1. An individual is characterized by βδ-preferences where $\beta = 0.7$ and $\delta = 0.9$ and his instantaneous utility function is $u(x) = \ln(10 + x)$ where x is the spending. At $t = 1$ Paul receives an endowment $W = 10$ to spend in $t = 2, t = 3$ and $t = 4$. (Assume R=1)

   a) Compute the optimal plan of spending from the perspective of $t = 1$.
   b) Compute the optimal plan of spending from the perspective of $t = 2$.
   c) Assume the agent forecasts correctly his future behaviour (i.e. he is sophisticated). Compute the optimal share of W invested in illiquid asset in $t = 1$

Solution

By c_i we denote the spending in period $i = 2, 3, 4$.
By w_i we denote the endowment in period $i = 1, 2, 3, 4$; then $w_1 = 10$

a) The problem is

$$\max_{\{c_2,c_3,c_4\}} \beta\delta u(c_2) + \beta\delta^2 u(c_3) + \beta\delta^3 u(c_4)$$
$$s.t.\, c_2 \leq w_1, c_3 \leq w_2, c_4 \leq w_3 \; and \; c_2, c_3, c_4 \geq 0$$

Note that $w_2 = w_1 - c_2$, $w_3 = w_2 - c_3$ and that in the solution must be $w_4 = 0$ (no resources left)

Then we can rewrite the problem as

$$\max_{\{w_2,w_3\}} \beta\delta u(w_1 - w_2) + \beta\delta^2 u(w_2 - w_3) + \beta\delta^3 u(w_3)$$
$$s.t.\, w_2, w_3 \geq 0$$

Using the assumption given in the text we have:

$$\max_{\{w_2,w_3\}} 0.7 \cdot 0.9 ln(10 + w_1 - w_2) + 0.7 \cdot 0.9^2 ln(10 + w_2 - w_3) + 0.7 \cdot 0.9^3 ln(10 + w_3)$$
$$s.t.\, w_2, w_3 \geq 0$$

FOCs are:

$$-\frac{0.7 \cdot 0.9}{10 + w_1 - w_2} + \frac{0.7 \cdot 0.9^2}{10 + w_2 - w_3} = 0$$
$$-\frac{0.7 \cdot 0.9^2}{10 + w_2 - w_3} + \frac{0.7 \cdot 0.9^3}{10 + w_3} = 0$$

Replacing $c_2 = w_1 - w_2$, $c_3 = w_2 - w_3$, $c_4 = w_3$ and simplifying

$$\frac{1}{10 + c_2} = \frac{0.9}{10 + c_3}$$
$$\frac{1}{10 + c_3} = \frac{0.9}{10 + c_4}$$

then

$$c_3 = 0.9c_2 - 1$$
$$c_4 = 0.9c_3 - 1$$

then replacing the first in the second we get

$$c_4 = 0.81c_2 - 1.9$$

To compute the spending of all periods we have to solve:

$$c_2 + c_3 + c_4 = 10$$

Replacing in the previous results we get:

$$c_2 + 0.9c_2 - 1 + 0.81c_2 - 1.9 = 10$$

and then

$$c_2 = 4.76$$
$$c_3 = 3.28$$
$$c_4 = 1.96$$

b) The problem is

$$\max_{\{c_2,c_3,c_4\}} u(c_2) + \beta\delta u(c_3) + \beta\delta^2 u(c_4)$$
$$s.t.\, c_2 \leq w_1, c_3 \leq w_2, c_4 \leq w_3 \; and \; c_2, c_3, c_4 \geq 0$$

Note that $w_2 = w_1 - c_2$, $w_3 = w_2 - c_3$ and that in the solution must be $w_4 = 0$ (no resources left)

Then we can rewrite the problem as

$$\max_{\{w_2,w_3\}} u(w_1 - w_2) + \beta\delta u(w_2 - w_3) + \beta\delta^2 u(w_3)$$
$$s.t.\, w_2, w_3 \geq 0$$

Using the assumption given in the text we have:

$$\max_{\{w_2,w_3\}} ln(10 + w_1 - w_2) + 0.7 \cdot 0.9 ln(10 + w_2 - w_3) + 0.7 \cdot 0.9^2 ln(10 + w_3)$$
$$s.t.\, w_2, w_3 \geq 0$$

FOCs are:

$$-\frac{1}{10 + w_1 - w_2} + \frac{0.7 \cdot 0.9}{10 + w_2 - w_3} = 0$$
$$-\frac{0.7 \cdot 0.9}{10 + w_2 - w_3} + \frac{0.7 \cdot 0.9^2}{10 + w_3} = 0$$

