A MODERN PHYSICS

PRIMER

FOR ENGINEERING STUDENTS

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Foreword

I am writing these notes to document a course on "Modern Physics" for engineering students. The main initial motivation was given by the opportunity to establish a physical background on semiconductors, for students of the Electronics curriculum, before they attended a course on solid-state electronic devices. The present scope is somewhat less well-defined, including the interests of engineers approaching the curriculum in Materials Science. Having clarified this, I have to admit that this is *not* a general introduction to Solid State Physics, *nor* a full-fledged introduction to Quantum Mechanics, but rather an attempt to introduce a minimum set of modern physics concepts, necessary to understand the potentiality and limits of several present-day technology developments.

The course will touch the following subjects:

Introduction : Particles and Waves : the crisis of Classical Physics.
Part I : An introduction to Quantum Mechanics
Part II : Some concepts in Solid State Physics
Part III : Semiconductors at equilibrium
Part IV : Transport phenomena in Semiconductors.
Conclusion : Case study : PV cells.

(c) A shortcut : energy conservation (conservative forces only)

$$E = K + U = \frac{1}{2}mv^2 + U = \frac{p^2}{2m} + U$$

kinetic energy potential energy

Ex.1 :

remember : $\vec{\mathcal{E}}$ = force/charge (numerically: force on unit charge)

V = potential energy/charge (...)

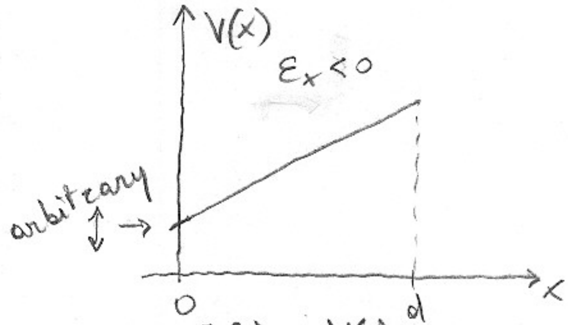
in one dimension :

$$V(x) = V(0) - \underbrace{\int_0^x \mathcal{E}_x \, dx}_{\text{work on unit charge from 0 to x}} \quad ; \quad \mathcal{E}_x = -\frac{dV}{dx}$$

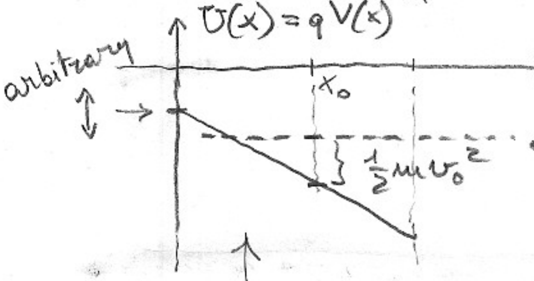
for the electron (charge $q = -|q| < 0$) : potential energy

$$U(x) = -|q|\,V(x) = -|q|\,V(0) - \underbrace{\int_0^x \underbrace{-|q|\,\mathcal{E}_x}_{\text{force}} \, dx}_{\text{work}}$$

in our example, constant $\mathcal{E}_x < 0$:



(independent of the charge! defined in general, for the unit charge)



$$E = U(x_0) + \frac{1}{2}mv_0^2 = \text{const.}$$

an invisible "potential wall" !

depending on the initial x_0 and v_0, the turning point (if any) of the trajectory is readily found.

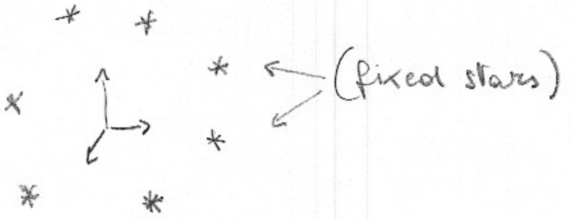
Lecture 1   Introduction, then: Particles and Waves classical point of view.

1.1. Motivations; frontiers in physics and technology; the electron: particle or wave? Practical issues.

(see slides)

1.2 Classical particles: mechanics (reminder)

Inertial frame



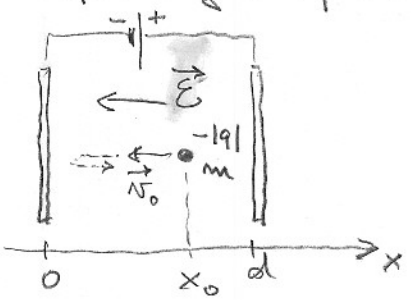
(a) Newton's laws : forces, initial conditions } ⇒ well defined trajectories by integration

$$\begin{cases} (I) \quad \dots \\ (II) \ \Sigma \vec{F} = \frac{d\vec{p}}{dt} \qquad \vec{p} = m\vec{v} \qquad d\vec{p} = \vec{F}_{net}\, dt \\ (III) \quad \dots \end{cases}$$

(example : electrostatic field acting on an electron ⇒ see slides)

(b) Standard approach : from the differential equation of motion with known forces and initial conditions, to the trajectory, that in principle can be determined with arbitrary precision, depending only on the knowledge of the initial conditions.

Ex. 1



electron in an electrostatic field $\vec{\mathcal{E}} = \mathcal{E}_x \hat{i}$ $(\hookrightarrow < 0)$

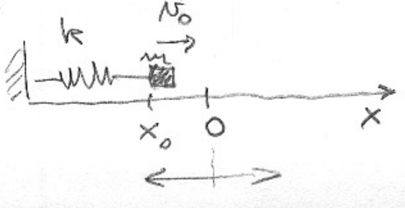
electric charge : $q = -|q| < 0$

$$\begin{cases} -|q|\,\mathcal{E}_x = m \frac{d^2x}{dt^2} \\ x(0) = 0, \ v_x(0) = v_0 \end{cases} \Rightarrow x(t) = x_0 + v_0 t + \frac{1}{2} a_x t^2$$

$$a_x = \boxed{-|q| \frac{\mathcal{E}_x}{m}}$$

another example : mass m, elastic force (Spring : k)

Ex. 2 : harmonic oscillator:



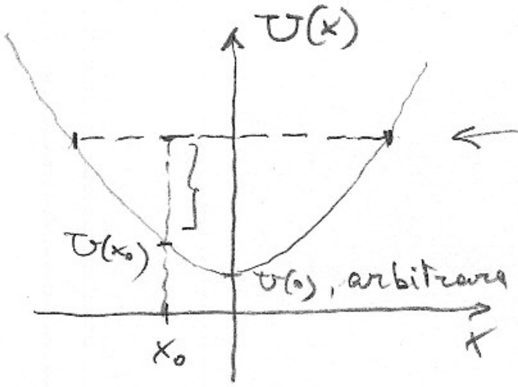
$$\begin{cases} -kx = m \frac{d^2x}{dt^2} \\ x(0) = x_0, \ v_x(0) = v_0 \end{cases} \Rightarrow x(t) = A\cos(\omega t + \varphi)$$

$$A(x_0, v_0) \qquad \omega = \sqrt{k/m} \qquad \varphi(x_0, v_0)$$

Similarly :

<u>Ex. 2</u> : the particle is confined within a "potential well"

Potential energy : $U(x) = U(0) - \underbrace{\int_0^x (-kx)\,dx}_{\text{work}} = U(0) + \frac{1}{2}kx^2$



$E = U(x_0) + \frac{1}{2}mv_0^2 = \text{const.}$

again : an invisible "potential well" !

(d) Note that, for a particle,

$$H \equiv \frac{p^2}{2m} + U \overset{\text{def.}}{\equiv} \text{"hamiltonian"}$$

is the basis of further developments in <u>classical mechanics</u>

for ex.: "Hamilton equations" re-formulation of mechanics

$$\left.\begin{array}{ll} q : & \text{generalized coordinates} \\ p : & \text{" momenta} \end{array}\right\} H = H(p,q)$$

if you are curious about that :

$$\begin{cases} \frac{dq}{dt} = \dot{q} = \frac{\partial H}{\partial p} \\ \frac{dp}{dt} = \dot{p} = -\frac{\partial H}{\partial q} \end{cases}$$

easy to check that this is an equivalent formulation in a specific example, like the harmonic oscillator.

