Notes on Special Relativity

1. Introduction

Two puzzles at the turn of XIX → XX century:

(a) The Newtonian (Galilean) relativity principle works fine for mechanics, not for Maxwell's equations and e.m. waves. (NB: "absolute time" for all observers!)

(b) If, by analogy with sound waves etc., one assumes that a "medium" (called "ether") must be there to support e.m. waves propagation, then one should detect different propagation velocities for observers moving with respect to the "ether": experiments contradict this (e.m. waves propagate with the same speed c with respect to all observers, in relative motion !?!)

Several attempts to fix both problems:

- "ether" and "ether drag" theories; "emission" theories ... are contradicted by experiments
- modifications "ad hoc" of Galilei coordinates transformations (Lorentz transformations) formally fix the invariance properties of Maxwell's equations, but lack a physical explanation and are not brought to their extreme consequences.

Einstein's solution.

- Revision of the concepts of time and simultaneity for "events", re-analyzing them from an operational (measurement) point of view. !!!

- Two postulates:

  (i) The laws of physics (all!) are the same in all inertial systems. No preferred inertial system (i.e. "ether") exists (or is detectable) "absolute motion cannot be detected"
  [PRINCIPLE OF RELATIVITY]

  (ii) The speed of light in free (empty) space has the same value c in all inertial system. "the speed of light does not depend on the motion of the source or the observer"
  [PRINCIPLE OF CONSTANCY OF THE SPEED OF LIGHT]

Note: Einstein's genius: with a "fresh mind":

⇒ extension of relativity for inertial observers is not revolutionary, but:

⇒ central role to measurement and experimental evidence

⇒ willingness to explore the consequence of general principles giving up philosophical assumptions ("absolute time equal for all observers")

- explicit derivation of:

kinematics { - Lorentz transformations and their consequences (kinematics) { relativity of simultaneity and clock synchronization, length contraction, time dilation, "new" invariants
- composition of velocities }

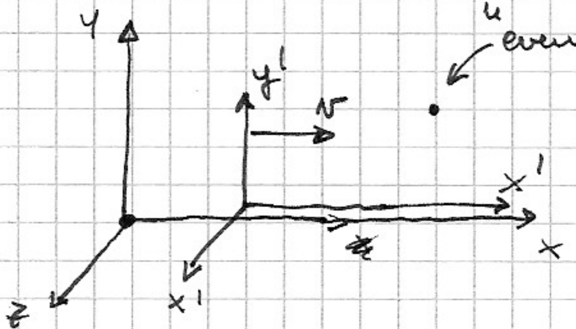
dynamics { - relativistic momentum, relationship with forces. (dynamics)
- relativistic energy }

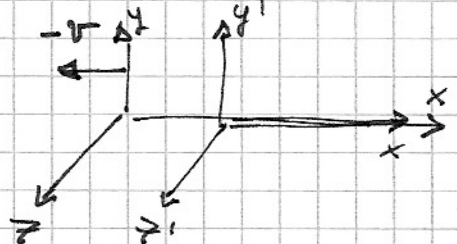
LL 1.10.2012 − 1 CF/SR/3

2. Galileo + Newton & relativity

inertial observers (1st principle) cannot decide who is moving : ~~both are~~ each are moving with respect to the other

"event" $\begin{cases} x, y, z; t & \text{measured by } S \\ x', y', z'; t' & \text{" " } S' \end{cases}$





v : velocity of S' with respect to S

$-v$: vel. of S wrt S'

(assuming that origins coincide at $t = t' = 0$)

$$\begin{cases} x' = x - vt \\ y' = y \\ z' = z \\ t' = t \end{cases} \qquad \begin{cases} x = x' + vt' \\ y = y' \\ z = z' \\ t = t' \end{cases}$$

Galileo transf. of coord.

velocity of an object seen by both observers : $\frac{d}{dt}$ in S , $\frac{d}{dt'}$ in S' u symbol for velocity (not to confuse with v)

$$\begin{cases} u'_x = u_x - v \\ u'_y = u_y \\ u'_z = u_z \end{cases} \qquad \begin{cases} u_x = u'_x + v \\ u_y = u'_y \\ u_z = u'_z \end{cases}$$

NB : this rule for "relative velocity" or "composition of velocities" works fine for $v \ll c$

$$\frac{v}{c} \ll 1$$

examples:

Sound $\frac{v}{c} \sim \frac{3 \times 10^2}{3 \times 10^8} \simeq 10^{-6}$

Earth around the Sun $\frac{v}{c} \sim \frac{3 \times 10^4}{3 \times 10^8} \simeq 10^{-4}$

NB : aereo $1000 \frac{km}{h} \simeq 10^3 \frac{10^3 m}{3.6 \times 10^3 s} \sim 3 \times 10^2 \, m/s$

terra attorno al sole: $30 \frac{km}{s}$

$$\frac{2\pi \cdot 150 \times 10^9 m}{352 \times 24 \times 3600 \, s} \simeq$$

$$\simeq \frac{6.3 \times 1.5 \times 10^6 \times 10^3 \times 10^2}{3.5 \times 2.4 \times 3.6 \times 10^6} \simeq 3 \times 10^4 \frac{m}{s}$$

LL 1.10.2012 - 2

NB: accelerations are the <u>same</u> for inertial observers!