Replacing $c_2 = w_1 - w_2$, $c_3 = w_2 - w_3$, $c_4 = w_3$ and simplifying

$$\frac{1}{10 + c_2} = \frac{0.63}{10 + c_3}$$
$$\frac{1}{10 + c_3} = \frac{0.9}{10 + c_4}$$

then

$$c_3 = 0.63c_2 - 3.7$$
$$c_4 = 0.9c_3 - 1$$

then replacing the first in the second we get

$$c_4 = 0.57c_2 - 4.33$$

To compute the spending of all periods we have to solve:

$$c_2 + c_3 + c_4 = 10$$

Replacing in the previous results we get:

$$c_2 + 0.63c_2 - 3.7 + 0.57c_2 - 4.33 = 10$$

and then

$$c_2 = 8.2$$
$$c_3 = 1.46$$
$$c_4 = 0.32$$

c) The optimal share invested in illiquid asset at time 1 is the endowment at time 1 minus the optimal consumption at time 2 computed from the perspective of time 1, then:

$$10 - 4.76 = 5.24$$

2. Solve all problems and examples in the slides of "doing it now or later"

Ex 3.
Immediate costs, $\beta = 0.5$, $\delta = 1$, $v = (12, 18, 18)$, $c = (3, 8, 13)$

Time consistent:

$$U_1(1) = 12 - 3 = 9$$
$$U_1(2) = 18 - 8 = 10$$
$$U_1(3) = 18 - 13 = 5$$

Then the PP strategy for a time consistent individual is (N, Y, Y) and $\tau_{tc} = 2$.

Naive:
From time perspective 1

$$U_1(1) = 0.5\ 12 - 3 = 3$$
$$U_1(2) = 0.5(18 - 8) = 5$$
$$U_1(3) = 0.5(18 - 13) = 2.5$$

Then $s_1 = N$

From time perspective 2

$$U_2(2) = 0.5\ 18 - 8 = 1$$
$$U_2(3) = 0.5(18 - 13) = 2.5$$

Then $s_2 = N$

Then the PP strategy for a naive individual is (N, N, Y) and $\tau_N = 3$.

Sophisticated
We go backward and we can use utilities computed for the naïve individual
$s_3 = Y$ (by definition)
Time perspective 2
$s_2 = N$
Time perspective 1
We compare only $U_1(1)$ with $U_1(3)$ then $s_1 = Y$

Then the PP strategy for a sophisticated individual is (Y, N, Y) and $\tau_s = 1$.

Immediate costs, $\beta = 0.5, \delta = 1, v = (1, 8, 0), c = (0, 5, 1)$

Time consistent:

$$U_1(1) = 1 - 0 = 1$$
$$U_1(2) = 8 - 5 = 3$$
$$U_1(3) = 0 - 1 = -1$$

Then the PP strategy for a time consistent individual is (N, Y, Y) and $\tau_{tc} = 2$.

Naive:
From time perspective 1

$$U_1(1) = 0.5\,1 - 0 = 0.5$$
$$U_1(2) = 0.5(8 - 5) = 1.5$$
$$U_1(3) = 0.5(0 - 1) = -0.5$$

Then $s_1 = N$

From time perspective 2

$$U_2(2) = 0.5\,8 - 5 = -1$$
$$U_2(3) = 0.5(0 - 1) = -0.5$$

Then $s_2 = N$

Then the PP strategy for a naive individual is (N, N, Y) and $\tau_N = 3$.

Sophisticated
We go backward and we can use utilities computed for the naïve individual
$s_3 = Y$ (by definition)
Time perspective 2
$s_2 = N$
Time perspective 1
We compare only $U_1(1)$ with $U_1(3)$ then $s_1 = Y$

Then the PP strategy for a sophisticated individual is (Y, N, Y) and $\tau_s = 1$.

Immediate rewards, $\beta = 0.5$, $\delta = 1$, $v = (0, 5, 1)$, $c = (1, 8, 0)$

Time consistent:

$$U_1(1) = 0 - 1 = -1$$
$$U_1(2) = 5 - 8 = -3$$
$$U_1(3) = 1 - 0 = 1$$

Then the PP strategy for a time consistent individual is (N, N, Y) and $\tau_{tc} = 3$.

Naive:
From time perspective 1

$$U_1(1) = 0 - 0.5\ 1 = -0.5$$
$$U_1(2) = 0.5(5 - 8) = -1.5$$
$$U_1(3) = 0.5(1 - 0) = 0.5$$

Then $s_1 = N$

From time perspective 2

$$U_2(2) = 5 - 0.5\ 8 = 1$$
$$U_2(3) = 0.5(1 - 0) = 0.5$$

Then $s_2 = Y$

Then the PP strategy for a naive individual is (N, Y, Y) and $\tau_N = 2$.

Sophisticated
We go backward and we can use utilities computed for the naïve individual
$s_3 = Y$ (by definition)
Time perspective 2
$s_2 = Y$
Time perspective 1
We compare only $U_1(1)$ with $U_1(2)$ then $s_1 = Y$

Then the PP strategy for a sophisticated individual is (Y, Y, Y) and $\tau_s = 1$.