1) Find the Perfect Bayesian equilibria of the signalling game of lecture 10 for $p < 0.5$

**solution**

The PBE **(**Challenger : *ready* if strong; *unready* if weak

Incumbent *: acquiesce* after *ready, fight* after *unready*

Incumbent' s beliefs:

Pr(*strong* | *ready*) =1   Pr(*weak* | *unready*) = 1)

is an equilibrium (see slide)

Then we have to study what happen if challenger chooses (*unready*, *unready*) but $p \leq 1/2$

- then *fight* is best response of incumbent after *unready*
- if incumbent chose *acquiesce* after *ready,* then strong challenger would prefer to deviate
- so incumbent must choose *fight* after *ready* (at least with probability $\geq 1/2$)
- this requires for incumbent's belief: Pr(*strong* | *ready*) $\leq 1/2$
- In equilibrium *ready* occurs with probability 0, so beliefs are not restricted
- So if $p \leq 1/2$, there are (pooling) PBE where both types of challengers choose *unready*, incumbent chooses *fight* after both *unready* and *ready* and believes the challenger to be strong with probability $\leq 1/2$ after both *unready* and *ready*

2) Three persons play the following game. Person $i$ receives a signal $s_i$ that can be either 0 or 1, $i \in \{1, 2, 3\}$. Signals are independently distributed and each signal's probability of being 1 is 0.6. Each person observes only his signal but not the signal of the other persons. The probability distributions of the signals are common knowledge. Each person has to guess the sum of the signals. If the guess is correct, the player gets £1, otherwise she receives 0. Person 1 guesses first, person 2 observes the guess of player 1 before making his guess. Finally person 3 observes the guesses of players 1 and 2 before making his guess. Find the Perfect Bayesian equilibria

**Solution**

Note that payoff of Person 1 does not depends on the strategies played by others.

By "Requirement 2" Person 1 has to choose the strategy with the higher expected value.

Person 1 has two information sets: $s_1 = 0$ and $s_1 = 1$.

Suppose $s_1 = 0$ and compute the expected value of each possible guess

$E(g_1 = 0) = Pr(s_2 + s_3 = 0) = 0.4^2 = 0.16$

$E(g_1 = 1) = Pr(s_2 + s_3 = 1) = 2\ 0.4\ 0.6 = 0.48$

$E(g_1 = 2) = Pr(s_2 + s_3 = 2) = 0.6^2 = 0.36$

$E(g_1 = 3) = Pr(s_2 + s_3 = 3) = 0$

Then in a PBE person 1 plays $g_1 = 1$ when $s_1 = 0$

Suppose $s_1 = 1$ and compute the expected value of each possible guess

$E(g_1 = 0) = Pr(s_2 + s_3 = -1) = 0$

$E(g_1 = 1) = Pr(s_2 + s_3 = 0) = 0.4^2 = 0.16$

$E(g_1 = 2) = Pr(s_2 + s_3 = 1) = 2\ 0.4\ 0.6 = 0.48$

$E(g_1 = 3) = Pr(s_2 + s_3 = 2) = 0.6^2 = 0.36$

Then in a PBE person 1 plays $g_1 = 2$ when $s_1 = 1$

Then in PBE person 1 plays the strategy $\{1, 2\}$ i.e he guesses by 1 when receive signal $s_1 = 0$ and by 2 when he receives signal $s_1 = 1$.

Person 2 has 8 information sets: all possible combination between $g_1$ and $s_2$. Let $(g_1, s_2)$ denotes an information set, then the person 2's information sets are:

$(0,0), (0,1), (1,0), (1,1), (2,0), (2,1), (3,0), (3,1)$

Only the following four information sets are on the equilibrium path $(1,0), (1,1), (2,0), (2,1)$, (because in equilibrium person 1 plays either 1 or 2).

Then in these information sets beliefs are determined by Bayes rule, in the others information sets beliefs can be arbitrary.

Let be $b_2(g_1, s_2)$ the belief of person 2 about the signal $s_1$, i.e. the probability person 2 assigns to the event $s_1 = 1$, in the information set $(g_1, s_2)$.

In the information sets on the equilibrium path the player 2's beliefs are:

$$b_2(1,0) = b_2(1,1) = 0$$

$$b_2(2,0) = b_2(2,1) = 1$$

Consider information set $(1,0)$(i.e. $g_1 = 1$ and $s_2 = 0$) and compute the expected value of each possible guess

$$E(g_2 = 0) = Pr(s_1 + s_3 = 0) = \big(1 - b_2(1,0)\big)0.4 = 0.4$$

$$E(g_2 = 1) = Pr(s_1 + s_3 = 1) = \big(1 - b_2(1,0)\big)0.6 + b_2(1,0)0.4 = 0.6$$
$$E(g_2 = 2) = Pr(s_1 + s_3 = 2) = b_2(1,0)0.6 = 0$$
$$E(g_2 = 3) = Pr(s_1 + s_3 = 3) = 0$$

Then $g_2 = 1$ produces the higher expected value

Consider information set (1,1)(i.e. $g_1 = 1$ and $s_2 = 1$) and compute the expected value of each possible guess

$$E(g_2 = 0) = Pr(s_1 + s_3 = -1) = 0$$
$$E(g_2 = 1) = Pr(s_1 + s_3 = 0) = \big(1 - b_2(1,0)\big)0.4 = 0.4$$
$$E(g_2 = 2) = Pr(s_1 + s_3 = 1) = \big(1 - b_2(1,0)\big)0.6 + b_2(1,0)0.4 = 0.6$$
$$E(g_2 = 3) = Pr(s_1 + s_3 = 2) = b_2(1,0)0.6 = 0$$

Then $g_2 = 2$ produces the higher expected value

Then in a BPE, Person 2 plays $s_2 + 1$ after observing $g_2 = 1$

Reasoning in a similar way we find that in a BPE, Person 2 plays $s_2 + 2$ after observing $g_2 = 1$ (because $b_2(2,0) = b_2(2,1) = 1$).