$\frac{d}{dt'}$, $\frac{d}{dt}$

accel. of point P:

$$\begin{cases} a'_x = \frac{du'_x}{dt'} = \frac{du_x}{dt} = a_x \\ a'_y = a_y \\ a'_z = a_z \end{cases}$$

$\sum \vec{F} = m\vec{a}$

$\sum \vec{F} = m\vec{a}'$

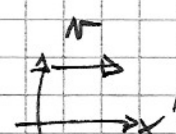
$\Downarrow$

forces and accelerations are the same for inertial observers, irrespective of their relative velocity v

NB: from the classical rule for the composition of velocities:

sound wave propagating with velocity $\vec{u}$

(source at rest with the medium (air)) in S



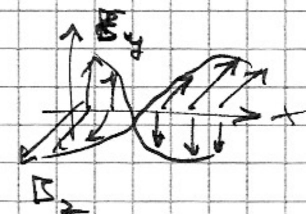
observer S' (moving at speed v in the medium) measures

$u' = u - v$

(OK, it works)

one would expect something similar to happen for e.m. waves

from Maxwell's equations one obtains:



$$\frac{\partial^2 \phi}{\partial x^2} = \varepsilon_0 \mu_0 \frac{\partial^2 \phi}{\partial t^2}$$

$\varepsilon_0\mu_0 = \frac{1}{c^2}$

$$\phi = \begin{cases} E_y(x,t) \\ B_z(x,t) \end{cases}$$

$c \simeq 3 \times 10^8$ m/s

with respect to what? "the ether"...?

changing coordinates does not leave the eq. invariant!

$\Rightarrow$ expect $\begin{cases} \text{(a) an "absolute" frame} \equiv \text{"ether"} \\ \text{(b) velocity } c - v \text{ if the observer is moving with vel. } v \text{ with respect to the ether...} \end{cases}$

highest velocity available: Earth around the Sun

$$\frac{v}{c} \sim \frac{3\times10^4\ m/s}{3\times10^8\ m/s} \sim 10^{-4} \quad \Rightarrow \quad c-v = c\underbrace{\left(1-\frac{v}{c}\right)}$$

small effect $\sim 10^{-4}$ but can be measured with interferometric methods.

Michelson-Morley expt, + others ...
Sensitive enough, but does not see any effect.

c seems to be the same for all inertial observer.
What is going on?

Several attempts (partial explanations); finally:

Einstein: special relativity, 1905

(1) "postulate of relativity" (extending Galilei/Newton)
- all inertial reference frames are equivalent (all phenomena, not only mechanics)
  $\Rightarrow$ equations describing physics must be invariant for transformations between inertial frames.

(2) "postulate of the constancy of the speed of light"
- the velocity of e.m. waves (light etc.) propagating in free space (empty) is the same for all inertial observers
  [irrespective of the motion of the source or the observer]

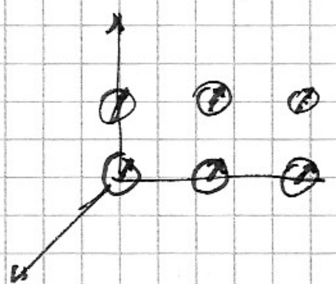
Contradiction with classical expectations!? How is it possible?
Einstein: there is something wrong in the classical assumptions about time and simultaneity of events.

Since ~~simultaneity~~ simultaneity is no longer an absolute concept, but depends on the observer, it ~~is not~~ should be a surprise if it turns out that observers disagree on measurements of lengths and time intervals

(but they agree if the measured object is brought to rest in their system!)

In each reference frame (for each inertial observer)

a) Measurement of "events":



network of ancillary equipment, at regular distances: "synchronized" clocks

local recording of $\begin{cases}\text{position}\\ \text{time}\end{cases}$ by a close-by recorder

(the information can be transmitted later on to the observer, located for instance at the origin)

b) clock synchronization: no signal can travel at infinite speed $\Rightarrow$ use light to check the synchronization of distant clocks

several equivalent methods:

synch. ok if $\begin{cases}\text{apparent time difference} = \dfrac{\text{distance}}{c}\\ \text{clocks appear to be exactly in time when seen from the mid point}\end{cases}$

NB: different observers do not agree on the synchronization of each other's clocks: to a moving observer they appear out of phase.

c) "simultaneity" of distant events: different observers do not agree; what is "simultaneous" for one observer, is not for another.

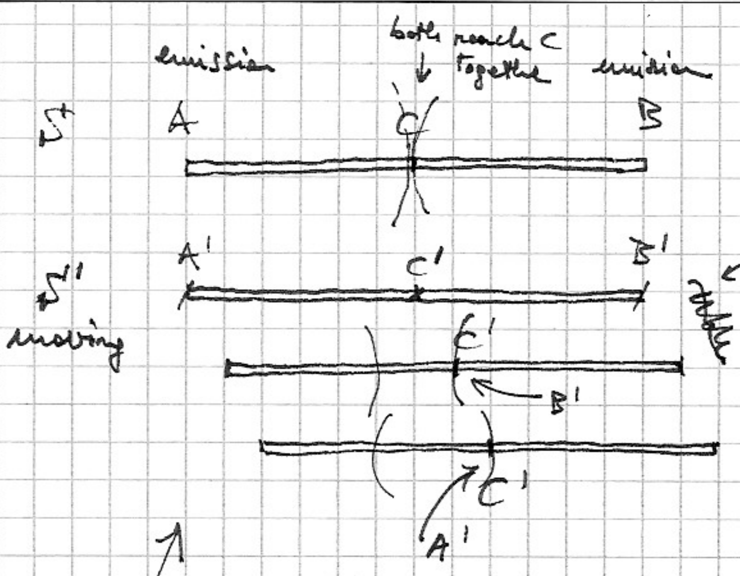
d) measurement of distances / lengths:

each observer applies the following rule: $\begin{cases}\text{object at rest} \leftarrow \text{obvious.}\\ \text{moving object}\end{cases}$ distance between end-points "simultaneous"

e) measurements of time intervals between events

difference in time $\begin{cases}\text{events occurring in the same place} \rightarrow \text{easy (the same clock!)}\\ \text{events occurring in different points.} \rightarrow \text{have to compare the readings of two different clocks located close to where the events}\end{cases}$

$\Longrightarrow$ in general: need to evaluate a "space-time" interval.



in S: two light flashes emitted in A, B are simultaneous if they reach together the midpoint C.

emission

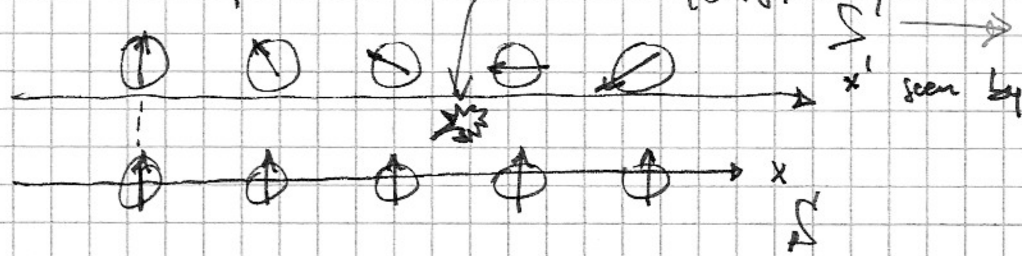
the flash from B' is seen in C' before the flash from A' } They are not simultaneous for S' !

