1. Consider the following game:

		Player 2	
		C	D
Player 1	A	x, x	z, w
	B	w, z	y, y

where y>z , w>x and x>y

This stage game is played an infinite number of times.

a. Find the Nash equilibrium in pure strategies of the stage game.
b. Find the value of the discount factor such that the strategy profile {(A), (C)} is played in each period of the repeated game

Solution

a) $y > z$ and $w > x$ imply that:
- strategy A is strictly dominated by strategy B
- strategy C is strictly dominated by strategy D

Then there is only one Nash equilibrium in pure strategies: {B, D}

b) Trigger strategy:
- Player 1: In period 1 he plays A; in the other periods he plays A only if Player 2 played C in all past periods, otherwise he plays B.
- Player 2: In period 1 he plays C; in the other periods he plays C only if Player 1 played A in all past periods, otherwise he plays D.

This strategy is an equilibrium if:

$$\frac{x}{1-\delta} \geq w + \frac{\delta y}{1-\delta}$$

That is satisfied for:

$$\delta \geq \frac{w-x}{w-y}$$

2. Find the pure strategy Nash equilibria of the following repeated game

		Player 2		
		L2	M2	R2
Player 1	L1	11, 0	0, 0	6, 6
	M1	10, 10	0, 0	0, 11
	R1	0, 0	8, 8	0, 0

This stage game is repeated 3 periods.

Solution:

In the stage game there are two Nash equilibrium: {R1, M2} and {L1, R2}.

The strategy profile {M1, L2} provides to both players higher payoffs than the Nash equilibria.

All strategy profiles where in each period any of the two Nash equilibrium of the stage game is played is a Nash equilibrium of the repeated game. For example, in the table there are the BIO sustained by strategy profiles that are NE in the stage game.

Period								
1	{L1, R2}	{R1, M2}	{L1, R2}	{L1, R2}	{R1, M2}	{R1, M2}	{L1, R2}	{R1, M2}
2	{L1, R2}	{L1, R2}	{R1, M2}	{L1, R2}	{R1, M2}	{L1, R2}	{R1, M2}	{R1, M2}
3	{L1, R2}	{L1, R2}	{L1, R2}	{R1, M2}	{L1, R2}	{R1, M2}	{R1, M2}	{R1, M2}

Then NE are

1) {(L1, L1, L1), (R2, R2, R2)}
2) {(R1, L1, L1), (M2, R2, R2)}
3) {(L1, R1, L1), (R2, M2, R2)}
4) {(L1, L1, R1), (R2, R2, M2)}
5) {(R1, R1, L1), (M2, M2, R2)}
6) {(R1, L1, R1), (M2, R2, M2)}
7) {(L1, R1, R1), (R2, M2, M2)}
8) {(R1, R1, R1), (M2, M2, M2)}

But there are other Nash equilibria. For example the following trigger strategy:

- Player 1: In period 1 he plays M1; in the other periods he plays R1 only if Player 2 played L2 in period1, otherwise he plays L1.

- Player 2: In period 1 he plays L2; in the other periods he plays M2 only if Player 1 played M1 in period 1, otherwise he plays R2

In according to this strategy, using backward induction, you can compute the reduced game in period 1

		Player 2		
		L2	M2	R2
Player 1	L1	23, 12	12, 12	**18, 18**
	M1	**26, 26**	12, 12	12, 23
	R1	12, 12	**20, 20**	12, 12

In bold the Nash equilibria.

Then {(M1, R1, R1), (L2, M2, M2)}) is a Nash equilibria of the stage game.

Note as this trigger strategy sustains other two Nash equilibria:

1) {(L1, L1, L1), (R2, R2, R2)}
2) {(R1, L1, L1), (M2, R2, R2)}

3. The following stage game is played an infinite number of times. Stage game: Two firms, 1and 2, produce an homogeneous good. Firms have no fixed cost and produce at constant marginal cost of 1. By q1 and q2 we denote the quantities produced, respectively, by firm 1 and 2. The inverse demand function is given by P(Q) = 100 – Q where Q = q1 + q2. Firms 1 and 2 simultaneously choose the quantities to produce.

Firms discount their future profits by a factor $0 < d < 1$.

In the infinitively repeated game find the set of values of the discount factor d such that the following strategy is an equilibrium:

To collude in period 1, then in periods t > 1, to collude only if the outcome in all previous periods was collusion, otherwise to play Nash equilibrium.

