energy is 2.00 eV? 2.00 keV? 2.00 MeV? 2.00 GeV? (*b*) For what kinetic energies will an electron and a proton each have a momentum of 5.00 MeV/*c*? (*c*) An electron, a muon, and a proton each have a kinetic energy of 10.0 MeV. What are their speed parameters? (*d*) An electron, a muon, and a proton each have a relativistic mass that is three times their rest mass. What are their kinetic energies?

74C. Using your calculator (II). For particles whose speed parameters are sufficiently close to unity, the program called for in Problem 72C will yield a meaningless result for β because of calculator overflow. In such cases it is useful to calculate $1 - \beta$, using an approximate formula appropriate to the extreme relativistic case. To study the transition from the relativistic to the extreme relativistic case, write a program for your hand-held calculator that will accept as an input the kinetic energy of an electron and will display as successive outputs: (1) the speed parameter β, calculated exactly; (2) the quantity $1 - \beta$, also calculated exactly; (3) the quantity $1 - \beta$, calculated using an approximate formula appropriate to the extreme relativistic case; and (4) the percent difference between these last two quantities. (*Hint:* Store the rest energy of the electron, which is 0.511003 MeV, in your calculator. For the extreme relativistic formula, use $(1 - \beta) = (1/2)(m_0c^2/K)^2$, in which K is the kinetic energy and m_0c^2 is the rest energy.)

75C. Checking it out (II). (*a*) Run the program that you have written in Problem 74 for electron kinetic energies extending from a few keV to several hundred GeV and get a feeling for the transition from the relativistic to the extremely relativistic case. (*b*) At what electron energy do the exact and the approximate formulas for $1 - \beta$ differ by 10 percent? By 1.0 percent? (*c*) At about what electron kinetic energy do the predictions of the exact formula for β break down totally because of calculator overflow?

REFERENCES

1. R. Kollath and D. Menzel, "Measurement of the Charge on Moving Electrons," *Z. Phys.* **134,** 530 (1953).
2. W. Bertozzi, "Speed and Kinetic Energy of Relativistic Electrons," *Am. J. Phys.*, **32,** 551 (1964).
3. R. R. Wilson, "The Batavia Accelerator," *Sci. Am.* (February 1974); R. R. Wilson, "The Tevatron," *Phys. Today* (October 1977).
4. R. T. Weidner, "On Weighing Photons," *Am. J. Phys.*, **35,** 443 (1967).
5. See *The Principle of Relativity* (Dover, New York, 1953), p. 29. This book is a collection of English translations of original papers by Einstein, Lorentz, and others.
6. Robert Resnick, *Introduction to Special Relativity* (Wiley, New York, 1968).
7. Edward M. Purcell, *Electricity and Magnetism* (McGraw-Hill, New York, 1965).
8. Mark P. Haugan and Clifford M. Will, "Modern Tests of Special Relativity," *Physics Today* (May 1987).

SUPPLEMENTARY TOPIC A

The Geometric Representation of Spacetime

Oh, that Einstein, always cutting lectures—I really would not have believed him capable of it.

Hermann Minkowski (ca. *1908*)

A-1 Spacetime Diagrams

We have seen that in classical physics it is proper to treat the space and time coordinates separately. In relativity, however, it is natural to treat them together, their intimate interconnection being clearly displayed in the Lorentz transformation equations; see Tables 2-2 and 2-3. The common use of the single word "spacetime" (without a hyphen) to represent the coordinate description of events is symbolic of the general acceptance of this view.

As we have learned, it was Einstein [1] who first set forth, in his special theory of relativity, the physical basis for the proper description of events in space and time. Shortly afterwards the mathematician Hermann Minkowski (who, incidentally, had formerly been Einstein's mathematics professor in Zurich) [2] presented a simple and symmetrical geometric representation of these ideas, a representation that permits a ready understanding in geometric terms of such matters as the relativity of simultaneity, the length contraction, and the time dilation, including their reciprocal nature.

In what follows, we shall consider only one space axis, the x axis, and shall ignore the y and z axes. We lose no generality by this algebraic simplification, and this procedure will enable us to focus more clearly on the interdependence of space and time and its geometric representation. The coordinates of an event are given, then, by x and t. All possible spacetime coordinates can be represented on a spacetime diagram in which the space axis is horizontal and the time axis is vertical. It is convenient to keep the dimensions of the coordinates the same; this is easily done by multiplying the time t by the universal constant c, the velocity of light. Let ct be represented by the symbol w. Then, the Lorentz transformation equations (see Table 2-2 and Problem 11 of Chapter 2) can be written as follows:

$$\begin{array}{llll} (a) & x' = \gamma(x - \beta w) & (a') & x = \gamma(x' + \beta w') \\ (b) & w' = \gamma(w - \beta x) & (b') & w = \gamma(w' + \beta x') \end{array} \qquad \text{(A-1)}$$

Notice the symmetry of this form of the equations.

To represent the situation geometrically, we begin by drawing the x and w axes of frame S at right angles to one another, as in Fig. A-1. If we want to represent a moving particle in this frame, we draw a curve, called the *world line* of the particle, which gives the loci of spacetime points corresponding to the motion. The tangent to the world line at any point makes an angle θ with the direction of the time axis that is given by $\tan\theta = dx/dw = (dx/dt)(1/c) = u/c$. Because we must have $u < c$ for a material particle, the angle θ at any point on its world line must always be less than 45°. If the particle is at rest, say, at position x_0 on the x axis of Fig. A-1, its world line is parallel to the w axis, with θ $(= \tan^{-1} u/c) = 0$ at all points. For a light ray traveling along the x axis we have $u = c$, so its world line is a straight line making an angle of 45° with the axes.

Consider now the primed frame (S'), which moves relative to S with a velocity $\mathbf{v}$ along the common x-x' axis. The equation of motion of the origin of S' relative to S can be obtained by setting $x' = 0$; from Eq. A-1a, we see that this corresponds to $x = \beta w$. We draw the line $x' = 0$ (that is, $x = \beta w$) on our diagram (Fig. A-2) and note that since $v < c$ and $\beta < 1$, the angle this line makes with the w axis, ϕ $(= \tan^{-1}\beta)$, is less than 45°. Just as the w axis corresponds to $x = 0$ and is the time axis in frame S, so the line $x' = 0$ gives the time axis w' in S'. Now, if we draw the line $w' = 0$ (giving the location of clocks that read $t' = 0$ in S'), we shall have the space axis x'. That is, just as the x axis corresponds to $w = 0$, so the x' axis corresponds to $w' = 0$. But, from Eq. A-1b, $w' = 0$ gives us $w = \beta x$ as the equation of this axis on our w-x diagram (Fig. A-2). The angle between the space axes is the same as that between the time axes. Note that, for simplicity, we have shown in Fig. A-2 only the quadrant in which both x and w are positive.

