1)
$$|f * g(x)| \leq \int e^{-a|x-y|^2 - b|y|^2}\, dy$$

Ora
$$-a|x-y|^2 - b|y|^2 = -a|x|^2 + 2a\, x\cdot y - (a+b)|y|^2$$

completiamo i termini $(a+b)|y|^2 - 2a\, x\cdot y$

in un quadrato perfetto
$$(a+b)|y|^2 - 2a\, x\cdot y = \left|\sqrt{a+b}\, y - \frac{a}{\sqrt{a+b}}\, x\right|^2 - \frac{a^2}{a+b}|x|^2$$

Quindi
$$-a|x-y|^2 - b|y|^2 = -\left(a - \frac{a^2}{a+b}\right)|x|^2 - \left|\sqrt{a+b}\, y - \frac{a}{\sqrt{a+b}}\, x\right|^2$$

$$|f * g(x)| \leq e^{-\frac{ab}{a+b}|x|^2} \int e^{-\left|\sqrt{a+b}\, y - \frac{a}{\sqrt{a+b}}\, x\right|^2} dy$$

Quindi $c = \frac{ab}{a+b}$ e

$$K = \int e^{-\left|\sqrt{a+b}\, y - \frac{a}{\sqrt{a+b}}\, x\right|^2} dy \underset{z = \sqrt{a+b}\, y - \frac{a}{\sqrt{a+b}}\, x}{=} \int e^{-|z|^2} (a+b)^{-n/2}\, dz =$$

$$= \left(\frac{\pi}{a+b}\right)^{n/2}.$$

2) Sia $F \subset \mathbb{S}^2 \times (0,\infty)$ l'insieme

$$F = \{(\xi, r) \mid \xi \in \mathbb{S}^2,\ 0 < r < g(\xi)\}.$$

F è il sottografico di una funzione misurabile.
Introducendo il cambio di coord. delle coord. polari

$$\Phi : \mathbb{S}^2 \times (0,\infty) \longrightarrow \mathbb{R}^3 \setminus \{0\}$$

$$\Phi(\xi, r) = r\xi$$

$$\mu_3(E) = \int_{\mathbb{R}^3 \setminus \{0\}} \chi_E(x)\, d\mu_3(x) =$$

$$= \int_{\mathbb{S}^2 \times (0,\infty)} \chi_F(\xi, r)\, r^2\, d\mu_1(r) \times d\mu_2(\xi) =$$

$$= \int_{\mathbb{S}^2} d\mu_2(\xi) \int_{(0,+\infty)} \chi_F(\xi, r)\, r^2\, d\mu_1(r) =$$

$$= \int_{\mathbb{S}^2} d\mu_2(\xi) \int_0^{g(\xi)} r^2\, d\mu_1(r) = \int_{\mathbb{S}^2} \frac{g^3(\xi)}{3}\, d\mu_2(\xi)$$

3) Siano s, s' esp. coniugati

$$\int |fg|^\alpha \leq \left(\int |f|^{\alpha s}\right)^{1/s} \left(\int |g|^{\alpha s'}\right)^{1/s'}$$

Cerchiamo α, s t.c. $\alpha s = p$, $\alpha s' = q$
cioè $s = p/\alpha$, $s' = q/\alpha$. Quindi

$$\frac{\alpha}{p} + \frac{\alpha}{q} = \frac{1}{s} + \frac{1}{s'} = 1 \implies \alpha \text{ è dato da}$$

$$\frac{1}{\alpha} = \frac{1}{p} + \frac{1}{q}$$