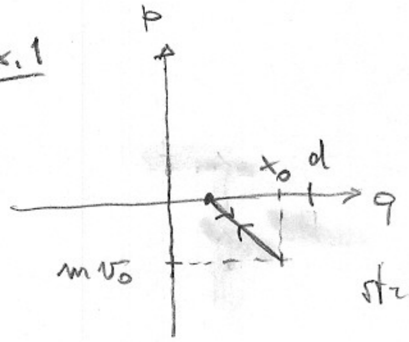
and becomes a fundamental "operator" in quantum mechanics, as we will shortly discover !

An aside : "phase space" , useful representation of the evolution of the state of the system under study, that will reappear later in this course .

In our two simple examples : $q \equiv x$ coordinate

$p = mv$ momentum

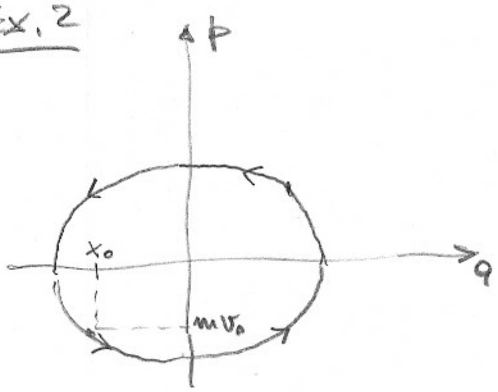
Ex. 1



straight line

electron in a constant $\vec{E}$ field

Ex. 2



$$q \equiv x = A \cos(\omega t + \varphi)$$

$$p \equiv mv = - m\omega A \sin(\omega t + \varphi)$$

it becomes a multi-dimensional space for systems described by a larger number of "generalized coordinates".

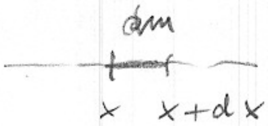
1.3. Classical Waves (in mechanics)

Newton's laws can be applied to continuous media (simplest example : a string) to obtain the equations of motion.

(a) equilibrium stretched position.

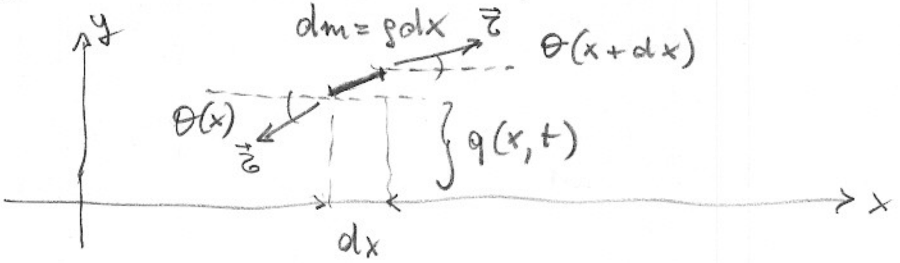
string characterized by :

density $\rho = \frac{dm}{dx}$ ← mass



tension τ

(b) displacement from equilibrium : (x, y) plane



string element, mass:
$dm = \rho dx$

$q(x,t)$: displacement in the y direction

Newton's law, projected along y, applied to the string element

$$\underset{\text{net force}}{\sum F_y} = \underset{\text{mass}}{\rho\, dx} \underset{\text{acceleration}}{\left[\frac{d^2q}{dt^2}\right]_{x \text{ fixed}}}$$

$$\parallel$$

$$\tau \cdot \left[\sin\theta(x+dx) - \sin\theta(x)\right]_{t \text{ fixed}}.$$

linearized assuming, for small θ, $\sin\theta \simeq \theta \simeq tg\,\theta$

$$\Rightarrow \sum F_y = \tau\left[tg\,\theta(x+dx) - tg\,\theta(x)\right]_{t \text{ fixed}} =$$

$$= \tau\left[\left.\frac{dq}{dx}\right|_{x+dx} - \left.\frac{dq}{dx}\right|_{x}\right]_{t \text{ fixed}} = \tau\, dx \left.\frac{d^2q}{dx^2}\right|_{x,\ t \text{ fixed}}$$

⇒ substituting, we obtain:

$$\tau \cancel{dx} \left[\frac{d^2 q}{dx^2}\right]_{t \text{ fixed}} = \rho \cancel{dx} \left[\frac{d^2 q}{dt^2}\right]_{x \text{ fixed}}$$

in other words:

(c) $$\frac{\partial^2 q(x,t)}{\partial x^2} = \frac{1}{u^2} \frac{\partial^2 q(x,t)}{\partial t^2}$$

with $u = \sqrt{\frac{\tau}{\rho}}$ is a <u>velocity</u>! (dimensions, easy to check)

very important result! The transverse motion of the string is governed by a 2nd order diff. equation (linearized for small displacements) relating the second partial derivatives w.r.to x and t.

and: "velocity" squared appearing as a coefficient;

$$u^2 \propto \tau \text{ (tension)}, \quad u^2 \propto \frac{1}{\rho} \text{ (density)}$$

Let us look at the solutions of this equation!

(d) "<u>D'Alembert solution</u>"

$f(\bar{x})$ arbitrary function of $\bar{x}$, twice differentiable

define $\bar{x} \equiv x - ut$

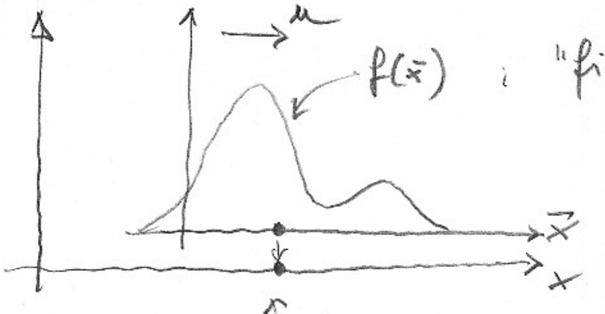
D'Alembert solution: $f(\bar{x}) = f(x - ut)$

easy to check :

$$\left.\begin{aligned} \frac{\partial f}{\partial x} &= \frac{df}{d\bar{x}}\frac{\partial \bar{x}}{\partial x} = \frac{df}{d\bar{x}} = f'(\bar{x})\\ \frac{\partial^2 f}{\partial x} &= \frac{d^2 f}{d\bar{x}^2}\cdot\frac{\partial \bar{x}}{\partial x} = \dots = f''(\bar{x})\\ \frac{\partial f}{\partial t} &= \frac{df}{d\bar{x}}\cdot\frac{\partial \bar{x}}{\partial t} = f'(\bar{x})\cdot(-u)\\ \frac{\partial^2 f}{\partial t^2} &= \dots = f''(\bar{x})\cdot(-u)(-u) = u^2 f''(x)\end{aligned}\right\} \Rightarrow \text{ok.}$$

⇒ any function (differentiable twice) $q(x,t) = f(x-ut)$ is a good solution!

Interpretation ?



$f(\bar{x})$: "fixed" in the reference frame $\bar{x}$,

→ $\bar{x} = x - ut$: moving with speed u with respect to x $(x = \bar{x} + ut)$

the "pulse" $f(\bar{x}) = f(x-ut)$ moves without change in shape with velocity u in the x direction.

Easy to check that changing $u \to -u$ simply reverses the direction of the motion $(x-ut) \to (x+ut)$

and : the equation is <u>linear</u> ⇒ accepts <u>linear combinations of solutions</u>

$$\Rightarrow \boxed{q(x,t) = \underset{\text{pulse}\rightarrow}{f(x-ut)} + \underset{\leftarrow\text{pulse}}{g(x+ut)}}$$ "D'Alembert solution"

Good to see in practice what this means:
demo with a real string/rope.

. . .