(remember in the collusion each firm produce half of the quantity that maximizes the profits in the case of monopoly)

Solution

In the Nash equilibrium $q_1 = q_2 = 33$

The profits are $\pi_1^e = \pi_2^e = 33(100 - 33 - 33 - 1) = 1089$

If the two firms collude they produce: $q_1 = q_2 = 24.75$

The profits are $\pi_1^c = \pi_2^c = 24.75(100 - 24.75 - 24.75 - 1) = 1225$

The best profitable deviation (best response) when firm 1 produce the collusion quantity is (for firm 2): $q_2 = 37.125$

$$\pi_2^d = 37.125(100 - 37.125 - 24.75 - 1) = 1378$$

The strategy

"To collude in period 1, then in periods t > 1, to collude only if the outcome in all previous periods was collusion, otherwise to play Nash equilibrium."

Is an equilibrium if

$$\frac{\pi_1^c}{1 - d} \geq \pi_1^d + \frac{d\pi_1^e}{1 - d}$$

$$\frac{1225}{1 - d} \geq 1378 + \frac{d\,1089}{1 - d}$$

That is satisfied for:

$$d \geq \frac{\pi_1^d - \pi_1^c}{\pi_1^d - \pi_1^e} = 0.529$$

4. Carrot-stick strategy: collusion quantity is played if in the previous period the outcome was either collusion or punishment. The punishment quantity is played in all other cases. Punishment: play a quantity x.

The payoff from punishment is: $\pi_1^p = (99 - 2x)x = 99x - 2x^2$
The best profitable deviation (best response) from punishment is:
$$q^{dp} = \frac{99 - x}{2}$$
And payoff is
$$\pi^{dp} = \left(99 - x - \frac{99 - x}{2}\right)\frac{99 - x}{2} = \frac{(99 - x)^2}{4}$$
Other relevant payoffs from previous exercise:
$$\pi_1^c = 1225 \quad \pi_1^d = 1378$$
Two types of subgames:
Collusive subgames: collusion quantity is played if in the previous period the outcome was either collusion or punishment.
Punishment subgames: the punishment quantity is played because in the previous period the outcome was neither collusion nor punishment.

Equilibrium in **Collusive subgames:**

$$\frac{\pi_1^c}{1 - \delta} > \pi_1^d + \delta\pi_1^p + \frac{\delta^2\pi_1^c}{1 - \delta}$$
$$\frac{1225}{1 - \delta} > 1378 + \delta(99x - 2x^2) + \frac{\delta^2 1225}{1 - \delta}$$
$$1225(1 + \delta) > 1378 + \delta(99x - 2x^2)$$
$$x > 24.75. + \sqrt{\frac{1224 + \delta}{16\delta}} \text{ or } x < 24.75 - \sqrt{\frac{1224 + \delta}{16\delta}} \quad [1]$$

Equilibrium in **punishment subgames:**

$$\pi_1^p + \frac{\delta\pi_1^c}{1 - \delta} > \pi^{dp} + \delta\pi_1^p + \frac{\delta^2\pi_1^c}{1 - \delta}$$
$$(99x - 2x^2) + \frac{\delta 1225}{1 - \delta} > \frac{(99 - x)^2}{4} + \delta(99x - 2x^2) + \frac{\delta^2 1225}{1 - \delta}$$
$$(99x - 2x^2)(1 - \delta) + \delta\, 1225 > \frac{(99 - x)^2}{4}$$
$$\frac{99.(-3. + 2.\delta)}{-9. + 8.\delta} - 2.\sqrt{\frac{1224.\delta + \delta^2}{(-9. + 8.\delta)^2}} < x < \frac{99.(-3. + 2.\delta)}{-9. + 8.\delta} + 2.\sqrt{\frac{1224.\delta + \delta^2}{(-9. + 8.\delta)^2}} \quad [2]$$

Fix $\delta = 0.8$. From the eq condition [1] we get
$$x < 14.97 \text{ or } x > 34.53$$
From the eq condition [2] we get
$$29.23 < x < 77.39$$
Then x satisfying both condition is
$$34.53 < x < 77.39$$

5. The following stage game is played three times. Stage game:
   3 players simultaneously contribute to a public good G where G is the sum of the contributions. The player i's payoff is given by 0.5 G – g_i where g_i is the contribution of player i. The individual contribution g_i ranges in the interval [0, 10].
   Find a Subgame Perfect outcome of the repeated game in which all players choose to contribute 10 at least in one stage.

Solution

The payoff of a player is decreasing with its contribution. Then for all strategies of the other players the best response is to contribute with g=0.
Therefore there is an unique Nash equilibrium of the stage game that is
$$\{(0),(0),(0)\}$$
Then, a Subgame Perfect outcome of the repeated game in which all players choose to contribute 10 at least in one stage does not exist. (see the first result in lecture 7)