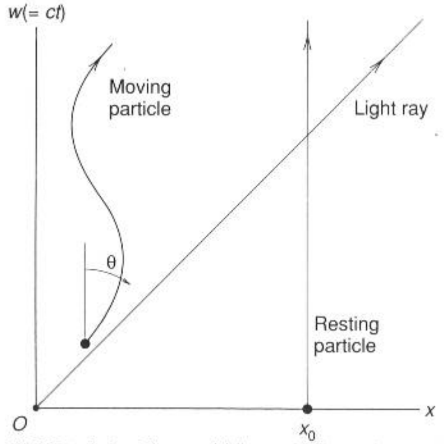

FIGURE A-1. The *world lines* of light and some particles.

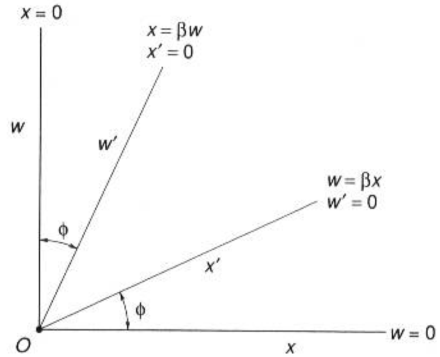

FIGURE A-2. The *Minkowski diagram* for frames S and S'.

You should compare Fig. A-2 carefully with the standard representation of Fig. 1-1, which we have used exclusively in the main body of the text. A point in the coordinate reference frames of Fig. 1-1 shows only the space coordinates of the event to which it corresponds; the time of occurrence of the event must be given separately. A point on the Minkowski diagram of Fig. A-2, however, shows both the space and the time coordinates of the event in a single geometric representation.

A-2 Calibrating the Spacetime Axes

Before we can make practical use of the spacetime diagram we must establish scales on its x, w and its x', w' axes. We can use the Lorentz transformation equations of Eq. A-1 for this purpose. Consider first point O, located at the common origin of the two pairs of axes in Fig. A-3. It has coordinates $x = w = 0$ and $x' = w' = 0$, and the event to which it corresponds is the coincidence in time of the origins of the S and S' reference frames.

Point P_1 on the x' axis of Fig. A-3 has been chosen as a point to which we wish to assign the value $x' = 1$, representing a unit of length on this axis. As for all points on the x' axis, the time coordinate w' of P_1 is zero. Putting $x' = 1$ and $w' = 0$ into Eq. A-1a' yields, by simple inspection, $x = \gamma$ for the x coordinate of P_1. With this information we can easily construct numerical scales for both the x and the x' axes, based on our initially assumed unit length.

Consider now point P_2 on the w' axis of Fig. A-3, to which we wish to assign the value $w' = 1$, representing a unit of time (measured in terms of ct', to be sure) on that axis. We wish the scales on both the x' and the w' axes to be based on the same unit length, so we choose to locate P_2 so that the line segment OP_2 is equal in length to the segment OP_1. As for all points on the w' axis, the space coordinate x' of P_2 is zero. Putting $w' = 1$ and $x' = 0$ into Eq. A-1b' yields, again by simple inspection, $w = \gamma$ for the w coordinate of P_2. We are now able to construct

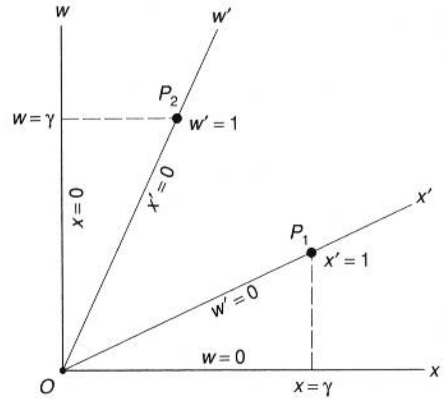

FIGURE A-3. Establishing the scales on the spacetime axes.

numerical scales for both the w and the w' axes, based on the same unit length as we assumed in calibrating the space axes.

To gain some physical familiarity with the Minkowski diagram, let us consider a clock at rest at the origin of the S' frame. For that clock we have $x' = 0$ (always), so events involving it must correspond to points along the w' axis of Fig. A-3. Point O, which is on that line, could represent the coincidence of the clock hand with a fiducial marker on the clock face, corresponding to zero time. Point P_2, whose time coordinate in the S' frame gives unit time ($w' = 1$) on that resting clock, is also on that line. The event represented by P_2 might correspond to a second coincidence of the clock hand with the fiducial marker. In frame S, however, the clock would be seen as a moving clock. We have seen above that $w' = 1$ in the S' frame corresponds to $w = \gamma$ in the S frame. Thus, by S-frame clocks, the unit time interval of the S' clock would be recorded as γ, corresponding exactly to the time dilation effect described by Eq. 2-14*b*.

In Fig. A-4 we show the calibration of the axes of the frames S and S', the unit time interval along w' being a longer line segment than the unit time interval along w and the unit length interval along x' being a longer line segment than the unit length interval along x. The first thing we must be able to do is to determine the spacetime coordinates of an event such as P directly from the Minkowski diagram. To find the space coordinate of the event, we simply draw a line parallel to the time axis from P to the space axis. The time coordinate is given similarly by a line parallel to the space axis from P to the time axis. The rules hold equally well for the primed frame as for the unprimed frame. In Fig. A-4, for example, the event P has the spacetime coordinates $x = 3.0$ and $w = 2.5$ in S (long dashed lines) and spacetime coordinates $x' = 2.0$ and $w' = 1.2$ in S' (short dashed lines). Figure A-4 was drawn assuming that $\beta = 0.50$, which yields $\gamma = 1.15$. Using these values for β and γ, you can readily derive the S-frame coordinates from the S'-frame

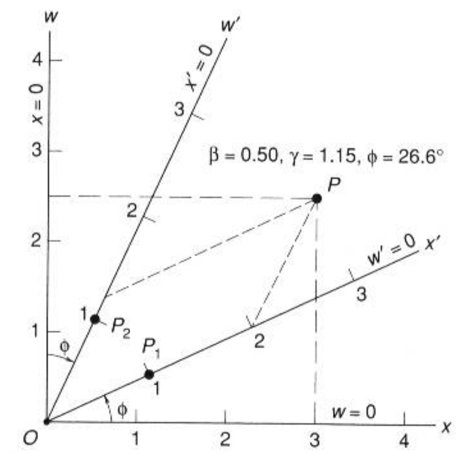

FIGURE A-4. Calibrating the axes of the frames S and S'.