Next lecture:

let's get comfortable with sinusoidal waves and their properties, and with superpositions of sinusoidal waves; standing waves in particular.

Then:
move on to a reminder of
- statistical mechanics (and thermodynamics?)
- electromagnetism.

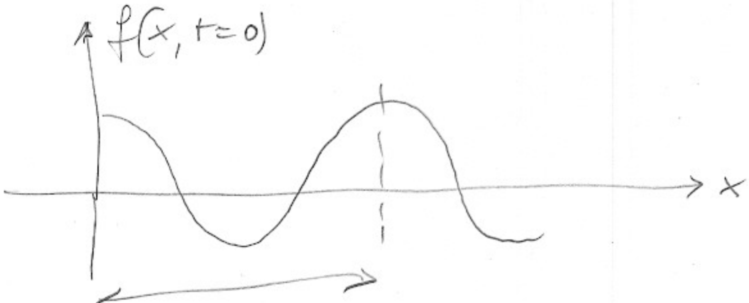
Waves: solutions of the wave equation.

sinusoidal waves $\left\{ \right.$ periodic solutions "excited by harmonic oscillators"

can reproduce arbitrary periodic solutions by superposition

$$f(\bar{x}) = \cos\Big(\underset{[L]^{-1}}{k}\,(\underbrace{x-ut}_{[L]})\Big) = \cos\Big(kx - \underbrace{\overbrace{ku}^{\omega}}_{[T]^{-1}}\,\underset{[T]}{t}\Big)$$

(a) snapshot at fixed t (for instance t=0)



$f(x,\overset{\overset{0}{\shortparallel}}{t}) = \cos kx$

"period" ≡

≡ "wavelenght"

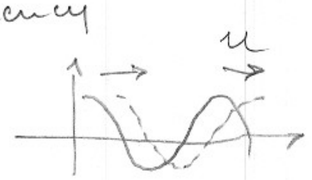
$$\lambda = \frac{2\pi}{k}$$

$$k = \frac{2\pi}{\lambda} \quad [m^{-1}]$$ "wave number

(b) Time evolution at fixed x (for instance at x=0)

$f(x=0,t) = \cos(\omega t)$

$$T = \frac{2\pi}{\omega} \quad ; \left\{ \begin{array}{l} \nu = \frac{1}{T} \text{ frequency} \\ \omega = \frac{2\pi}{T} \end{array} \right.$$



(c) u : "phase velocity" (for instance: displacement of max in unit time)

$$k u = \omega \Rightarrow u = \frac{\omega}{k} = \frac{2\pi}{T}\cdot\frac{\lambda}{2\pi} = \frac{\lambda}{T} = \nu\lambda$$

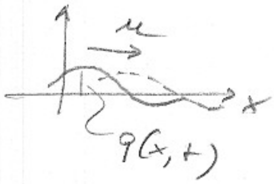
$$\lambda = uT \quad , \quad T = \frac{\lambda}{u}$$

$\left. \right\}$ useful relations easy to remember (dim.)

OK. we found that mechanical on a string waves propagate according to:

d'Alembert solution

$$\frac{\partial^2 q}{\partial x^2} = \frac{1}{u^2}\frac{\partial^2 q}{\partial t^2} \longrightarrow f(x-ut) + g(x+ut) = q(x,t)$$



sinusoidal waves: example

$$q(x,t) = A\cos(kx - \underset{ku}{\underset{\uparrow}{\omega}} t) \qquad u = \frac{\omega}{k} = \lambda\nu = \frac{\lambda}{T} \qquad \begin{cases} \lambda = \frac{2\pi}{k} \\ T = \frac{2\pi}{\omega} = 1/\nu \end{cases}$$

[complex exponential form : $A e^{i(kx-\omega t)}$]

numerical example for the string: u is fixed by $\begin{cases} \rho \text{ density} \\ \tau \text{ tension} \end{cases}$

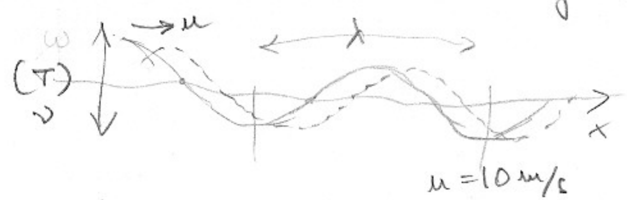
$$u = \sqrt{\frac{\tau}{\rho}}$$

$$[u] = [\cancel{M}]^{1/2}[L]^{1/2}[T]^{-2/2}\cdot[\cancel{M}]^{-1/2}[L]^{1/2} = [L][T]^{-1}$$

$$\left.\begin{array}{l}\rho = 0{,}10 \text{ kg/m} \\ \tau = 10 \text{ N}\end{array}\right\} \Rightarrow u \simeq \sqrt{10^2} \simeq 10 \text{ m/s}$$

in this model, <u>u is constant</u> (fixed) for fixed ρ and τ ; does <u>not</u> depend on ω or k.

⟹ if I shake one end of the string with a sinusoidal time dependence, I will generate sinusoidal waves, propagating at phase velocity $u \simeq 10$ m/s, and wavelength $\lambda = u . T$



$$\Longrightarrow \begin{array}{lll} T = 1 \text{ s.} \longrightarrow & \lambda = 10 \text{ m} & (\nu = \frac{1}{T} = 1 \text{ Hz}) \\ T = 0{,}1 \text{ s} \longrightarrow & \lambda = 1 \text{ m} & (\quad 10 \text{ Hz}) \\ T = 0{,}01 \text{ s} \longrightarrow & \lambda = 0{,}1 \text{ m} & (\quad 100 \text{ Hz}) \end{array}$$

NB u is different if I change the tension τ of the same rope (fixed ρ)

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L03/2 bis

from Taylor - Zafiratos . Dubson , p. 205

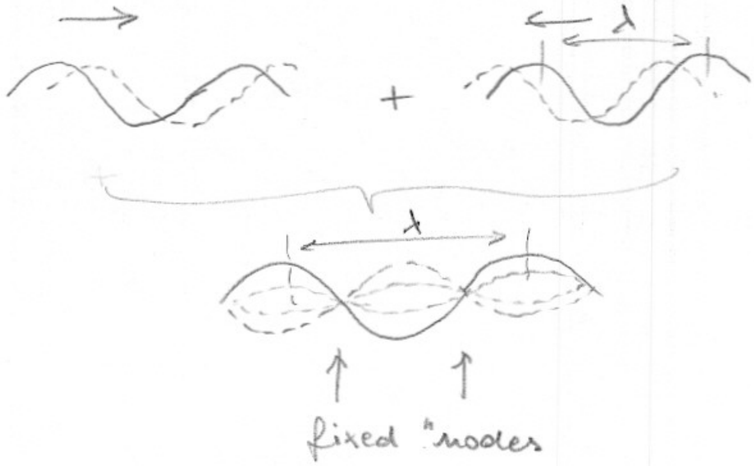
$$\varphi(x,t) = \underset{\uparrow}{B}\sin\overset{\rightarrow}{(kx-\omega t)} + \underset{\uparrow}{B}\sin\overset{\leftarrow}{(kx+\omega t)} =$$

same amplitude

trigonometry : $\sin a + \sin b = 2\sin\frac{a+b}{2}\cos\frac{a-b}{2}$

$$= \underset{A}{\boxed{2B}}\ \underbrace{\sin kx}_{\text{"envelope" of amplitudes}}\ \cos\omega t$$

NB:



two oppositely travelling waves (same k, ω, λ, ...)

make a "standing wave"

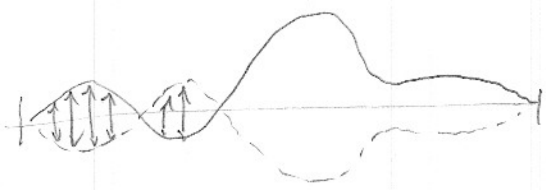
This is a typical configuration realized when a string of given length L is clamped at its ends.