(c) disagreement on synchronization.

NB The network of ~~d~~ synchronized clocks of a different (moving) observer appears out-of-time ! (and with a characteristic time shift from one clock to the next)

qualitatively :

event : $\begin{cases}(x,y,z,t)\\(x',y',z',t')\end{cases}$

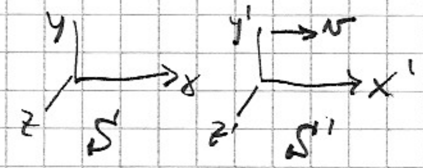


$t \neq t'$

(but : also the spacing between clocks ~~seems to~~ is measured to be shorter ! in S)

3. Lorentz transformations for event coordinates

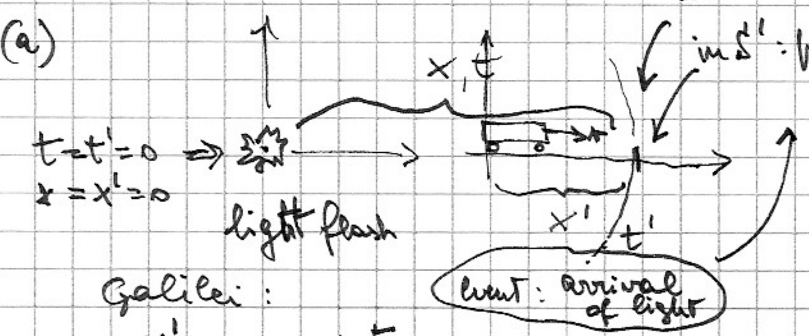
homogeneity of space $\begin{cases} \text{transf. are linear.} \\ y = y',\ z = z' \end{cases}$



relativity of simultaneity $\Rightarrow t \neq t'$!!!

how should we relate event coordinates, for $\begin{cases} x, t \\ x', t' \end{cases}$ for instance in these 2 examples?

(a)



(classically $t = t'$ now expect $t \neq t'$!)

Galilei:

$$\begin{cases} x' = x - vt \\ x = x' + vt' \end{cases}$$

$t' = t$ in this case

Lorentz

$$\begin{cases} x' = \gamma (x - vt) & (1) \\ x = \gamma (x' + vt') & (2) \end{cases}$$

(*)

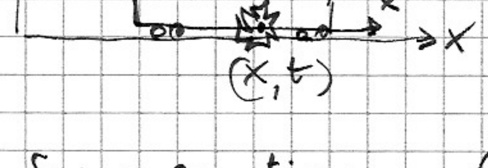
$\gamma = ?$ should be 1 if $v \ll c$

$t \overset{?}{\underset{\neq}{\leftrightarrow}} t'$?

NB the same γ!

(b) event: little explosion, leaving a mark on both the wagon and the track $\Rightarrow$ localised in space coordinates $\begin{cases} x \\ x' \end{cases}$

the close-by clocks record the times $\begin{cases} t \\ t' \end{cases}$



Same equations. (no light prop. involved in this case)

~~[(*) $x' = \gamma x - \gamma v t$~~
~~$\gamma x = x' + \gamma v t \Rightarrow x = \frac{1}{\gamma}(x' + \gamma v t) = \frac{x'}{\gamma} + vt$??]~~

$$\text{system}\begin{cases} S' : & x' = ct' \\ & \updownarrow \text{ same } c \,!! \text{ (2}^{nd}\text{ postulate)} \\ S : & x = ct \end{cases}$$

Classically the velocity would be different!

2 equations, 2 unknowns
$\gamma = ?$ $\quad t \leftrightarrow t'$? $(t \neq t')$

$$\begin{cases} (1) \Rightarrow ct' = \gamma(ct - vt) = \gamma t(c-v) = \gamma c t(1 - \underbrace{\tfrac{v}{c}}_{\beta}) & (1') \\ (2) \Rightarrow ct = \gamma(ct' + vt') = \gamma t'(c+v) = \gamma c t'(1+\beta) & (2') \end{cases}$$

$$\left.\begin{aligned} (1') \quad & \frac{t'}{t} = \gamma(1-\beta) \\ (2') \quad & \frac{t'}{t} = \frac{1}{\gamma(1+\beta)} \end{aligned}\right\} \quad \gamma(1-\beta) = \frac{1}{\gamma} \cdot \frac{1}{1+\beta}$$

$$\gamma^2 = \frac{1}{1-\beta^2}$$

Ok, first result ! $\Longrightarrow$ $\boxed{\gamma = \frac{1}{\sqrt{1-\beta^2}}}$ $\quad \gamma \to 1$ if $\frac{v}{c} = \beta \to 0$ $\quad \frac{v}{c} \ll 1$

2nd unknown : t' in terms of t (but also of x ... !)
remember synchronization problem

from (1) : $t' = \gamma(t - \frac{v}{c}\underbrace{t}_{\frac{x}{c}}) = \boxed{\gamma\left(t - \frac{v \cdot x}{c^2}\right)}$

(*) CONCLUSION : LORENZ TRANSFORMATIONS AS A CONSEQUENCE OF S. RELATIVITY POSTULATES

→ see next page

$$\begin{cases} x' = \gamma(x - vt) \\ y' = y \\ z' = z \\ t' = \gamma\left(t - \frac{v}{c}\frac{x}{c}\right) \end{cases} \qquad \begin{cases} x = \gamma(x' + vt') \\ y = y' \\ z = z' \\ t = \gamma\left(t' + \frac{v}{c}\frac{x'}{c}\right) \end{cases}$$

(more symmetric form: measure time in length units: ct)

$$\begin{cases} ct' = \gamma(ct - \beta x) \\ x' = \gamma(x - \beta ct) \end{cases} \qquad \beta \equiv \frac{v}{c} \qquad \begin{cases} ct = \gamma(ct' + \beta x') \\ x = \gamma(x' + \beta ct') \end{cases}$$

NB : now <u>both</u> space <u>and</u> <u>time</u> coordinates are mixed together in the transformation ! (for $\beta \to 0$ we go back to good ol' Galilei transf.)

consequences ?

kinematics
(a) Space and time intervals are no longer separately invariant when changing observer (but their combination will still be an invariant !)
- → "length contraction"
- → "time dilation" (⇒ clock synchronization ...)