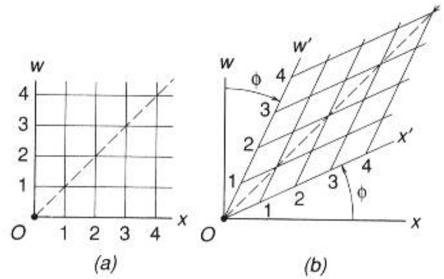

FIGURE A-5. An orthogonal reference frame, (*a*), transforms into a nonorthogonal one, (*b*).

coordinates—or conversely—by means of the Lorentz transformation equations (Eq. A-1), thus verifying the graphical relationships displayed in the Minkowski diagram.

In using the Minkowski diagram it is almost as if the rectangular grid of coordinate lines of S (Fig. A-5*a*) became squashed toward the 45° bisecting line when the coordinate lines of S' are put on the same graph (Fig. A-5*b*). In more formal language, we say that the Lorentz transformation equations transform an orthogonal (perpendicular) reference frame into a nonorthogonal one. Note that as $\beta \to 1$, corresponding to $v \to c$, the angle ϕ in Fig. A-5b ($= \tan^{-1} \beta$) approaches 45°, thus compressing the S'-frame coordinate space into a thinner and thinner wedge of the S-frame coordinate space. Alternatively, as $\beta \to 0$, corresponding to an approach to

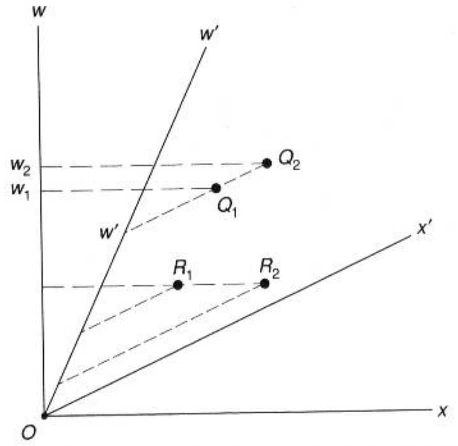

FIGURE A-6. Showing the relativity of simultaneity.

classical conditions, the angle ϕ between corresponding S and S' axes becomes very small. Even for a speed as high as that of a typical earth satellite ($\sim$17,000 mi/h), we note that $\beta = 2.5 \times 10^{-5}$, which yields a value of only 0.0015° for ϕ; relativistic mechanics is not much different from classical mechanics in these circumstances.

A-3 Simultaneity, Contraction, and Dilation

Now we can easily show the relativity of simultaneity. As measured in S', two events will be simultaneous if they have the same time coordinate w'. Hence, if the events lie on a line parallel to the x' axis, they are simultaneous to S'. In Fig. A-6, for example, events Q_1 and Q_2 are simultaneous in S'; they obviously are not simultaneous in S, occurring at different times w_1 and w_2 there. Similarly, two events R_1 and R_2, which are simultaneous in S, are separated in time in S'.

As for the space contraction, consider Fig. A-7a. Let a meter stick be at rest in the S frame, its end points being at $x = 3$ and $x = 4$, for example. As time goes on, the world line of each end point traces out a vertical line parallel to the w axis. The length of the stick is defined as the distance between the end points measured simultaneously. In S, the rest frame, the length is the distance in S between the intersections of the world lines with the x axis, or any line parallel to the x axis, for these intersecting points represent simultaneous events in S. The rest length is one meter. To get the length of the stick in S', where the stick moves, we must obtain the distance in S' between end points measured simultaneously. This will be the separation in S' of the intersections of the world lines with the x' axis, or any line parallel to the x' axis, for these intersecting points represent simultaneous events in S'. The length of the (moving) stick is clearly less than one meter in S' (see Fig. A-7a).

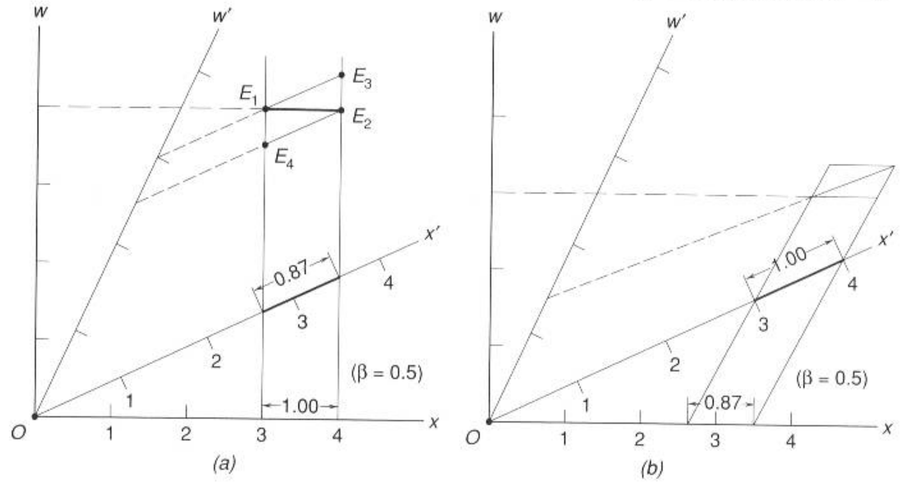

FIGURE A-7. Showing the space contraction, (a), and its reciprocal nature, (b).

Notice how very clearly Fig. A-7a reveals that it is a disagreement about the simultaneity of events that leads to different measured lengths. Indeed, the two observers do not measure the same pair of events in determining the length of a body (for example, the S observer uses E_1 and E_2, say, whereas the S' observer would use E_1 and E_3, or E_2 and E_4) for events that are simultaneous to one inertial observer are *not* simultaneous to the other. We should also note that the x' coordinate of each end point decreases as time goes on (simply project from successive world-line points parallel to w' onto the x' axis), consistent with the fact that the stick that is at rest in S moves towards the left in S' .