Let's analyze this more quantitatively

Let's discuss now the "normal modes" ("standing waves" see L∅3/2 bis)

① ⇒ special case: oscillation with all points oscillating with the same frequency

(but possibly with a different phase)



Solutions with factorized x and t dependence
(practical examples: "stationary, standing waves on a string")

② $$q(x,t) = q(x)\underbrace{\cos(\omega t}_{} + \delta)$$

initial phase, could be different for different points

amplitude is different for different points

all points oscillate "in time"

③ substituting:

$$\frac{\partial^2 q}{\partial x^2} = \frac{1}{u^2}\frac{\partial^2 q}{\partial t^2}$$

$$\frac{d^2 q(x)}{dx^2}\cancel{\cos(\omega t + \varphi)} = \frac{1}{u^2}\, q(x)\,(-\omega^2)\cancel{\cos(\omega t + \varphi)}$$

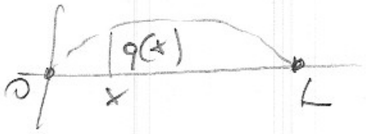
④ $$\boxed{\frac{d^2 q}{dx^2} = -\underbrace{\frac{\omega^2}{u^2}}_{k^2}\, q(x)}$$

$q = q(x)$!

"Helmoltz equation"

① Interesting case: <u>fixed endpoints</u>

$q(0)=q(L)=0$



general solutions:

$$q(x) = A\sin kx + B\cos kx$$

$$q(0)=0 \Rightarrow B=0$$

$$q(L)=0 \Rightarrow A\sin kL = 0 \iff k_m L = m\pi$$

$$\boxed{k_n = n\frac{\pi}{L}} \quad n=1,2,\dots\infty$$

"eigenvalues"

② $\boxed{q_m(x) = A\sin k_m x}$ "eigenfunctions" (of the operator $\frac{d^2}{dx^2}$)

$\sqrt{\frac{2}{L}}$ to obtain $\int_0^L q_n(x)\,dx = 1$

$$\Rightarrow q_n(x) = \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi}{L}x\right)$$

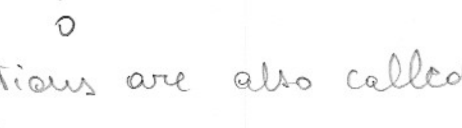
$$\omega_n = u k_n = n\frac{\pi}{L}u$$

phase velocity

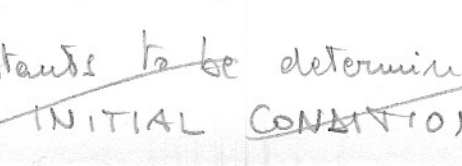
③ $\Rightarrow q_m(x,t) = \sqrt{\frac{2}{L}}\sin\left(m\frac{\pi}{L}x\right)\cos\left(\omega_n t + \delta_n\right)$

$\boxed{\text{"normal modes"}}$

Examples: $m=1$



$m=2$



[$(\delta_n = 0)$ ← NB !?]

④ These "factorized" solutions are also called "normal modes" of the string with fixed end points.

(ω_n, δ_n are constants to be determined by the INITIAL CONDITIONS of the motion)

① $\Longrightarrow$ generic solution : linear superposition of "normal modes"

$$q(x,t) = \sum_1^\infty{}_n A_n \sqrt{\frac{2}{L}} \sin(\underbrace{n\frac{\pi}{L}}_{k_n} x) \cos(\omega_n t + \delta_n)$$

$\hookrightarrow (n\frac{\pi}{L})u = k_n u$

(satisfies Huygens eq. with the given boundary conditions, fixed endpoints)

② specific solution : initial conditions for

$$\begin{cases} q(x, t=0) \\ \left.\frac{\partial q}{\partial t}\right|_{t=0} \end{cases} \Longrightarrow \text{fixes } A_n , \delta_n$$

initial "shape" of the string

initial transverse velocities

That is:

$$q(x,0) = \sum_1^\infty{}_n \underbrace{A_n \cos\delta_n}_{a_n} \underbrace{\sin(n\frac{\pi}{L}x)\cdot\sqrt{\frac{2}{L}}}_{\phi_n(x)} = f(x)$$

$$\left.\frac{\partial q}{\partial t}\right|_{t=0} = -\sum_1^\infty{}_n \underbrace{\omega_n A_n \sin\delta_n}_{\omega_n a_n} \underbrace{\sin(n\frac{\pi}{L}x)\cdot\sqrt{\frac{2}{L}}}_{\phi_n(x)}$$

Fourier series !!!

given functions "initial conditions"

it can be shown that the coeff. A_n, δ_n are fixed by these conditions.

③ To solve for A_n, δ_n one can use the following property:

$$\int_0^L \phi_n(x)\phi_m(x)\,dx = \delta_{nm}\begin{cases}0 & n\neq m\\ 1 & n=m\end{cases}$$

$$\Longrightarrow \int_0^L \phi_m(x)\cdot f(x)\,dx = \sum_1{}_n a_n \underbrace{\int_0 \phi_m(x)\phi_n(x)}_{\delta_{mn}} = a_m$$

④ in practice, this means that the recipe to compute A_m and δ_m is:

$$\Longrightarrow \forall m \begin{cases} a_m = A_m\cos\delta_m = \int_0^L \underbrace{q(x,0)}_{f(x)} \cdot \underbrace{\sqrt{\frac{2}{L}}\sin(m\frac{\pi}{L}x)}_{\phi_m(x)}\,dx = \ldots \\ -\omega_m a_m = -\omega_m A_m \cos\delta_m = \int_0^L \left.\frac{\partial q}{\partial t}\right|_{t=0} \cdot \sqrt{\frac{2}{L}}\sin(m\frac{\pi}{L}x)\,dx = \ldots \end{cases}$$

number that can be computed for the given $q(x,0)$, ...

explicit solution!

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① In summary:

every specific solution can be written as an (infinite) sum of "normal modes", with coefficients that can be computed from the initial configuration ($q(x,t=0)$ and $\frac{\partial q}{\partial t}\Big|_{t=0}$)

② for a long string: $L \to \infty$

$$\Longrightarrow \quad k_n = \frac{\pi}{L} n \quad : \Delta k = k_n - k_{n-1} = \frac{\pi}{L} \longrightarrow 0$$

the spacing of wave numbers becomes vanishing small.

the "Fourier sum" $\left(\sum_{1}^{\infty}{}_{n}\right)$ becomes a "Fourier integral" $\frac{1}{\pi}\int_0^\infty dk *$

③ why?

for any function $f(k_n)$

$$\sum_{1}^{\infty}{}_{n} f(k_n) = \frac{L}{\pi}\underbrace{\sum_{1}^{\infty}{}_{n} \cdot \frac{\pi}{L} \; f\left(\frac{\pi n}{L}\right)}_{\begin{array}{c} \| \\ \sum_{1}^{\infty}{}_{n} \, \Delta k \; f(k_n) \\ \big\downarrow L \to \infty \\ \int_0^\infty dk \; f(k) \end{array}}$$

④ in other terms:

$$\sum_{1}^{\infty}{}_{n} \xrightarrow[L\to\infty]{} \frac{L}{\pi}\int_0^\infty dk$$

and if $f(k) = f(-k)$, even function, for $L \to \infty$

$$\sum_{1}^{\infty}{}_{n} \xrightarrow[L\to\infty]{} \frac{L}{2\pi}\int_{-\infty}^{+\infty} dk$$

"Fourier integrals"

A quick reminder of electrodynamics

① We use mainly S.I. units.

mass : kg
length : m
time : s
Force : 1 N = 1 kg · m/s²

charge : C
voltage : V
current : A   1A = 1C/s

② Basic principles :

$q_1 \bullet \longrightarrow \bullet\, q_2$ $\vec{r} = \vec{r}_2 - \vec{r}_1$

$$\vec{F}_{21} = \underbrace{\frac{1}{4\pi\varepsilon_0}}_{8{,}988\times 10^9 \frac{N\,m^2}{C^2}} \frac{q_1 q_2}{r^2} \overbrace{\left(\frac{\vec{r}}{r}\right)}^{\hat{r}}$$