(b) composition of velocities is modified !

dynamics
(c) Newton's law must be modified
⇝ momentum and momentum conservation

(d) Energy – mass equivalence, relativistic expressions for kinetic energy and energy - momentum - mass relation.

Let's examine these consequences one by one.

NB : please note that now time is crucial : it is also used to define lengths in each ref. frame !

ABOUT THE DERIVATION OF LORENZ TRANSFORMATIONS

(*) NB : exactly the same conclusion is reached thinking in terms of example (b), where no light is directly involved (but is involved in the synchronization of clocks ..!) separately in each reference frame.

$\boxed{\text{and : is now used in the S.I. system of units to define the unit of length !}}$

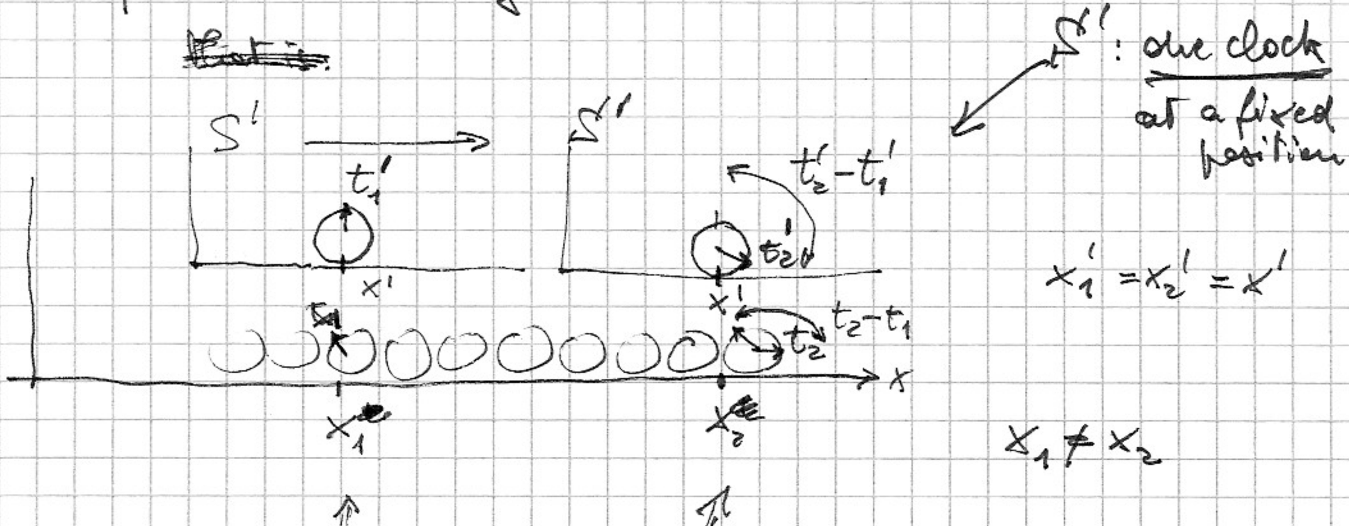
↳ equivalent to using relations such as:

$$\frac{x'}{c} = \gamma\left(\frac{x}{c} - \frac{v}{c}t\right) = \gamma\left(\frac{x}{c} - \beta t\right) \quad \text{etc.}$$

4. TIME DILATION

One well-known (but difficult to understand ...) consequence : "moving clocks run slow"

~~that is~~



S must compare the readings (and recordings) t_1, t_2 of two clocks at different positions $x_1 \neq x_2$

how do the readings compare ? from Lorentz transf. :

$$t' = \gamma\left(t - \frac{v}{c^2}x\right) \qquad t = \gamma\left(t' + \frac{v}{c^2}x'\right)$$

~~$t_2' - t_1' = \gamma(t_2 - t_1) \ldots$~~

$$\underbrace{t_2 - t_1}_{\Delta t} = \underbrace{\gamma}_{>1} \underbrace{(t_2' - t_1')}_{\Delta t' \equiv \Delta\tau} + \cancel{\frac{v}{c^2}(x_2' - x_1')}_{0} \qquad x_2' = x_1'$$

$\Delta t' \equiv \Delta\tau$ "proper time" (measured at rest)

$$\Delta t = \gamma \Delta\tau > \Delta\tau$$

The same conclusion is reached by S' looking at S clocks !

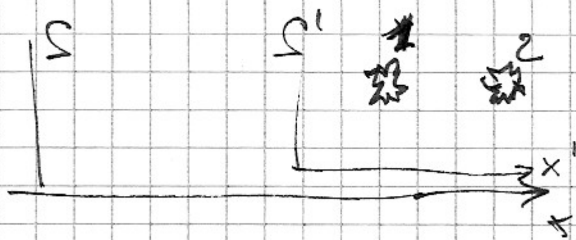
The paradox is only apparent ! The situation is clearly asymmetric because only one of the two observers needs 2 clocks for the comparison ~~...~~ with a single moving clock

(we will see a physical example to get a feeling ...)