The reciprocal nature of this result is shown in Fig. A-7b. Here, we have a meter stick at rest in S', and the world lines of its end points are parallel to w' (the end points are always at $x' = 3$ and $x' = 4$, say). The rest length is one meter. In S, where the stick moves to the right, the measured length is the distance in S between intersections of these world lines with the x axis, or any line parallel to the x axis. The length of the (moving) stick is clearly less than one meter in S (Fig. A-7b).

It remains now to demonstrate the time-dilation result geometrically. For this purpose consider Fig. A-8. Let a clock be at rest in frame S, ticking off units of time there. The solid vertical line in Fig. A-8, at $x = 2.3$, is the world line corresponding to such a single clock. T_1 and T_2 are the events of ticking at w ($= ct$) $= 2$ and w ($= ct$) $= 3$, the time interval in S between ticks being unity. In S', this clock is moving to the left so that it is at a different place there each time it ticks. To measure the time interval between events T_1 and T_2 in S', we use two different clocks, one at the location of event T_1 and the other at the location of event T_2. The difference in reading of these clocks in S' is the difference in times between T_1 and T_2 as measured in S'. From the graph, we see that this interval is greater than unity.

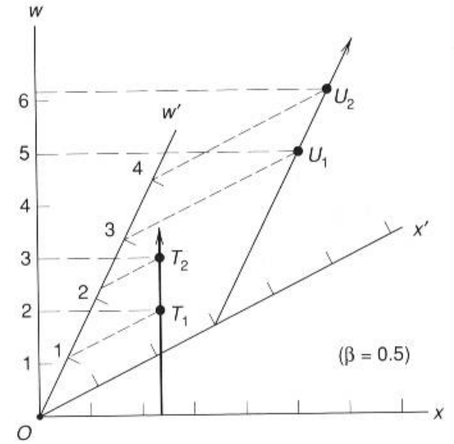

FIGURE A-8. Showing time dilation.

Hence, from the point of view of S', the moving S clock appears slowed down. During the interval that the S clock registered unit time, the S' clock registered a time greater than one unit.

The reciprocal nature of the time-dilation result is also shown in Fig. A-8. You should construct the detailed argument. Here a clock at rest in S' emits ticks U_1 and U_2 separated by unit proper time. As measured in S, the corresponding time interval exceeds one unit.

A-4 The Time Order and Space Separation of Events

We can also use the geometric representation of spacetime to gain further insight into the concepts of simultaneity and the time order of events that we discussed in Chapter 2. Consider the shaded area in Fig. A-9, for example. Through any point P in this shaded area, bounded by the world lines of light waves, we can draw a w' axis from the origin; that is, we can find an inertial frame S' in which the events O and P occur at the same place ($x' = 0$) and are separated only in time.* As shown in Fig. A-9, event P follows event O in time (it comes later on S' clocks), as is true wherever event P is in the upper half of the shaded area. Hence, events in the upper half (region 1 on Fig. A-10) are absolutely in the future relative to O, and this region is called the Absolute Future. If event P is at a spacetime point in the lower half of the shaded area (region 2 on Fig. A-10), then P will precede event O in time. Events in the lower half are absolutely in the past relative to O, and this region is called the Absolute Past. In the shaded regions, therefore, there is a definite time order of

*We cannot draw an x' axis through points such as P in Fig. A-9 because the angle ϕ in Fig. A-2 would then exceed 45°, which requires that $\beta > 1$ (or, equivalently, that $v > c$). For the same reason, we cannot draw a w' axis through points such as Q in Fig. A-9.

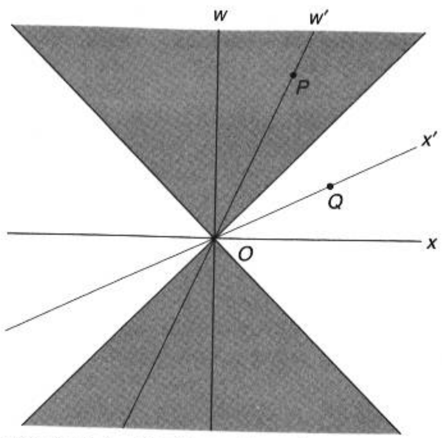

FIGURE A-9. The time order and space separation of events.

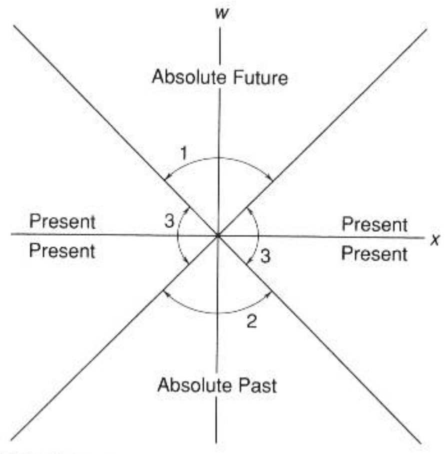

FIGURE A-10. Location in time of events relative to the origin.

events relative to O, for we can always find a frame in which O and P occur at the same place; a single clock will determine absolutely the time order of the event at this place.

Consider now the unshaded regions of Fig. A-9. Through any point Q we can draw an x' axis from the origin; that is, we can find an inertial frame S' in which the events O and Q occur at the same time ($w' = ct' = 0$) and are separated only in space. We can always find an inertial frame in which events O and Q appear to be simultaneous for spacetime points Q that are in the unshaded regions (region 3 of Fig. A-10), so that this region is called the Present. In other inertial frames, of course, O and Q are not simultaneous, and there is no absolute time order of these events but a relative time order, instead.

If we ask about the space separation of events, rather than their time order, we see that events in the Present are absolutely separated from O, whereas those in the Absolute Future or Absolute Past have no definite space order relative to O. Indeed, region 3 (Present) is said to be "spacelike" whereas regions 1 and 2 (Absolute Past or Future) are said to be "timelike." That is, a world interval such as OQ is spacelike and a world interval such as OP is timelike.