2: on particle 2
1: by particle 1

Coulomb's law

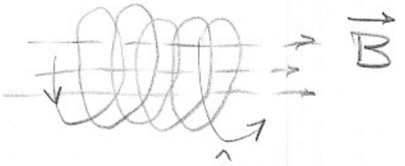
electrostatic field $\vec{\mathcal{E}}$

$$\boxed{\vec{\mathcal{E}}_1 = \frac{\vec{F}_{21}}{q_2} = \frac{1}{4\pi\varepsilon_0}\frac{q_1}{r^2}\hat{r}} \longleftarrow \frac{N}{C} = \frac{V}{m}$$

③

magnetostatic field $\vec{B}$

⟹ Simpler expression for the field generated by a solenoid :



$$\boxed{B = \underbrace{\mu_0}_{\text{defined as}} n\, i} \Longleftarrow$$

"infinite solenoid" { n turns/m
current i (A)

defined as

$$\frac{\mu_0}{2\pi} = 1{,}000\times 10^{-7}\,\frac{N\,s^2}{C^2} = 1{,}000\times 10^{-7}\,\frac{T\,m}{A}$$

B measured in Tesla (T)

$$\left(1T = \frac{1\,kg}{C\,s} = 10^4 \text{ gauss}\right)$$

④

Force on a charged particle

"Lorentz" : $\boxed{\vec{F} = q\left(\vec{\mathcal{E}} + \vec{v}\times\vec{B}\right)}$

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(a) An <u>example</u> of how a charged particle interacts with $\vec{E}$ and $\vec{B}$ fields in an experimental apparatus:

① 

$\vec{B}$ ⊥ drawing (entering) ; $\vec{E} = 0$

uniform

$q > 0$ (for instance : proton)

$$\vec{F} = q(\vec{v} \times \vec{B}) = \frac{d\vec{p}}{dt} = m\vec{a} \quad \text{(non relativistic)}$$

$$qvB = m\frac{v^2}{r}$$

⊛ relativ. : $\vec{p} = \gamma m_0 \vec{u}$

→ <u>circular motion</u>, radius: $r = \frac{mv}{qB}$ ⟹ $r \propto p$

②

orders of magnitude (non-relativistic)

proton :
$m_p = 1.6 \times 10^{-27}$ kg
$v = 3{,}0 \times 10^{7}$ m/s (1/10 speed of light)
$q = 1{,}6 \times 10^{-19}$ C

mag. field : $B = 0{,}1\text{T} = 10^3$ gauss

$$r \simeq \frac{\cancel{1{,}6} \times 10^{-27} \times 3 \times 10^{7}}{\cancel{1{,}6} \times 10^{-19} \times 10^{-1}} = 3 \times \frac{10^{-20}}{10^{-20}} \simeq 3 \text{ m}$$

($r \simeq 0{,}3$ m if $B \simeq 1$ T)

NB : in this example
$v = 0{,}1\ c$
↑ speed of light in vacuum

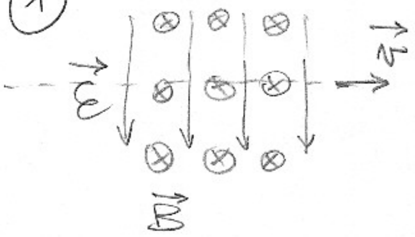
$$\Rightarrow \quad p = \gamma m v = \frac{m}{\sqrt{1-(0{,}1)^2}}\, v \simeq mv$$

↑ $\gamma \simeq 1$

for relativistic particles ($v \simeq c$) $\vec{F} = \frac{d\vec{p}}{dt}$ remains the same, but $\vec{p} = \gamma m \vec{v}$, $\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$

(b) Another example: a "velocity selector" with static fields $\vec{\mathcal{E}} \perp \vec{B}$

①



a particle (m, q) entering this space region is subject to a force

$$\vec{F} = q\left(\vec{\mathcal{E}} + \vec{v}\times\vec{B}\right)$$

② in this configuration, the two contributions can be opposite

$$\left(\underset{\downarrow}{q\vec{\mathcal{E}}} = -\underbrace{q\vec{v}\times\vec{B}}_{\uparrow}\right) \Longrightarrow \vec{F} = 0 \iff \mathcal{E} = vB$$

$q>0$

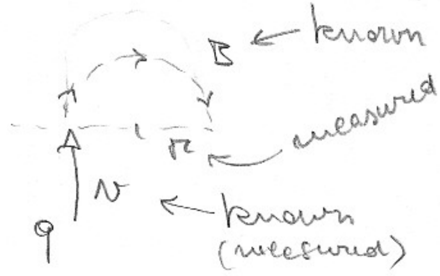
$$v = \frac{\mathcal{E}}{B}$$

③ for instance: $\begin{cases} \mathcal{E} = 10^3\ \mathrm{V/cm} = 10^5\ \mathrm{V/m} \\ B = 1.\ \mathrm{T} \end{cases}$ selects $v = 10^5\ \mathrm{m/s}$

("small" with respect to $c = 3\times10^8\ \mathrm{m/s}$)

(c) a combination of $\left\{\begin{array}{l}\text{(b) velocity selector} \\ \text{(a) magnetic field..}\end{array}\right\}$ makes a "mass spectrograph"

④



$$m = \underbrace{\left(\frac{qB}{v}\right)}_{\substack{\uparrow \\ \text{known}}}\ \underbrace{r}_{\substack{\uparrow \\ \text{measured}}} \longleftarrow \text{experimental mass determination.}$$

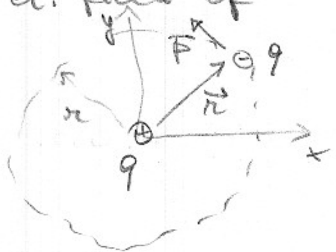
(d) classic circular motion of an electron in the Coulomb el. field of a nucleus:

① Newton's law → $m\vec{a} = \vec{F}$

kinematics → $m\frac{v^2}{\cancel{r}} = \frac{q^2}{4\pi\varepsilon_0}\frac{1}{r^{\cancel{2}}}$

② angular momentum $\vec{\ell} = \vec{r}\times\vec{p} = (r m v)\,\hat{z}$

$$\frac{m^2v^2r^2}{m} = \frac{\ell^2}{m} = \left(\frac{q^2}{4\pi\varepsilon_0}\right) r \qquad \boxed{r \propto \ell^2}$$



Maxwell's equations

The description of the interactions between charged particles (static or in motion → currents) is much simpler if we think in terms of :

"Sources" ⟶ fields ⟶ forces on particles/currents

ch. density $\rho(\vec{r},t)$ $\vec{\mathcal{E}}$, $\vec{B}$

curr. density $\vec{J}(\vec{r},t)$

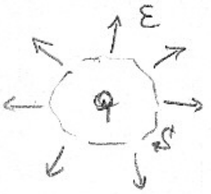
[divergence]

$$\begin{cases} \vec{\nabla}\cdot\vec{\mathcal{E}} = \dfrac{\rho}{\varepsilon_0} \\ \vec{\nabla}\cdot\vec{B} = 0 \\ \text{[curl]} \\ \vec{\nabla}\times\vec{\mathcal{E}} = -\dfrac{\partial\vec{B}}{\partial t} \\ \vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\varepsilon_0\dfrac{\partial\vec{E}}{\partial t} \end{cases}$$

integral form

[flux (closed surface)]

$$\oint_S \vec{\mathcal{E}}\cdot\hat{n}\,dS = \frac{Q}{\varepsilon_0}$$



$$\oint_S \vec{B}\cdot\hat{n}\,dS = 0$$

[line integral, closed path]

$$\oint_\Gamma \vec{\mathcal{E}}\cdot d\vec{l} = -\frac{d}{dt}\int_S \vec{B}\cdot\hat{n}\,dS$$

$$\oint_\Gamma \vec{B}\cdot d\vec{l} = \mu_0\int_S \vec{J}\cdot\hat{n}\,dS + \mu_0\varepsilon_0\underbrace{\int_S \frac{\partial\vec{E}}{\partial t}\cdot\hat{n}\,dS}_{u}$$