5. Relativity of simultaneity

as anticipated using an example based on light propagation (see CF/SR/7)

events that are simultaneous (but at different positions) in S' are not simultaneous in S and vice-versa.



$$t_2 - t_1 = \gamma \underbrace{(t_2' - t_1')}_{0 \text{ if } t_2'-t_1'=0} + \frac{\gamma v}{c^2}(x_2' - x_1') \neq 0 \text{ in general}$$

(only if $x_2' - x_1' = 0$ they will be simultaneous in S)

NB: S and S' may also disagree on the time sequence!

for instance:

S' sees ev. 2 after 1 : $t_2' > t_1'$

but: if x_1' large enough, one can have $t_2 - t_1 < 0$

$t_2 < t_1$

Relativity of Synchronization

$$t = \gamma t' + \frac{\gamma v}{c^2} x'$$

$\overset{\shortparallel}{\text{const.}}$

t' decreases as x' increases!

compare clocks in S at a given time t (for S) with corresponding clocks at different positions

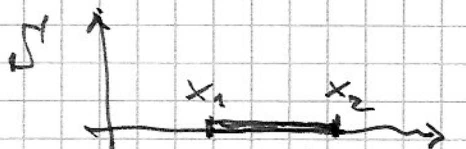
clocks of S' are out of synch according to S, with increasing time shift as the coord. increases.

LL · 2.10.2012 - ~~#~~ 3 CF/SR/13

6. LENGTH CONTRACTION.

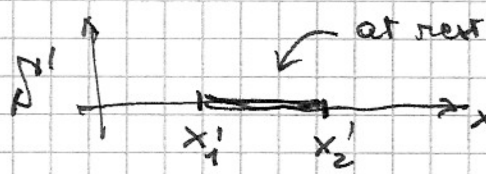
we must first define : length of a rigid body (a rod) = ?

(a) at rest : no problem, we expect that all observers will agree on the same length $\ell_0 = x_2 - x_1$ when the rod is brought at rest in their own system! (ref. frame



(implicit : $t_1 = t_2$)

at rest in S
at rest in S' } same length ℓ_0 "at rest")



(implicit : $t_1' = t_2'$)

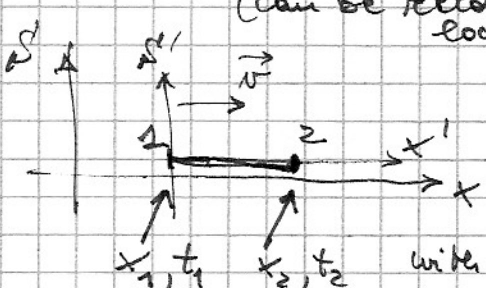
(b) what about moving objects?
for instance : rod at rest in S', moving at speed v with respect to S ?

must agree on a measurement procedure :

in S measure the space separation of the following events :

position x_1 → of first end
" x_2 → of second end
} simultaneous, i.e. $\underline{\underline{t_1 = t_2}}$ ← NB: crucial!

in x_1, x_2 clocks must record the same time
(can be recorded locally and checked afterwards)



with $t_1 = t_2 = t$

- observer at x_2 notes the time $t_2 = t$ ~~[illegible]~~ simultaneous with the passage of 2
- look for the position x_1 that recorded the passage of 1 with a local clock showing the same $t_1 = t_2 = t$

how is $x_2 - x_1$ related to the length at rest ℓ_0 measured by observer S' ?

(a) what do the equations tell us?

$$\begin{cases} x_2' = \gamma(x_2 + v t_2) \\ x_1' = \gamma(x_1 + v t_1) \end{cases} \qquad t_1 = t_2 = t$$

coordinates in S' of the two rod ends corresponding to the events that are simultaneous in S (not in S')

solving for $x_2 - x_1$: (subtracting)

$$\underbrace{x_2 - x_1}_{l} \underset{\text{if } t_2 = t_1}{=} \frac{1}{\gamma}\underbrace{(x_2' - x_1')}_{l_0} < l_0$$

"length contraction" Lorentz-Fitzgerald

l length measured in S

l_0 : length in S' (at rest) PROPER LENGTH or REST LENGTH

$$\frac{1}{\gamma} = (1-\beta^2) < 1$$

in the direction // to the phys. velocity

NB (1) both observers see the same effect (contraction) for objects (systems) that are moving in their ref. frame independently of the direction ⇄

Apparent asymmetry: observers with objects at rest are "privileged" PROPER LENGTH

NB (2) one can think of the "length at rest" as the intrinsic length, on which everybody agrees

⇒ the contraction can be seen as an effect of the measurement process!

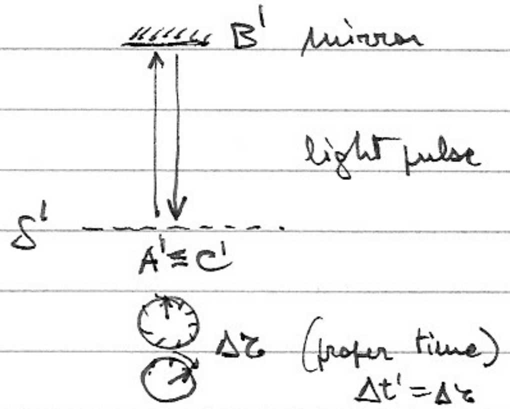
(analogous for instance to Doppler shifts etc.)

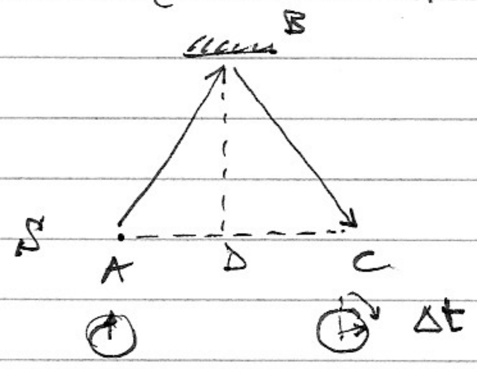
NB (3) No need to think of the contraction as a physical "squeezing" as proposed by Lorentz and Fitzgerald to explain the null results of the Michelson-Morley experiment.