The geometric considerations that we have presented are connected with the invariant nature of the spacetime *interval*, described in Section 2.3. As presented there, the interval involves a pair of events. For our purposes we can choose as one universal member of this pair the standard reference event represented by point O in Fig. A-9. It corresponds to the coincidence in time of the origins of the two reference frames, S and S', and has the spacetime coordinates $x = w = 0$ and $x' = w' = 0$. The other member of the event pair can then be a generalized event represented by points such as P or Q in Fig. A-9. In this way we can associate the spacetime interval with P and Q alone, and can write (from Eq. 2-16, recalling that $w = ct$),

$$s^2 = w^2 - x^2 = w'^2 - x'^2. \tag{A-2}$$

We have seen that s^2, which has the same numerical value in all reference frames, can be either positive, negative, or zero, depending on the relative magnitudes of w and x (or of w' and x'). If $w > x$, as it is for points such as P in Fig. A-9, then s^2 is positive and s is a real quantity; we write it as $c\tau$, where τ is the *proper time interval* associated with the event pairs such as OP; see Eq. 2-17. If $w < x$, as it is for points such as Q in Fig. A-9, then $-s^2$ is a positive quantity; we call its square root σ, the *proper distance interval* for the event pairs such as OQ. We have then two relations,

$$c^2\tau^2 = w^2 - x^2 \tag{A-3a}$$

and

$$\sigma^2 = x^2 - w^2. \tag{A-3b}$$

Now consider Fig. A-10. In regions 1 and 2 we have spacetime points for which $w > x$, so the proper time is a real quantity, $c^2\tau^2$ being positive; see Eq. A-3*a*. In regions 3 we have spacetime points for which $x > w$, so the proper distance σ is a real quantity; see Eq. A-3*b*. Hence either τ or σ is real for any two events (that is, the event at the origin and the event elsewhere in spacetime) and either τ or σ may be called the spacetime interval between the two events. When τ is real the interval is called "timelike"; when σ is real the interval is called "spacelike." Because σ and τ are invariant properties of two events, it does not depend at all on what inertial frame is used to specify the events whether the interval between them is spacelike or timelike.

In the spacelike region we can always find a frame S' in which the two events are simultaneous, so that σ can be thought of as the spatial interval between the events in that frame. (That is, $\sigma^2 = x^2 - w^2 = x'^2 - w'^2$. But $w' = 0$ in S', so $\sigma = x'$.) In the timelike region we can always find a frame S' in which the two events occur at the same place, so that τ can be thought of as the time interval between the events in that frame. [That is, $\tau^2 = t^2 - (x^2/c^2) = t'^2 - (x'^2/c^2)$. But $x' = 0$ in S', so $\tau = t'$.]

What can we say about points on the 45° lines? For such points, $x = w$. Therefore, the proper time interval between two events on these lines vanishes, for $c^2\tau^2 = w^2 - x^2 = 0$ if $x = w$. We have seen that such lines represent the world lines of light rays and give the limiting velocity ($v = c$) of relativity. On one side of these 45° lines (shaded regions in Fig. A-9), the proper time interval is real; on the other side (unshaded regions), it is imaginary. An imaginary value of τ would correspond to a velocity in excess of c. But no signals can travel faster than c. All this is relevant to an interesting question that can be posed about the unshaded regions.

In this region, which we have called the Present, there is no absolute time order of events; event O may precede event Q in one frame but follow event Q in another frame. What does this do to our deep-seated notions of cause and effect? Does relativity theory negate the causality principle? To test cause and effect, we would have to examine the events at the same place so that we could say absolutely that Q followed O, or that O followed Q, in each instance. But in the Present, or spacelike, region these two events occur in such rapid succession that the time difference is less than the time needed by a light ray to traverse the spatial distance between two events. We cannot fix the time order of such events absolutely, for no signal can travel from one event to the other faster than c. In other words, no frame of reference exists with respect to which the two events occur at the same place; thus, we simply cannot test causality for such events even in principle. Therefore, there is no violation of the law of causality implied by the relative time order of O and events in the spacelike region. We can arrive at this same result by an argument other than this operational one. If the two events, O and Q, are related causally, then they must be capable of interacting physically. But no physical signal can travel faster than c, so events O and Q cannot interact physically. Hence, their time order is immaterial, for they cannot be related causally. Events that can interact physically with O are in regions other than the Present. For such events, O and P, relativity gives an unambiguous time order. Therefore, relativity is completely consistent with the causality principle.

QUESTIONS AND PROBLEMS

1. **Interpreting events on a spacetime diagram (I).** Draw a spacetime diagram and on it locate an event P whose coordinates are $x = 450$ m and $t = 1.00$ μs ($w = ct = 300$ m). With respect to the standard reference event O at the origin, (*a*) does P represent an event in the future? The present? The past? (*b*) Is the interval OP spacelike? Timelike?

$$\Delta\tau = \int_P^Q d\tau = \int_P^Q \sqrt{(dt)^2 - (\frac{dx}{c})^2}. \qquad \text{(B-2)}$$

Both *dt* and *dx* in the above are differential spacetime path elements as measured by the observer in the inertial reference frame of Fig. B-1*b*. We are not surprised that the two paths shown in this figure differ as far as *x* is concerned (odometer readings), and we have learned not to be surprised that clock readings vary in much the same way. Simple inspection of Eq. B-2 shows that the quantity depends not only on the initial and final points but also on the path taken between them.

In Fig. B-1*c* we let one of these paths be a straight line, corresponding to the simple passage of time for a stationary particle; the other path remains arbitrary. From Eq. B-2 we have, for the straight path,

$$\Delta\tau_s = \int_P^Q \sqrt{(dt)^2 - \left(\frac{dx}{c}\right)^2} = \int_P^Q dt = t_Q - t_P,$$

in which the subscript on $\Delta\tau$ refers to the stationary clock. In such a case *dx* is zero along the path, and the proper time coincides with the time interval, $t_Q - t_P$, recorded by the stationary clocks of the inertial reference frame. Along the second world line, however, the elapsed power time is

$$\Delta\tau_t = \int_P^Q \sqrt{(dt)^2 - \left(\frac{dx}{c}\right)^2},$$

in which the subscript refers to the traveling clock; we see that $\Delta\tau_t$ will *not* equal $\Delta\tau_s$. In fact, since $(dx)^2$ is always positive, we find that

$$\Delta\tau_t < \Delta\tau_s. \qquad \text{(B-3)}$$

The clocks will read different times when brought back together, the traveling clock running behind (recording a smaller time difference than) the stay-at-home clock. Figure B-1*d* is a special case of Fig. B-1*c* in that the traveling clock moves with constant velocity over most of its path, its motion being accelerated only near its "turnaround point." Note that, although the turnaround may occupy only a small fraction of the total travel time, it is vitally necessary to the motion if the two clocks are to reconvene.