+ "charge conservation"

$$\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\vec{J} = 0$$

$$\oint_S \vec{J}\cdot\hat{n}\,dS = -\frac{dQ}{dt}$$

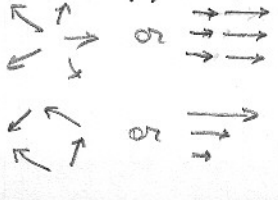
$$= -\frac{d}{dt}\int_V \rho\,dV$$

in cartesian coordinates :

$$\begin{cases} \vec{v} = (v_x\hat{\imath} + v_y\hat{\jmath} + v_z\hat{k} \quad \text{any vector} \\ \vec{\nabla} = \left(\dfrac{\partial}{\partial x}\hat{\imath} + \dfrac{\partial}{\partial y}\hat{\jmath} + \dfrac{\partial}{\partial z}\hat{k}\right) \end{cases}$$

$$\begin{cases} \vec{\nabla}\cdot\vec{v} = \dfrac{\partial v_x}{\partial x} + \dfrac{\partial v_y}{\partial y} + \dfrac{\partial v_z}{\partial z} \quad \text{"divergence" (scalar)} \\ \vec{\nabla}\times\vec{v} = \left(\dfrac{\partial v_y}{\partial z} - \dfrac{\partial v_z}{\partial y}\right)\hat{\imath} + \left(\dfrac{\partial v_z}{\partial x} - \dfrac{\partial v_x}{\partial z}\right)\hat{\jmath} + \left(\partial v \quad\right)\hat{k} \quad \text{"curl" (vector)} \end{cases}$$

locally ≠ 0 if :



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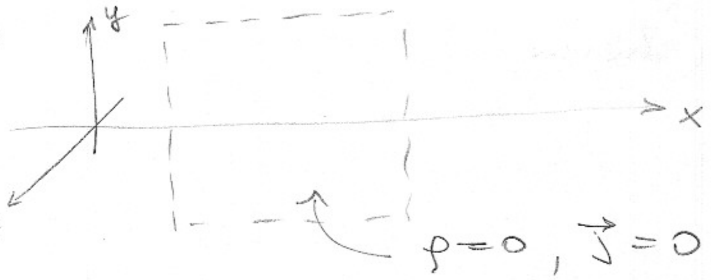
e.m. waves : from the Maxwell equations:

[NB: can explain in reverse order]

①



it can be shown that in an empty region (no charges, no currents)

② e.m. fields generated outside can propagate for instance in the x direction, obeying the following wave equation, derived from Maxwell's equations that connect $\vec{\mathcal{E}}$ and $\vec{B}$ (time-varying $\vec{B}$ field is a source of $\vec{\mathcal{E}}$ field and vice-versa)

③ wave equations for $\mathcal{E}_y$, B_z

$$\begin{cases} \dfrac{\partial^2 \mathcal{E}_y}{\partial x^2} = \boxed{\mu_0 \varepsilon_0} \dfrac{\partial^2 \mathcal{E}_y}{\partial t^2} \\ \dfrac{\partial^2 B_z}{\partial x^2} = \boxed{\mu_0 \varepsilon_0} \dfrac{\partial^2 B_z}{\partial t^2} \end{cases}$$

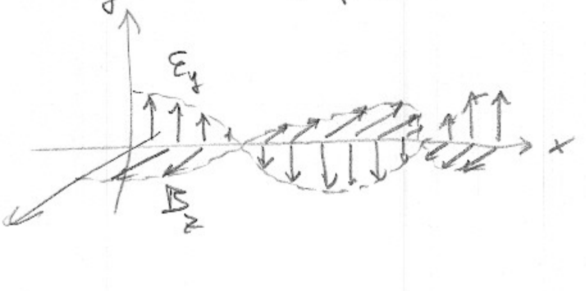
$$\frac{1}{c^2} = \mu_0 \varepsilon_0$$

$$c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = 2.998 \times 10^8 \ m/s$$

④ solutions ("plane wave" of Maxwell's and wave equation)

$$\begin{cases} \mathcal{E}_y(x,t) = A \cos k(x - ct) \\ B_z(x,t) = \frac{1}{c} \mathcal{E}_y(x,t) \end{cases}$$

t fixed



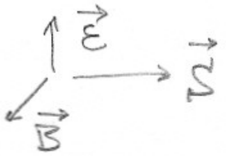
NB emphasize physical meaning and speed $c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$

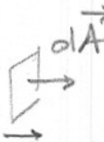
① E.m. waves carry energy and momentum.

(explain in intuitive physical terms...) → static fields → propagating fields

② energy: $\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B}$ units: $\frac{J}{m^2 s}$

"energy flux"





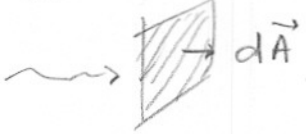
$(d\vec{A} = \hat{n}\,dA)$

$\vec{S}\cdot d\vec{A}$: energy flowing through $d\vec{A}$ in unit time (s)

③ momentum: $\vec{P} = \frac{\vec{S}}{c}$

"momentum flux" : momentum through $d\vec{A}$ in unit time

$\vec{P}\cdot d\vec{A}$



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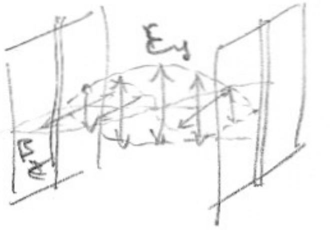
④ "Normal modes" for an e.m. field

it is an interesting exercise to verify that there are "standing wave" solutions for $\vec{E}$ and $\vec{B}$ between perfectly conducting parallel plates, in a vacuum



$$E_y = A\sin\left(n\frac{\pi}{L}x\right)\cos\left(n\frac{\pi}{L}ct\right) \qquad n = 1, 2, \ldots \infty$$

$$-cB_z = A\cos\left(n\frac{\pi}{L}x\right)\sin\left(n\frac{\pi}{L}ct\right)$$

↖ "normal modes" → $\omega_n = k_n c = n\frac{\pi}{L}c$

⑤ Radiation sources

see next.

What are the possible sources of classical "radiation" (e.m. waves)

?

Accelerated charges!

amplitude of e.m. radiation

(intensity (energy) $\propto A^2$)

$$A \propto q \left(\frac{d^2\vec{r}}{dt}\right)_\perp$$

proportional

charge

component of the acceleration $\perp$ direction of observation.

examples:

(1) dipole antenna

harmonic motion of charges in the z direction : $\frac{d^2z}{dt^2} = -\omega^2 z$

(2) storage ring, synchrotron radiation

$\vec{a}$ pointing to the center

$\Rightarrow$ radiation focussed tangentially.

see also: J.R. Taylor, C.D. Zafiratos, M.A. Dubson
Modern Physics for Scientists and Engineers.

chapter 11, Atomic Transitions and Radiation

L03/12 bis

in more detail : (Taylor - Zafiratos - Dubson, p. 335)

power radiated by a single charge moving non-relativistically

$$P = \frac{1}{4\pi\varepsilon_0} \cdot 2 \frac{q^2 a^2}{3c^2}$$

$\frac{1}{4\pi\varepsilon_0} \equiv k$

useful when discussing Bohr's atom : $\left(q = 1.6 \times 10^{-19}\ C \right.$

$\left. a = \frac{F}{m} = \frac{1}{4\pi\varepsilon_0} \frac{q^2}{r^2} \cdot \frac{1}{m} \right)$