⟶ physical example showing that this can be seen as a consequence of the "time dilation"

LL - 3.10.2012 - 1 CF/SR/15

TIME INTERVALS : COMPARISON (specific example)





↑ the same clock

2 synchronized clocks

$$\Delta\tau = 2\,\frac{BD}{c} \qquad \Delta t = \frac{AB + BC}{c} = 2\,\frac{AB}{c}$$

$$BD^2 = AB^2 - AD^2$$

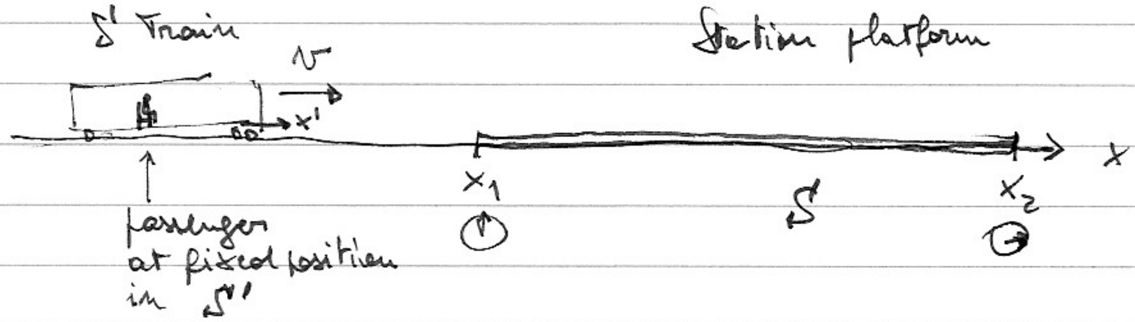
$$\frac{\Delta\tau}{\Delta t} = \frac{BD}{AB} = \frac{\sqrt{AB^2 - AD^2}}{AB} = \sqrt{1 - \frac{AD^2}{AB^2}} = \sqrt{1 - \frac{v^2}{c^2}} = \frac{1}{\gamma}$$

$$\boxed{\Delta t = \gamma \cdot \Delta\tau} = \frac{\Delta\tau}{\sqrt{1-\beta^2}} > \Delta\tau$$

↑ proper time

NB : the situation is asymmetric because the observer S' measuring the "proper time" uses only 1 clock, while observer S uses <u>2 synchronized clocks</u>!

LENGTHS PARALLEL TO THE DIRECTION OF RELATIVE MOTION



S { platform length in S: proper length $\ell_0 = x_2 - x_1$

the passenger covers this length in a time $\Delta t = \frac{\ell_0}{v} = t_2 - t_1$

(non-proper time!) timed by 2 clocks localized in x_1, x_2

passenger in S':

S' { sees the beginning and the end of the platform (passage by)

as two events timed by the same clock (wristwatch!)

$\Rightarrow$ proper time $\Delta\tau = \frac{1}{\gamma}\Delta t = \frac{1}{\gamma}\frac{\ell_0}{v} = \frac{\ell_0}{v}\sqrt{1-\beta^2}$

(see time dilation in S) $\Delta t = \frac{\ell_0}{v}$

sees the platform moving with respect to her at vel. $|-v| = v$

$\Rightarrow$ distance measured between beginning and end of platform:

$$\ell = v\,\Delta\tau = \ell_0\sqrt{1-\beta^2} = \frac{\ell_0}{\gamma} \leq \ell_0$$

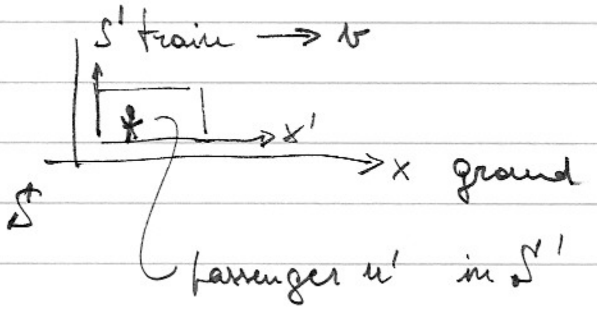
<u>length contraction!</u>

NB: asymmetry established by:

observer S' uses <u>only one clock</u>! (measuring proper time)

while observer S uses <u>two clocks</u> at different positions.

6. RELATIVISTIC ADDITION OF VELOCITIES



Lorentz.

$$\begin{cases} x' = \gamma(x - vt) \\ t' = \gamma\left(t - \frac{vx}{c^2}\right) \end{cases} \qquad \begin{cases} x = \gamma(x' + vt') \\ t = \gamma\left(t' + \frac{vx'}{c^2}\right) \end{cases}$$

classically : $u = u' + v$ $\quad (u' = u - v)$

now?

$u_x = \frac{dx}{dt}$; $u' = \frac{dx'}{dt'}$

$$dx = \gamma(dx' + v\,dt')$$
$$dt = \gamma\left(dt' + \frac{v}{c^2}dx'\right)$$

$$u_x = \frac{dx}{dt} = \frac{dx' + v\,dt'}{dt' + \frac{v}{c^2}dx'} = \frac{\frac{dx'}{dt'} + v}{1 + \frac{v}{c^2}\frac{dx'}{dt'}} = \frac{u'_x + v}{1 + \frac{v \cdot u'_x}{c^2}}$$

similarly :

$$u_y = \frac{dy}{dt} \overset{dy = dy'}{=} \frac{dy'}{\gamma\left(dt' + \frac{v\,dx'}{c^2}\right)} = \frac{dy'/dt'}{\gamma\left(1 + \frac{v}{c^2}\frac{dx'}{dt'}\right)} = \frac{u'_y}{\gamma\left(1 + \frac{v u'_x}{c^2}\right)}$$

$$u_z = \dots = \frac{u'_z}{\gamma\left(1 + \frac{v u'_x}{c^2}\right)}$$

They can be inverted (...)

$$\begin{cases} u'_x = \frac{u_x - v}{1 - \frac{v u'_x}{c^2}} \\ u'_y = \frac{u_y}{\gamma\left(1 - \frac{v u'_x}{c^2}\right)} \\ u'_z = \frac{u_z}{\gamma\left(1 - \frac{v u'_x}{c^2}\right)} \end{cases}$$