We have noted that the reference frame whose axes are drawn in Fig. B-1 is an inertial frame. The motion of the traveling clock is represented in this frame by a curved world line, for this clock undergoes accelerated motion rather than motion with uniform velocity. It could not return to the stationary clock, for example, without reversing its velocity. The special theory of relativity can predict the behavior of accelerated objects as long as, in the formulation of the physical laws, we take the view of the inertial (unaccelerated) observer. This is what we have done so far. A frame attached to the clock traveling along its round-trip path would not be an inertial frame. We could reformulate the laws of physics so that they have the same form for accelerated (noninertial) observers—this is the program of general relativity theory—but it is unnecessary to do so to explain the twin paradox. All we wish to point out here is that the situation is *not* symmetrical with respect to the clocks (or twins); one is always in a single inertial frame and the other is not.

B-2 Spacetime Diagram of the Twin Paradox

In our earlier discussions of time dilation, we spoke of "moving clocks running slow." What is meant by that phrase is that a clock moving at a constant velocity **u** relative to an inertial frame containing synchronized clocks will be found to run slow by the factor $\sqrt{1 - u^2/c^2}$ *when timed by those clocks*. That is, to time a clock moving at constant velocity relative to an inertial frame, we need at least *two* synchronized clocks in that frame. We found this result to be reciprocal in that a single S' clock is timed as running slow by the many S clocks, and a single S clock is timed as running slow by the many S' clocks.

The situation in the twin paradox is different. If the traveling twin traveled always at a constant speed in a straight line, he would never get back home. And each twin would indeed claim that the other's clock runs slow compared to the synchronized clocks in his own frame. To get back home—that is, to make a round trip—the traveling twin would have to change his velocity. What we wish to compare in the case of the twin paradox is a single moving clock with a *single* clock at rest. To do this we must bring the clocks into coincidence twice—they must come back together again. It is not the idea that we regard one clock as moving and the other at rest that leads to the different clock readings, for if each of two observers seems to the other to be moving at constant speed in a straight line, they cannot absolutely assert who is moving and who is not. Instead, it is because one clock has *changed* its velocity and the other has not that makes the situation unsymmetrical.

Now you may ask how the twins can tell who has changed his velocity. This is clearcut. Each twin can carry an accelerometer. If he changes his speed or the direction of his motion, the acceleration will be detected. We may not be aware of an airplane's motion, or a train's motion, if it is one of uniform velocity; but let it move in a curve, rise and fall, speed up or slow down, and we are our own accelerometer as we get thrown around. Our twin on the ground watching us does not experience these feelings—his accelerometer registers nothing. Hence, we can tell the twins apart by the fact that the one who makes the round-trip experiences and records accelerations whereas the stay-at-home does not.

A numerical example, suggested by C. G. Darwin [2], is helpful in fixing the ideas. We imagine that, on New Year's Day, Bob leaves his twin brother Dave, who is at rest on a spaceship, fires rockets that get him moving at a speed of $0.8c$ relative to Dave, and by his own clock travels away at this constant speed toward a distant star, which he reaches after three years of travel. He then fires more powerful rockets that exactly reverse his motion and gets back to Dave after another three years by his clock. By firing rockets a third time, he comes to rest beside Dave and compares clock readings. Bob's clock says he has been away for six years (the $\Delta\tau_t$ of Eq. B-3), but Dave's clock says that ten years have elapsed (the $\Delta\tau_s$ of Eq. B-3). Let us see how this comes about.

First, we can simplify matters by ignoring the effect of the accelerations on the traveling clock. Bob can turn off his clock during the three acceleration periods, for example. The error thereby introduced can be made very small compared to the total time of the trip, for we can make the trip as far and as long as we wish without

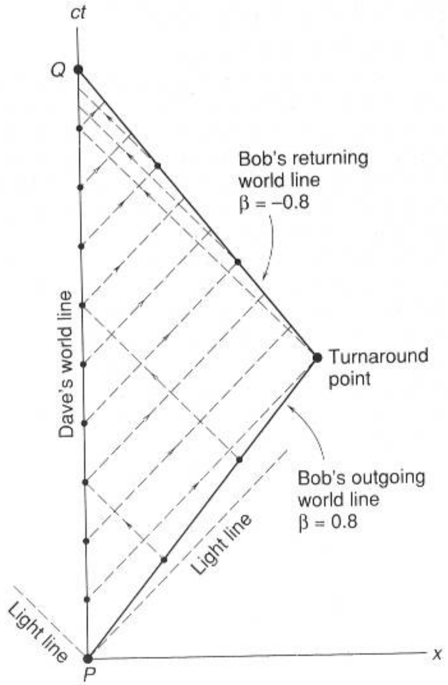

FIGURE B-2. Spacetime diagram of the twin paradox.

changing the acceleration intervals. It is the total time that is at issue here in any case.* We do not destroy the asymmetry, for even in the ideal simplification of Fig. B-2 (where the world lines are straight lines rather than curved ones), Dave is always in one inertial frame whereas Bob is definitely in two different inertial frames—one going out ($0.8c$) and another coming in ($-0.8c$).

Let the spaceships be equipped with identical clocks that send out light signals at one-year intervals. Dave receives the signals arriving from Bob's clock and records them against the annual signals of his own clock; likewise, Bob receives the signals from Dave's clock and records them against the annual signals of his clock.

In Fig. B-2, Dave's world line is straight along the ct-axis; he is at $x = 0$ and we mark off ten years (in terms of ct), a dot corresponding to the annual New Year's Day signal of his clock. Bob's world line at first is a straight line inclined to the ct axis, corresponding to a ct' axis of a frame moving at $+0.8c$ relative to Dave's frame. We mark off three years (in terms of ct'), a dot corresponding to the annual New Year's Day signal of his clock. After three of Bob's years, he switches to another inertial frame whose world line is a straight line inclined to the ct axis, corresponding to the ct'' axis of a frame moving at $-0.8c$ relative to Dave's frame. We mark off three years (in terms of ct''), a dot corresponding to the annual New Year's Day signal of his clock. Note the dilation of the time interval of Bob's clock compared to Dave's.

*An analogy is that the total distance traveled by two drivers between the same two points, one along the hypotenuse of a right triangle and the other along the other two sides of the triangle, can be quite different. One driver always moves along a straight line, whereas the other makes a right turn to travel along two straight lines. We can make the distance between the two points as long as we wish without altering the fact that only one turn must be made. The difference in mileage traveled by the drivers certainly is not acquired at the turn that one of them makes.