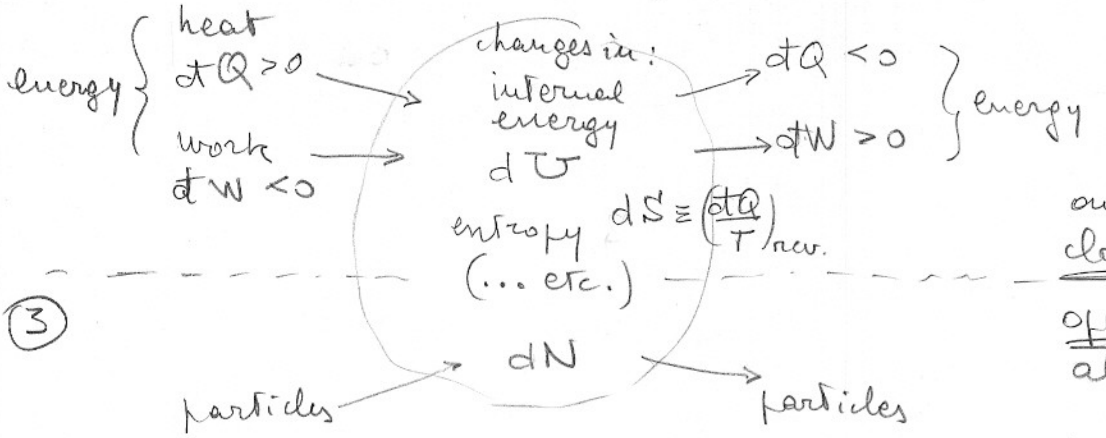
one can compute $P = 2.9 \times 10^{11}$ eV/s

$\longrightarrow$ classically : quick collapse !

$r \simeq 0{,}5 \times 10^{-10}$ m

① Let's recall now some results from <u>Thermodynamics</u> and <u>Statistical Mechanics</u> (also called sometimes "Thermal Physics")

② Thermodynamics ("macroscopic")



Environment

System $\begin{cases} P, T, \mu & \text{"intensive"} \\ V, U, S, \ldots & \text{"extensive"} \end{cases}$

I) $dU = \text{đ}Q - \text{đ}W$
$\quad \doteq TdS - pdV$

II) $dS \geq 0$
for <u>isolated</u> systems

only <u>energy</u> exchanged
<u>closed</u> systems

---

③

<u>open</u> systems
also <u>particles</u> exchanged

I) $dU = \underset{\text{đ}Q}{\underset{\uparrow}{TdS}} - \underset{\text{đ}W}{\underset{\uparrow}{PdV}} + \underset{\text{particles}}{\underset{\uparrow}{\mu dN}}$

④ Statistical Mechanics

"macrostate" : "ensemble" of "microstates" $\begin{cases}\text{very large number } \Omega \\ \text{equiprobable} \\ \text{all explored, given enough time}\end{cases}$

(typical example : ○●●○○●○●○○
$r=4$ quanta in $n=10$ atoms: $\Omega = \binom{n}{r} = \frac{n!}{(n-r)!\,r!} = 210$) (*) [see back]

equilibrium state of a large system : $\begin{cases} \text{entropy } S = k_B \ln \Omega \\ dS = 0 \ : \ \text{maximum entropy !} \end{cases}$

etc... (average properties ... probabilities)

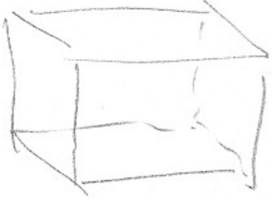
⑤ $\Longrightarrow$ just recall a few results :

- meanings of T (k_BT), P, μ
- equipartition theorem : energy and degrees of freedom.

LL 04. 10. 2010

Meaning of T, the Boltzmann constant k_B, and the equipartition theorem.

Ideal gas



experimentally:

$$\boxed{PV = \bar{n} R T} \quad (1)$$

"absolute Temperature" in K

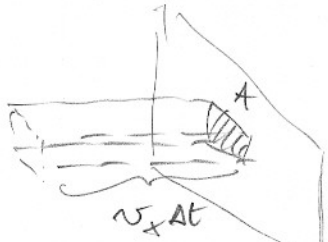
pressure — volume — n. of moles $\bar{n} N_A = N$

$R \simeq 8.3\ J/mol \cdot K$

universal constant for rarified gases at high Temperatures

number of particles

Avogadro



P : force/area

$n = \frac{N}{V}$

$n = \frac{N}{V}$ density of particles.

Kinetic model

$\Delta n = (v_x \Delta t) \cdot A \cdot n \times \frac{1}{2}$

number of particles hitting the wall in a time Δt

$2mv_x$: momentum transferred to the wall per particle (elastic scattering)

momentum " " "

$$\Delta p = \Delta n \cdot 2mv_x = \cancel{2}mv_x \cdot v_x \Delta t \cdot A \cdot \frac{1}{\cancel{2}} n$$

''

$F \Delta t$ ← impulse

average force on A

$$\Rightarrow F = mv_x^2 \cdot A \cdot n$$

n ← $\frac{N}{V}$

$$\Rightarrow \langle P \rangle = \frac{F}{A} = \frac{N}{V} \cdot \underbrace{mv_x^2}$$

pressure

$$\frac{1}{3}mv^2 = \frac{2}{3} \cdot \frac{1}{2}mv^2 = \frac{2}{3} \langle \text{kinetic energy} \rangle$$

$\Updownarrow$ internal energy

$U = N \langle \frac{1}{2}mv^2 \rangle$

$$\Rightarrow PV = N \cdot \frac{2}{3} \langle \frac{1}{2}mv^2 \rangle = \frac{2}{3} U$$

$$\boxed{PV = \bar{n} N_A \cdot \frac{2}{3} \langle \frac{1}{2}mv^2 \rangle} \quad (2)$$

kinetic model

① Comparing the exp. tal results with the kinetic model

(experiment)

$$pV = \begin{cases} \bar{n}\;RT = \bar{n}\cdot N_A \cdot \underbrace{\frac{R}{N_A}}_{k_B}\,T = \bar{n}\,N_A \cdot k_B T \\ \bar{n}\;N_A \cdot \frac{2}{3}\langle \frac{1}{2} m v^2 \rangle \end{cases}$$

(theory)

can be identified

② $\Longrightarrow$ "internal energy" $\langle \frac{1}{2} m v^2 \rangle = 3 \times \frac{1}{2} k_B T$ ← one particle

$$U = \bar{n} N_A \langle \tfrac{1}{2} m v^2 \rangle = 3\,\underbrace{\bar{n} N_A}_{N} \cdot \tfrac{1}{2} k_B T$$ ← N particles.

Severel important conclusions:

③ T : is proportional to "average kinetic energy" (at least for ideal gases)

④ k_B : Boltzmann constant:
Conversion factor | from temperature (K)
| to energy (J)

$$k_B = \frac{R}{N_A} = \frac{8{,}3\ J/mol\cdot K}{6.0 \times 10^{23}\ mol^{-1}} = 1.4 \times 10^{-23}\ J/K$$

1,38

⑤ Special case of the "equipartition theorem" of statistical mechanics:

there is a contribution of $\frac{1}{2} k_B T$
to the mean energy of the system
for every "quadratic degree of freedom" in the Hamiltonian.

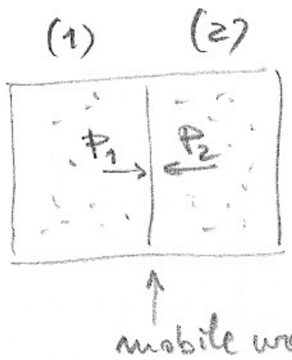
in this case: ideal monoatomic gas

$$U = 3\,\overbrace{\bar{n} N_A}^{N} \cdot \frac{1}{2} k_B T = \dots = \frac{3}{2}\bar{n} R T$$

Next: meaning of $\begin{cases} P & \text{pressure} \\ T & \text{temperature} \\ \mu & \text{chemical potential} \end{cases}$

as parameters indicating "equilibrium" conditions.

