force of mortality by age on a logarithmic scale. Between ages 60 and 90 mortality increases in a roughly linear way, i.e. exponentially increasing mortality on the natural scale. Below age 60 this linearity breaks down as the non-age-related component of mortality makes itself felt. Above age 95 there is evidence of data-quality problems, which is common for life-office data sets such as this.

3. Mortality laws and distributions for future lifetime

A major advantage of fitting a formula for the force of mortality is that smoothness is built-in and there is no need to separately graduate (smooth) the resulting fitted rates. In this paper we will look at some actuarial mortality laws listed in Table 1. The parameterisations in Table 1 are often different from those used by the original authors, such as Gompertz (1825) who gave his law as $\mu_x = Bc^x$, with $B > 0$ and $c > 0$. The more modern exponential parameterisations mean we can dispense with any constraints on the range of parameters, allowing them to vary over the entire real line. This has practical advantages in optimising log-likelihood functions using computers.

The naming convention in Table 1 follows Richards (2008) and is different from what might be seen elsewhere. For example, the model labelled as Makeham–Beard was proposed by Perks (1932). We have opted: (1) to use the term Makeham wherever the constant e^ϵ appears; (2) to name the logistic form $\frac{e^a}{1+e^a}$ after Perks; and (3) to use the term Beard wherever the logistic form has a so-called heterogeneity parameter, ρ.

Some of the models in Table 1 are related to the proportional hazards model of Cox (1972). For example, the Gompertz model can be expressed as a proportion of a baseline hazard, albeit as a time- or age-varying proportion. The Makeham model, however, cannot be expressed in terms of a baseline hazard due to the non-multiplicative e^ϵ term. The models in Table 1 are mainly non-linear in their nature, although this does not cause

Table 1. Some actuarial mortality laws and their corresponding integrated hazard functions, $H_x(t)$.

Mortality law	μ_x	$H_x(t)$
Gompertz (1825)	$e^{\alpha+\beta x}$	$\frac{(e^{\beta t}-1)}{\beta}e^{\alpha+\beta x}$
Makeham (1859)	$e^\epsilon + e^{\alpha+\beta x}$	$te^\epsilon + \frac{(e^{\beta t}-1)}{\beta}e^{\alpha+\beta x}$
Perks (1932)	$\frac{e^{\alpha+\beta x}}{1+e^{\alpha+\beta x}}$	$\frac{1}{\beta}\log\left(\frac{1+e^{\alpha+\beta(x+t)}}{1+e^{\alpha+\beta x}}\right)$
Beard (1959)	$\frac{e^{\alpha+\beta x}}{1+e^{\alpha+\rho+\beta x}}$	$\frac{e^{-\rho}}{\beta}\log\left(\frac{1+e^{\alpha+\rho+\beta(x+t)}}{1+e^{\alpha+\rho+\beta x}}\right)$
Makeham–Perks (1932)	$\frac{e^\epsilon + e^{\alpha+\beta x}}{1+e^{\alpha+\beta x}}$	$te^\epsilon + \frac{(1-e^\epsilon)}{\beta}\log\left(\frac{1+e^{\alpha+\beta(x+t)}}{1+e^{\alpha+\beta x}}\right)$
Makeham–Beard (1932)	$\frac{e^\epsilon + e^{\alpha+\beta x}}{1+e^{\alpha+\rho+\beta x}}$	$te^\epsilon + \frac{(e^{-\rho}-e^\epsilon)}{\beta}\log\left(\frac{1+e^{\alpha+\rho+\beta(x+t)}}{1+e^{\alpha+\rho+\beta x}}\right)$