	$2p+(1-p)$	$p+2(1-p)$	1

type 1 by prob p		Player 2	
		C	D
Player 1	C	3,2	0,3
	D	2,0	1,1

type 2 by prob 1-p		Player 2	
		C	D
Player 1	C	3,3	0,2
	D	2,1	1,0

Suppose Player 2 plays (C, C); Player 1's best response is C; Player 2's best response to C is (D, C)

(C, D) → C (if p>0.5) → (D, C)

(C, D) → D (if p<0.5) → (D, C)

We repeat this procedure for all possible strategy of a player

Nash equilibria: {C,(D,C)} if p <1/2 and {D,(D,C)} if p >1/2

Solution 2

		Player 2 $(a_2(t_1), a_2(t_2))$			
		(C,C)	(C,D)	(D,C)	(D,D)
Player 1	C	3	$3p$	$3(1-p)$	0
	D	2	$2p+(1-p)$	$p+2(1-p)$	1

- We see that for type 1 of Player 2, D is dominant, and for type 2, C is dominant.
- Then for player 2 is optimal to play D when he is of type 1 and to play C when he is of type 2. This strategy is denoted by (D, C)
- Player 1's best response to (D,C) is C if $p < 1/2$ and D if $p > 1/2$ (he compares $3(1-p)$ with $p+2(1-p)$)
- Nash equilibria: {C,(D,C)} if $p < 1/2$ and {D,(D,C)} if $p > 1/2$
- For $p = 1/2$, Player 1 is indifferent, so we get arbitrary mixing

Solution 3

type 1 by prob p		Player 2	
		C	D
Player 1	C	3,2	0,3
	D	2,0	1,1

type 2 by prob 1-p		Player 2	
		C	D
Player 1	C	3,3	0,2
	D	2,1	1,0

We can represent this game in the following equivalent normal form where we label the rows and columns with strategies and payoff are computed ex-ante, i.e. before that player 2 knows his/her type.

		Player 2			
		(C,C)	(C,D)	(D,C)	(D,D)
Player 1	C	3, 3-p	3p, 2	3(1-p), 3	0, 2+p
	D	2, 1-p	1+p, 0	2-p, 1	1, p

type 1 by prob p		Player 2 C	Player 2 D
Player 1	C	3,2	0,3
	D	2,0	1,1

type 2 by prob 1-p		Player 2 C	Player 2 D
Player 1	C	3,3	0,2
	D	2,1	1,0

$E_1(C|(CC) = 3p + 3(1-p)$

$E_2((CC)|(C) = 2p + 3(1-p)$

$E_1(C|(DC) = 0p + 3(1-p)$

$E_2((DC)|(C) = 3p + 3(1-p)$

$E_1(C|(CD) = 3p + 0(1-p)$

$E_2((CD)|(C) = 2p + 2(1-p)$

$E_1(C|(DD) = 0p + 0(1-p)$

$E_2((DD)|(C) = 3p + 2(1-p)$

		Player 2 (C,C)	Player 2 (C,D)	Player 2 (D,C)	Player 2 (D,D)
Player 1	C	3, 3-p	3p, 2	3(1-p), 3	0, 2+p
	D	2, 1-p	1+p, 0	2-p, 1	1, p

		Player 2			
		(C,C)	(C,D)	(D,C)	(D,D)
Player 1	C	3, 3-p	3p, 2	3(1-p), 3	0, 2+p
	D	2, 1-p	1+p, 0	2-p, 1	1, p

For Player 2 strategy (D, C) is dominant, i.e. it represents a best response to all strategies of player 1.

For player 1 the best response to strategy (D, C) is:

a) C if $3(1-p) > 2 - p \rightarrow 1 - 2p > 0 \rightarrow p < 0.5$

b) D if $3(1-p) < 2 - p \rightarrow 1 - 2p < 0 \rightarrow p > 0.5$

NE is {C, (D, C)} if $p < 0.5$;

{D, (D, C)} if $p > 0.5$

Example 2

$G = \{A_1, A_2; T_1, T_2; p_1, p_2; u_1, u_2\}$ 2 players, 1 and 2

$A_1 = \{T, B\}, A_2 = \{L, R\}$

$t_1 \in \{1, 2\}$ $t_2 \in \{1, 2\}$

$t_1 = 1$ *by probability 0.5,* $t_2 = 1$ *by probability 0.5*

t_1 and t_1 are i.i.d.

		Player 2, $t_2 = 1$	
		L	R
Player 1 $t_1 = 1$	T	2, 2	0, 0
	B	0, 0	1, 1

		Player 2, $t_2 = 2$	
		L	R
Player 1 $t_1 = 1$	T	2, 1	0, 0
	B	0, 0	1, 2

		Player 2, $t_2 = 1$	
		L	R
Player 1 $t_1 = 2$	T	1, 2	0, 0
	B	0, 0	3, 1

		Player 2, $t_2 = 2$	
		L	R
Player 1 $t_1 = 2$	T	1, 2	0, 0
	B	0, 0	3, 2

		Player 2, t_2= 1	
		L	R
Player 1 $t_1 = 1$	T	2, 2	0, 0
	B	0, 0	1, 1

		Player 2, t_2= 2	
		L	R
Player 1 $t_1 = 1$	T	2, 1	0, 0
	B	0, 0	1, 2

		Player 2, t_2= 1	
		L	R
Player 1 $t_1 = 2$	T	1, 2	0, 0
	B	0, 0	3, 1

		Player 2, t_2= 2	
		L	R
Player 1 $t_1 = 2$	T	1, 1	0, 0
	B	0, 0	3, 2

Player 1's strategies: {(T, T), (T, B), (B, T), (B, B)}

Player 2's strategies: {(L, L), (L, R), (R, L), (R, R)}

Solution 1

		Player 2			
		LL	LR	RL	RR
Player 1 $t_1 = 1$	T	2	1	1	0
	B	0	0.5	0.5	1

		Player 2			
		LL	LR	RL	RR
Player 1 $t_1 = 2$	T	1	0.5	0.5	0
	B	0	1.5	1.5	3

		Player 1			
		TT	TB	BT	BB
Player 2 $t_1 = 1$	L	2	1	1	0
	R	0	0.5	0.5	1

		Player 1			
		TT	TB	BT	BB
Player 2 $t_1 = 2$	L	1	0.5	0.5	0
	R	0	1	1	2

1. Suppose Player 2 plays (L,L), Player 1 best response is (T,T) ; Player 2 best response to (T, T,) is (L,L)

   Then (T, T) (L, L) is BNE
2. Suppose Player 2 plays (L,R), Player 1 best response is (T,B) ; Player 2 best response to (T, B,) is (L,R)

   Then (T, B) (L, R) is BNE
3. Suppose Player 2 plays (R,L), Player 1 best response is (T,B) ; Player 2 best response to (T, B,) is (L,R)

   Then (T, B) (R, L) is **not** a BNE
4. Suppose Player 2 plays (R,R), Player 1 best response is (B,B) ; Player 2 best response to (B, B,) is (R,R)

   Then (B, B) (R, R) is BNE

Solution 3

		Player 2			
		LL	LR	RL	RR
Player 1	TT	1.5, 1.5	0.75, 1	0.75, 0.5	0, 0
	TB	1, 075	1.25, 1	1.25, 0.5	1.5, 0.75
	BT	0.5, 0.75	0.5, 1	0.5, 0.5	0.5, 0.75
	BB	0, 0	1, 1	1, 0.5	2, 1.5

Applications

1. Mixed strategies
2. Auction