Now let us draw the light signals from Bob's clock on the spacetime diagram of Fig. B-2. Recall (see Fig. A-1) that such signals are drawn at 45° to the spacetime axes, corresponding to their speed of c. Thus from each dot on Bob's world line we draw such a 45°-line headed back to Dave on the line at $x = 0$. There are six signals, the last one emitted when Bob returns home to Dave. Likewise, the signals from Dave's clock are straight lines, from each dot on Dave's world line, inclined 45° to the axes and headed out to Bob's spaceship. We see that there are ten signals, the last one emitted when Bob returns home to Dave.

How can we confirm this spacetime diagram numerically? Simply by the Doppler effect. As the clocks recede from each other, the frequency ν of their signals is reduced from the proper frequency ν_0 by the Doppler effect. From Eq. 2-30*b* we thus have

$$\frac{\nu}{\nu_0} = \sqrt{\frac{1-\beta}{1+\beta}} = \sqrt{\frac{1-0.8}{1+0.8}} = \frac{1}{3}.$$

Hence, Bob receives the first signal from Dave after three of his years, just as he is turning back. Similarly, Dave receives messages from Bob on the way out once every three of his years, receiving three signals in nine years. As the clocks approach one another, the frequency ν of their signals is increased from the proper frequency ν_0 by the Doppler effect. In this case (see Eq. 2-30*a*) we have

$$\frac{\nu}{\nu_0} = \sqrt{\frac{1+\beta}{1-\beta}} = \sqrt{\frac{1+0.8}{1-0.8}} = 3.$$

Thus, Bob receives nine signals from Dave in his three-year return journey. Altogether, Bob receives ten signals from Dave. Similarly, Dave receives three signals from Bob in the last year before Bob is home. Altogether, Dave receives six signals from Bob.

Figure B-3 shows the signal logs for Dave's and Bob's spaceships. Signals sent are indicated below the time axis in each case and signals received are shown above that axis. There is no disagreement about the signals: Bob sends six and Dave receives six; Dave sends ten and Bob receives ten. Everything works out, each seeing the correct Doppler shift of the other's clock and each agreeing to the number of signals that the other sent. The different total times recorded by the twins corresponds to the fact that Dave sees Bob recede for nine years and return in one year, although Bob both receded for three of his years and returned for three of his years. Dave's records will show that he received signals at a slow rate for nine years and at a rapid rate for one year. Bob's records will show that he received signals at

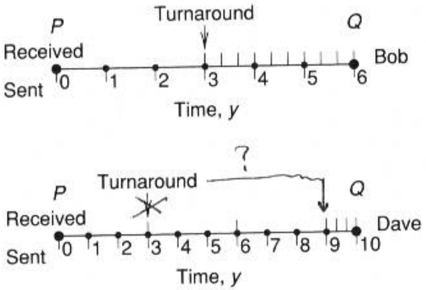

FIGURE B-3. The signal logs for the twins.

a slow rate for three years and at a rapid rate for another three years. The essential asymmetry is thereby revealed by a Doppler effect analysis. When Bob and Dave compare records, they will agree that Dave's clock recorded ten years and Bob's recorded only six. Ten years have passed for Dave during Bob's six-year round trip.

B-3 Some Other Considerations

Will Bob really be four years younger than his twin brother? Since for the word "clock" we could have substituted any periodic natural phenomena, such as heartbeat or pulse rate, the answer is yes. We might say that Bob lived at a slower rate than Dave during his trip, his bodily functions proceeding at the same slower rate as his physical clock. Biological clocks behave in this respect the same as physical clocks. There is no evidence that there is any difference in the physics of organic processes and the physics of the inorganic materials involved in these processes. If motion affects the rate of a physical clock, we expect it to effect a biological clock in the same way.

It is of interest to note the public acceptance of the idea that human life processes can be slowed down by refrigeration, so that a corresponding different aging of twins can be achieved by temperature differences. What is paradoxical about the relativistic case, in which the different aging is due to the difference in motion, is that since (uniform) motion is relative, the situation appears (incorrectly) to be symmetrical. But, just as the temperature differences are real, measurable, and agreed upon by the twins in the foregoing example, so are the differences in motion real, measurable, and agreed upon in the relativistic case—the changing of inertial frames, that is, the accelerations, are not symmetrical. The results are absolutely agreed upon.

Although there is no need to invoke general relativity theory in explaining the twin paradox, the student may wonder what the outcome of the analysis would be if we knew how to deal with accelerated reference frames. We could then use Bob's spaceship as our reference frame, so that Bob is the stay-at-home, and it would be Dave who, in this frame, makes the round-trip space journey. We would find that we must have a gravitational field in this frame to account for the accelerations that Bob

feels and the fact that Dave feels no accelerations even though he makes a round trip. If, as required in general relativity, we then compute the frequency shifts of light in this gravitational field, we come to the same conclusion as in special relativity [3].

B-4 Experimental Tests

Testing the conclusions we have reached with actual twins and clocks in spaceships moving with speeds close to the speed of light is, of course, more than can be managed at present. However, totally equivalent high-speed tests can be carried out using as clocks unstable elementary particles such as muons or pions or one of the hundreds of varieties of radioactive atoms available to us. At lower speeds—those of jet planes, for example—tests can be carried out with macroscopic atomic clocks, thanks to impressive improvements in the stability and time-keeping ability of such clocks.

In Section 2-7 we described the precise measurements of Kundig [Ref. 9, Chapter 2] on the transverse Doppler effect. This effect, as we noted in that section, is a direct measure of the time dilation, and we can use it to illustrate the twin paradox. From the point of view of the observer on the rotor axis (the stay-at-home twin), the absorbing foil on the perimeter of the rotor (the traveling twin) has a characteristic resonant absorption frequency that matches the source frequency only when the rotor is *not* turning. When the rotor *is* turning, the resonant frequency of the moving foil drops, just as predicted by Eqs. 2-32. Put another way, the round-trip twin ages less than his stay-at-home brother and (to within 1 percent) by exactly the amount predicted by relativity theory.

In 1968 a careful measurement of time dilation was reported from CERN (the European Nuclear Research Center, located near Geneva) in which laboratory-generated 1.18-GeV muons, for which the corresponding speed is $0.9966c$, served as high-speed traveling clocks [4]. These muons were constrained to circulate in an orbit 5.0 m in diameter in the muon storage ring in that laboratory. Thus, like the traveling twin (and also like the resonant absorbing foil in the experiment described above), they traverse a closed path and undergo (centripetal) acceleration during their journey. Their mean life for decay in flight can then be compared with the mean life observed when muons are brought to rest in an absorbing block. Many experiments give the accepted value of 2.200 ± 0.0015 μs for the decay of resting muons (the stay-at-home twin); the CERN experimenters measured 26.15 ± 0.03 μs for the mean decay time for the traveling muons (the traveling twin). This agrees within about 2 percent with the lifetime predicted for these traveling muons because of the time dilation, namely, 26.72 μs. The time dilation phenomenon is universally accepted and is, in fact, turned to specific advantage in the design of certain high-energy particle experiments. As one high-energy physicist has written [5]: "We frequently transport beams of unstable particles over long distances such that no particles would be left without the help of Einstein's factor."