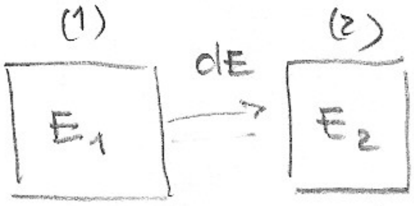
(a)



Systems (1) and (2) can exchange <u>work</u>

$P_1 = P_2$: <u>mechanical equilibrium</u>

(b)



Systems (1) and (2) can only exchange energy among themselves (<u>heat</u>)

$\Omega_1(E_1)$ $\Omega_2(E_2)$ microstates: number $\Omega(\underset{\substack{\uparrow \\ \text{depends on energy}}}{E})$

most probable division of energy between (1) and (2):

max. $\Omega_1(E_1) \cdot \Omega_2(E_2)$ total number of microstates

$$\Longleftrightarrow \frac{d}{dE_1}\left(\quad " \quad \right) =$$

$$\Omega_2(E_2)\frac{d\Omega_1}{dE_1} + \Omega_1(E_1)\cdot\frac{d\Omega_2}{dE_2}\underset{\substack{\uparrow \\ = -1}}{\frac{dE_2}{dE_1}} = 0 \quad \Big/ \frac{1}{\Omega_1 \Omega_2}$$

$(dE_2 = -dE_1$: total energy unchanged.

$$\frac{1}{\Omega_1}\frac{d\Omega_1}{dE_1} - \frac{1}{\Omega_2}\frac{d\Omega_2}{dE_2} = 0$$

$$\underset{\uparrow}{\frac{d \ln\Omega_1}{dE_1}} = \underset{\uparrow}{\frac{d \ln\Omega_2}{dE_2}}$$ equilibrium: "equal temperature"

we identify these expression with the def. of temp.

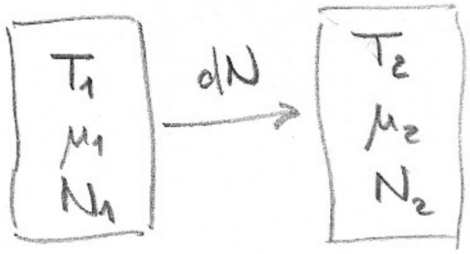
$$\left(k_B = 1,38 \times 10^{-23}\ \mathrm{J\,K^{-1}} \right)$$

$$\frac{1}{k_B T} = \frac{d \ln \Omega}{dE}$$

$$\frac{1}{T} = \frac{d\left(k_B \ln \Omega\right)}{dE} = \frac{dS}{dE}$$

$$\frac{1}{T_1} = \frac{1}{T_2} \quad \text{or} \quad T_1 = T_2$$: thermal equilibrium

(c)



Systems (1) and (2) can exchange particles.

what happens? see next.

Blundell – Blundell, p. 244

1. The chemical potential

Generalization of thermodynamic potentials when the number of particles may change:

$$dU = TdS - PdV + \mu\, dN$$

← NB: for instance
$N \sim 10^{23}$
$dN \sim 10^{6}$ or 10^{9}
is a "small" change

$$\boxed{\mu = \left(\frac{\partial U}{\partial N}\right)_{S,V}}$$

S, V not easy to fix in practice
⇒ use rather:

$$F = U - TS \Rightarrow dF = \cancel{TdS} - PdU - \cancel{TdS} - SdT + \mu dN$$

$$= -PdV - SdT + \mu dN \Rightarrow \boxed{\mu = \left(\frac{\partial F}{\partial N}\right)_{V,T}}$$

$$G = H - TS =$$

$$= U + PV - TS \Rightarrow dG = \cancel{TdS} - \cancel{pdV} + \cancel{pdV} + VdP - \cancel{TdS} - SdT + \\ + \mu dN$$

$$= VdP - SdT + \mu dN$$

$$\Rightarrow \mu = \left(\frac{\partial G}{\partial N}\right)_{P,T}$$

⇒ chemical potential
"Gibbs free energy per particle"
↳ constant P, T

2. Meaning of the chemical potential ?

different values of the chemical potential drives particles to move from one place to another, much in the same way as a difference in temperature drives the transfer of energy (heat)

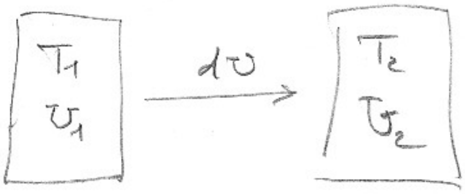
→ equilibrium ? max. entropy

$$dU = TdS - PdV + \mu dN$$

$$dS = \frac{1}{T}\left(dU + pdV - \mu dN\right)$$

$$\Longrightarrow \quad \left(\frac{\partial S}{\partial U}\right)_{V,N} = \frac{1}{T} \qquad \left(\frac{\partial S}{\partial V}\right)_{U,N} = \frac{p}{T} \qquad \left(\frac{\partial S}{\partial N}\right)_{U,V} = -\frac{\mu}{T}$$

(a) **heat flow:** (constant N) from system 1 to system 2



$dU_1 = -dU \qquad dU_2 = dU$

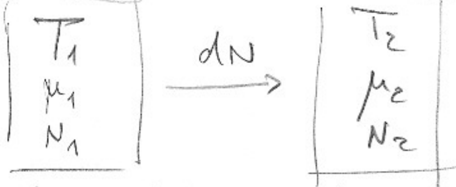
$$dS = \underbrace{\left(\frac{\partial S_1}{\partial U_1}\right)_{N,V}}_{\frac{1}{T_1}} \overset{dU_1}{(-dU)} + \underbrace{\left(\frac{\partial S_2}{\partial U_2}\right)_{N,V}}_{\frac{1}{T_2}} dU$$

$$dS = \left(-\frac{1}{T_1} + \frac{1}{T_2}\right) dU \geq 0$$

$dS = 0$ if $T_1 = T_2$ (thermal equilibrium : no entropy increase)
or : reversible heat exchange at $T_1 \approx T_2 \approx T$

$$\left.\begin{matrix} dS > 0 \\ dU > 0 \end{matrix}\right\} \Leftrightarrow -\frac{1}{T_1} + \frac{1}{T_2} > 0 \Leftrightarrow \frac{1}{T_2} > \frac{1}{T_1} \Leftrightarrow T_1 > T_2$$

(b) **particle exchange** (dN) from system 1 to system 2



$dN_1 = -dN \qquad dN_2 = dN$

$$dS = \left(\frac{\partial S_1}{\partial N_1}\right)_{U,V}(-dN) + \left(\frac{\partial S_2}{\partial N_2}\right)_{U,V} dN =$$

$$= \left(\frac{\mu_1}{T_1} - \frac{\mu_2}{T_2}\right) dN \geq 0$$

for $T_1 = T_2 = T \qquad dS = 0 \Leftrightarrow \mu_1 = \mu_2$

$$dS \geq 0 \Leftrightarrow \frac{\mu_1 - \mu_2}{T} dN \geq 0 \overset{dN>0}{\Longleftrightarrow} \mu_1 > \mu_2$$

3. "Total" chemical potential

If the particles have interactions with conservative forces, with a potential energy (for instance:

$$\begin{cases} U(z) = mgz & \text{: gravitational energy, or:} \\ U = q\mathcal{V} & \text{: electrostatic energy} \\ \vdots & \\ \text{etc.} & \end{cases}$$

Total internal energy:

chemical potential μ electrical potential

$$dU = TdS - p\,dV + \underbrace{\mu\, dN + q\mathcal{V}}_{\text{Total electro-chemical potential}}$$

Gibbs function:

$$dG = \underset{0}{\cancel{SdT}} + \underset{0}{\cancel{Vdp}} \boxed{+ \mu\, dN + q\, d\mathcal{V}} = 0$$

T = const
p = const.

$$\mu\, dN = -q\, d\mathcal{V}$$

NB: equilibrium requires the "total chemical potential", in this case the "electro-chemical potential", to be a constant (we will see this property is very useful in discussing semiconductors)

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from here: Powerpoint slides.