NB different from the classical

$$\boxed{\begin{cases} u_x = u'_x + v \\ u_y = u'_y \\ u_z = u'_z \end{cases}}$$

NB

if $\frac{v}{c} \ll 1$

classical limit

$$\boxed{\begin{cases} u_x \to u'_x + v \\ u_y \to u'_y \\ u_z \to u'_z \end{cases}}$$

if $v = c$

$v/c = 1$

$$u_x = \frac{u'_x + c}{1 + \frac{vc}{c^2}} = \frac{\overset{c}{u'_x} + c}{\underbrace{1 + \frac{c}{c}}_{=2}} = \frac{2c}{2} = c$$

7. RELATIVISTIC MOMENTUM.

Up to now: Lorentz transformations and their impact on "kinematics" (measurement of time intervals, length, velocity and addition of velocities)

Now: Dynamics?

one should expect modifications in:

- definition of momentum (involves mass and velocity)
- Newton's Second Law (invariant for Galilei transf., not for Lorentz)
- energy...

what is a "good" definition of momentum? (good = useful)

remember: $\vec{P} = \vec{p}_1 + \vec{p}_2 + \ldots$ for a system of particles

conserved if no external forces

$\vec{p} = m\vec{u}$? not "good" because

$\vec{P}$ conserved for one inertial observer $\not\Rightarrow$ $\vec{P}$ conserved for all inertial observer

can be seen by just one contrary example
⇝ violation of relativity principle!

Solution (skip demonstration)

redefine momentum of a moving particle:

$$\boxed{\vec{p} = \gamma m_0 \vec{u} = \frac{m_0 \vec{u}}{\sqrt{1 - u^2/c^2}}}$$

m_0: mass of the particle measured at rest

$$\begin{cases} p_x = \gamma m_0 u_x \\ p_y = \gamma m_0 u_y \\ p_z = \gamma m_0 u_z \end{cases}$$

with this definition, one can see that momentum conservation is OK in all reference ~~#~~ frames. (skip explicit demo, not enough time...)

NB. one may be tempted to consider:

$$m_{rel} = \gamma m_0 = \frac{m_0}{\sqrt{1-u^2/c^2}} \qquad \left(\begin{matrix} m > m_0 \\ \text{increasing with } u \end{matrix}\right)$$

as the "relativistic mass". This may be useful for some purpose, like understanding why the speed of a particle cannot increase indefinitely but is limited to c (some textbooks on it)

However this is <u>not</u> a "good" idea, because one cannot use it to generalise other quantities like kinetic energy

(relativistic $K \neq \frac{1}{2}\underbrace{(\gamma m_0)}_{m} u^2$! etc.)

NB. an equivalent definition

$$p_x = m_0(\gamma u_x) = m_0 \frac{dx}{dt/\gamma} = m_0 \frac{dx}{d\tau}$$

$\frac{dt}{\gamma}$ = proper time

Generalization of Newton's law:

$$\boxed{\Sigma \vec{F} = \frac{d\vec{p}}{dt} = \frac{d}{dt}\left(\gamma m_0 \vec{u}\right) = \frac{d}{dt}\left(\frac{m_0 \vec{u}}{\sqrt{1-u^2/c^2}}\right)}$$

this form is invariant for Lorentz transformations, and has the needed properties:

a) classical limit for $u/c \to 0$

b) a constant force cannot indefinitely accelerate a particle to infinite velocity : c is an upper limit.

8. RELATIVISTIC ENERGY

Remember: starting point of energy definitions in classical mechanics: "kinetic energy theorem" for a particle

$$\Delta K = K_f - K_i = W^{(tot.)} = \int_i^f (\Sigma\vec{F})\cdot d\vec{r}$$

simpler expression in one dimension (along x) starting with a particle at rest ($u_x = 0$) up to a vel. u_x

$$\Delta K = K_f = \int_{u'_x=0}^{u'_x=u_x} \underbrace{m_0 \frac{du'_x}{dt}}_{\text{II law}} u'_x \, dt = m_0 \int_0^{u_x} u'_x \, du'_x = \frac{1}{2} m u_x^2$$

How is this modified by relativity?

~~next lecture~~!

$$K_f - \cancel{K_i}^{u_i=0} = \int_0^{u_f} \underbrace{\frac{d}{dt}(\gamma m u)}_{\Sigma F_x} \cdot \underbrace{u\, dt}_{dx} = \int_0^{u_f} u \cdot d(\gamma m u)$$

$$d(\gamma m u) = \frac{d}{du}\left(\frac{mu}{(1-u^2/c^2)^{1/2}}\right) du =$$

$$= \frac{m(1-u^2/c^2)^{1/2} - mu\left(\frac{1}{\cancel{2}}\right)\left(1-\frac{u^2}{c^2}\right)^{-1/2}\left(-\cancel{2}\frac{u}{c^2}\right)}{\left[(1-u^2/c^2)^{1/2}\right]^2} du$$

$$= \left(m \frac{1}{(1-u^2/c^2)^{1/2}} + m\frac{u^2}{c^2}\frac{1}{(1-u^2/c^2)^{3/2}}\right) du$$

$$= m \frac{1 - \cancel{\frac{u^2}{c^2}} + \cancel{\frac{u^2}{c^2}}}{(1-u^2/c^2)^{3/2}} du$$

$$\Rightarrow \quad K_f = \int_0^{u_f} \frac{m\,u}{(1-u^2/c^2)^{3/2}}\,du =$$

$\downarrow$ primitive

$$= \left[m_0 c^2 \left(1-\frac{u^2}{c^2}\right)^{-1/2} \right]_0^{u_f} =$$

$$= \frac{m_0 c^2}{(1-u_f^2/c^2)^{1/2}} - m_0 c^2$$

$\Rightarrow$ in general (K_f corresponds to u_f)
$\quad K \qquad u$

$$K = \frac{m_0 c^2}{(1-u^2/c^2)^{1/2}} - m_0 c^2 \qquad \text{kinetic energy}$$

total energy — "energy at rest"

$$E = \gamma m_0 c^2 = m_0 c^2 + K$$

if $u^2/c^2 \ll 1$: K reduces to the non-relativistic expression:

Taylor:

$$f(x) = (1-x)^{-1/2} = 1 + \left(-\tfrac{1}{2}\right)(1-0)^{-3/2}(-1) + \dots \cong 1 + \tfrac{1}{2}x$$
$x \ll 1$

$$\Rightarrow \quad K = m_0 c^2 \left(\left(1-\frac{u^2}{c^2}\right)^{-1/2} - 1 \right) =$$

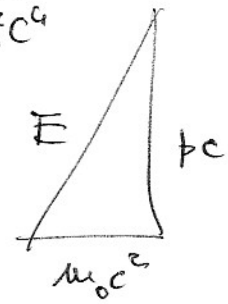
$$= m_0 c^2 \left(\cancel{1} + \frac{1}{2}\frac{u^2}{c^2} - \cancel{1} \right) = \frac{1}{2} m_0 u^2$$

as expected, <u>indeed</u>!

derive a very useful relation between E, p, m_0 :

$$E^2 = p^2c^2 + m_0^2c^4$$

$$E = \sqrt{p^2c^2 + m_0^2c^4}$$



$$\begin{cases} p = \gamma m_0 u = \dfrac{m_0 u}{\sqrt{1 - u^2/c^2}} & (1) \\[2ex] E = \gamma m_0 c^2 = \dfrac{m_0 c^2}{\sqrt{1 - u^2/c^2}} & (2) \end{cases}$$

eliminate u from these realions: dividing

$$\frac{p}{E} = \frac{u}{c^2} \Rightarrow \frac{pc}{E} = \frac{u}{c} \Rightarrow u = \frac{pc^2}{E}$$

substituting for instance in (1):

$$p = \frac{m_0 \frac{pc^2}{E}}{\sqrt{1 - \frac{p^2c^2}{E^2}}} \Rightarrow \cancel{p^2} = \frac{m_0^2 \cancel{p^2} c^4 / \cancel{E^2}}{(E^2 - p^2c^2)/\cancel{E^2}}$$

$$\Rightarrow E^2 - p^2c^2 = m_0^2c^4 \Rightarrow E^2 = p^2c^2 + m_0^2c^4$$

NB:

(a) "ultrarelativistic particles" $pc \gg m_0c^2$
$(\beta \to 1,\ \gamma \gg 1)$

$E \simeq pc$

$p \simeq \frac{E}{c}$

!?!

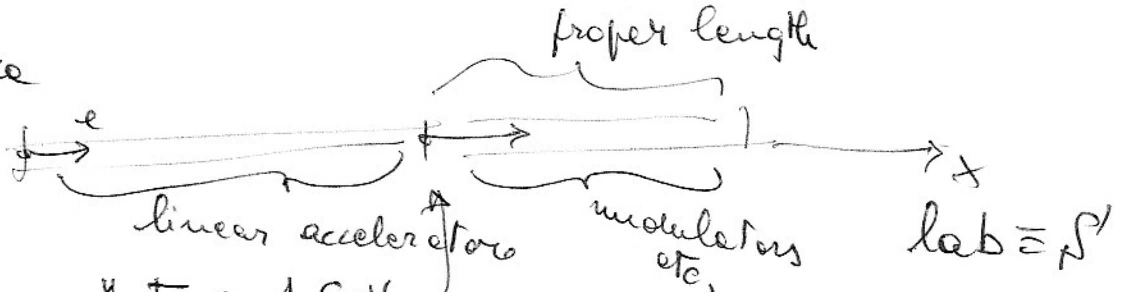
(b) if rest mass = zero $m_0 = 0$, still $E \neq 0$

$E = pc$ $\quad p = E/c \quad \underline{\beta = 1}$

such massless particles exist! (photons ....)
they must have $\beta = 1$ $u = c$

Numerical examples.

(1) Fermi @ Elettra



e "electrons" $E \simeq 1\,\mathrm{GeV}$

$\Longrightarrow$ compute : $p = ?$ $\beta = ?$ $u = ?$
at the end of the linac.

$\Longrightarrow$ undulator: assume proper length (in the lab.)
$l_0 = 10\,\mathrm{m}$

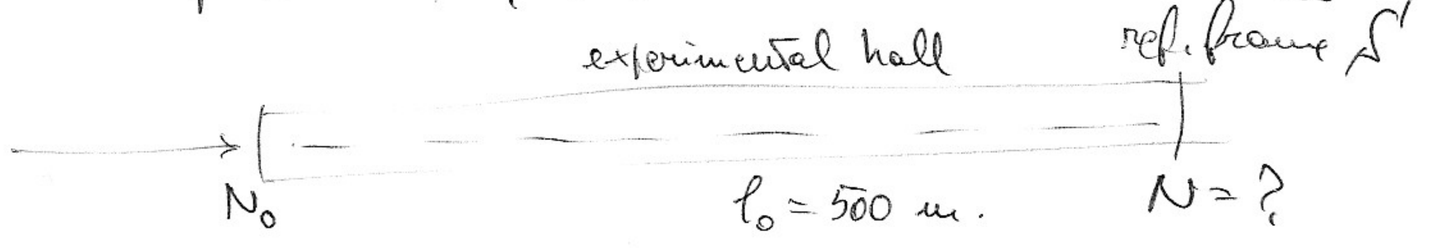
$\Longrightarrow$ what is the length l "seen" by electrons in their moving ref. frame S' in which they are at rest?

(2) beam of unstable particles: "muons" μ

<u>proper</u> lifetime: $\tau \sim 2{,}2 \times 10^{-6}\,\mathrm{s}$; $m_0 = 105\,\mathrm{MeV}/c^2$

assume $p_\mu = 100\,\mathrm{GeV}/c$; $N = N_0\, e^{-t/\tau}$

compute E ; β ; γ



- how many muons reach the end of the hall before decaying?
- how long is the hall in the muon's rest ref. frame S'?