Refinements of atomic clocks have so improved the accuracy of timekeeping that time-dilation effects can be detected at speeds as low as those of jet planes. In

TABLE B-1
Round-the-World Atomic Clocks [6] (The numbers shown are time differences, in nanoseconds, with respect to reference clocks at the U.S. Naval Observatory.)

	Eastward	Westward
Predicted:		
Special relativity	−184 ± 18	96 ± 10
General relativity	144 ± 14	179 ± 18
Net predicted:	− 40 ± 23	275 ± 21
Observed:	− 59 ± 10	273 ± 7

October 1977, Joseph Hafele and Richard Keating [6] carried four cesium-beam atomic clocks around the world on commercial airline flights, " . . . to test Einstein's theory of relativity with macroscopic clocks." They took their clocks around once each way, that is, once eastward and once westward, comparing the traveling clocks to those that stayed at home at the Naval Observatory on the Earth, rotating (eastward) below them. The calculations must take into account not only the kinematic time-dilation effect (which is related only to the speed of the traveling clocks), but also relativistic frequency shifts associated with changes encountered in the strength of the earth's gravitational field (see Supplementary Topic C, Section C-2). Table B-1 shows the predictions of relativity theory along with the experimental findings. Hafele and Keating conclude: "There seems to be little basis for further arguments about whether clocks will indicate the same time after a round trip, for we find that they do not."

Today, when precision clocks move from one location to another, cumulative time corrections with respect to a stay-at-home clock are made routinely [7]. Such considerations enter, for example, when precision clocks are moved for comparison purposes between Washington, D.C., and the National Bureau of Standards Laboratory at Boulder, Colorado. Similarly, relativistic time-dilation effects must be considered in the design and operation of the Global Positioning System (GPS/NAVSTAR), a precision navigation system in which it is planned to employ 24 orbiting satellites.

QUESTIONS AND PROBLEMS

1. **The shortest distance between two points is a straight line (?).** Comparison of Fig. B-1*c* and Eq. B-3 shows us that, in terms of elapsed proper time in units of ct on a spacetime diagram, a straight line is not the shortest distance between two points but the *longest*. Is this statement still true if one of the two particles involved is not stationary (as in Fig. B-1*c*) but moves with constant speed? Draw a spacetime diagram to represent this situation.

2. **Einstein on the clock "paradox."** Einstein, in his first paper on the special theory of relativity, wrote the following: "If one of two synchronous clocks at A is moved in a closed curve with constant velocity until it returns to A, the journey lasting t seconds, then by the clock that has remained at rest the travelled clock on its arrival at A will be $tv^2/2c^2$ seconds slow." Prove this statement. (*Note:* Elsewhere in his paper Einstein indicated that this result is an approximation, valid only for $v << c$.)

3. **Do you really want to do it?** You wish to make a round trip from earth in a spaceship, traveling at constant speed in a straight line for six months and then returning at the same constant speed. You wish further, on your return, to find the earth as it will be a thousand years in the future. (*a*) How fast must you travel? (*b*) Does it matter whether or not you travel in a straight line on your journey? If, for example, you traveled in a circle for one year, would you still find that a thousand years had elapsed by earth clocks when you returned?

4. **Synchronizing clocks.** Consider two clocks fixed along the x axis of an inertial reference frame, one at $x = x_1$ and the other at $x = x_2$. In Section 2-1 we saw how to synchronize such clocks, using light signals. Here is another proposed method that, at first glance, may seem quite reasonable: Let a traveler move out along the x axis with constant speed v, wearing a wristwatch. Let the traveler then set each of the two x axis clocks to agree with the wristwatch as she passes them. What is wrong with this method of synchronization?

5. **Bob and Dave.** (*a*) In the spacetime diagram of Fig. B-2, how far apart are Bob and Dave when Bob turns around? (*b*) Suppose that Dave did not know beforehand when Bob was planning to turn around. When (by his own clocks and calendars) would Dave find out that Bob had done so? (*c*) If Bob's clock runs slow on the outbound trip (as it does), then why does it not run fast on the inbound trip, for which his velocity is reversed in sign?

6. **Bob changes his mind.** Suppose that Bob, after noting the passage of three years by his on-board clock, decides not to return to Dave but simply stops. He compares his on-board clock with one of the local clocks belonging to the synchronized array of stationary clocks fixed in Dave's inertial frame. (*a*) What will this local clock read? (*b*) Draw Bob's world line for this new situation on a spacetime diagram.

7. **Bob is older than Dave this time.** Bob, once started on his outward journey from Dave, keeps on going at his original uniform speed of $0.8c$. Dave, knowing that Bob was planning to do this, decides, after waiting for three years, to catch up with Bob and to do so in three additional years. (*a*) To what speed must Dave accelerate to do so? (*b*) What will be the elapsed time by Bob's clock when they meet? (*c*) How far will they each have traveled when they meet, measured in Dave's original inertial reference frame? (*d*) Draw the world lines for Bob and Dave on a spacetime diagram and compare it with Fig. B-2. Notice that the present scenario is the mirror image of the one discussed in connection with that figure; there Dave turned out to be four years older than Bob when they reconvened; here Bob will turn out to be four years older than Dave [8].

8. **Bob and Dave are twins again.** Suppose that Bob and Dave each agree to follow the scenario described by Fig. B-2 for three years, each counting the years by his own onboard clock. Then Bob will come to rest and Dave will accelerate to $0.8c$ and eventually catch up with Bob. (*a*) What will be the total elapsed times on each of their clocks when they meet? (*b*) Draw a spacetime diagram and compare it carefully with that of Fig. B-2. Note the total symmetry of the situation. In the scenario as given originally, Dave turned out to be four years older than Bob when they met; in Problem 7, the reverse turned out to be true; in this case they turn out to be the same age at the end of their journeys [8].

9. **The twins talk it over.** Explain (in terms of heartbeats, physical and mental activities,