Out of equilibrium path there are arbitrary beliefs and the strategy must be optimal given the (arbitrary) beliefs.

Person 3 has 32 information sets: all possible combinations between $g_1$, $g_2$ and $s_3$. Let $(g_1, g_2, s_3)$ denotes the information sets of person 3. The following information sets are on the equilibrium path:

(1,1,0) (1,2,0) (2,2,0) (2,3,0)
(1,1,1) (1,2,1) (2,2,1) (2,3,1)

Let be:

i) $b_{3,1}(g_1, g_2, s_3)$ the belief of person 3 about the signal $s_1$, i.e. the probability person 3 assigns to the event $s_1 = 1$, in the information set $(g_1, g_2, s_3)$ .

ii) $b_{3,2}(g_1, g_2, s_3)$ the belief of person 3 about the signal $s_2$, i.e. the probability person 3 assigns to the event $s_2 = 1$, in the information set $(g_1, g_2, s_3)$ .

In the information sets on the equilibrium path the player 3's beliefs are:

$$b_{3,1}(1,1,0) = b_{3,1}(1,1,1) = b_{3,1}(1,2,0) = b_{3,1}(1,2,1) = 0$$
$$b_{3,1}(2,2,0) = b_{3,1}(2,2,1) = b_{3,1}(2,3,0) = b_{3,1}(2,3,1) = 1$$

$$b_{3,2}(1,1,0) = b_{3,2}(1,1,1) = b_{3,2}(2,2,0) = b_{3,2}(2,2,1) = 0$$
$$b_{3,2}(1,2,0) = b_{3,2}(1,2,1) = b_{3,2}(2,3,0) = b_{3,2}(2,3,1) = 1$$

Then computing the expected payoff for each possible guess (as above), we find the optimal action for each information set on the equilibrium path. Optimal guesses are

| Info set | (1,1,0) | (1,1,1) | (1,2,0) | (1,2,1) | (2,2,0) | (2,2,1) | (2,3,0) | (2,3,1) |
|---|---|---|---|---|---|---|---|---|
| Optimal guess | 0 | 1 | 1 | 2 | 1 | 2 | 2 | 3 |

Then in all PBE

Person 1 plays $s_1 + 1$

Person 2 plays:

i) $s_2 + 1$ when $g_1 = 1$ and

ii) $s_2 + 2$ when $g_1 = 2$.

iii) When $g_1 = 0$ or $g_1 = 3$ (out of equilibrium path) beliefs can be arbitrary and Person 2 has to play optimally given these beliefs.

Person 3 plays:

i) $s_3 + 0$ when $g_1 = 1$ and $g_2 = 1$

ii) $s_3 + 1$ when $g_1 = 1$ and $g_2 = 2$

iii) $s_3 + 1$ when $g_1 = 2$ and $g_2 = 2$

iv) $s_3 + 2$ when $g_1 = 2$ and $g_2 = 3$

v) In the information sets out of equilibrium path beliefs can be arbitrary and Person 3 has to play optimally given these beliefs.

3) Consider the following game. ***Nature*** determines if payoffs are as in G1 or in G2 by equal probability

| | | Player 1 | | |
|---|---|---|---|---|
| | **G1** | L | C | R |
| Player 2 | T | 2,0 | 0, 1 | 4, 2 |
| | M | 3,4 | 1, 2 | 2, 3 |
| | B | 1, 3 | 0, 2 | 3, 0 |

| | | Player 1 | | |
|---|---|---|---|---|
| | **G2** | L | C | R |
| Player 2 | T | 2,0 | 0, 1 | 4, 0 |
| | M | 3,4 | 1, 2 | 2, 3 |
| | B | 1, 3 | 0, 2 | 3, 0 |

Players 1 knows which game ***Nature*** has chosen, but Player 2 does not. Player 1 moves first. Player 2 observes the choice of player 1 before to move. Find the Perfect Bayesian equilibria

**Solution**

Player 1 has the following 9 strategies:

$$(L, L), (L, C), (L, R), (C, L), (C, C), (C, R), (R, L), (R, C), (R, R)$$

If Player 1 plays L, Player 2's best response is to play M for all possible beliefs
If Player 1 plays C, Player 2's best response is to play M for all possible beliefs
If Player 1 plays R, Player 2's best response is to play T for all possible beliefs

Then player 1 anticipates player 2 strategy:

i) Player 1 knows that if he plays L player 2 will play M and his payoff will be 4 in both games
ii) Player 1 knows that if he plays C player 2 will play M and his payoff will be 2 in both games
iii) Player 1 knows that if he plays R player 2 will play T and his payoff will be 0 or 2

Then the optimal strategies for player 1 is to play $(L, L)$

Let $b$ denote the probability that player 2's assigns at the event *game=G1*

PBE

Player 1: $(L, L)$;
Player 2 $(M, M, T)$;
$$b = \begin{cases} 0.5 \; if \; player \; 1 \; plays \; L \\ [0,1] \; if \; player \; 1 \; plays \; C \; or \; R \end{cases}$$