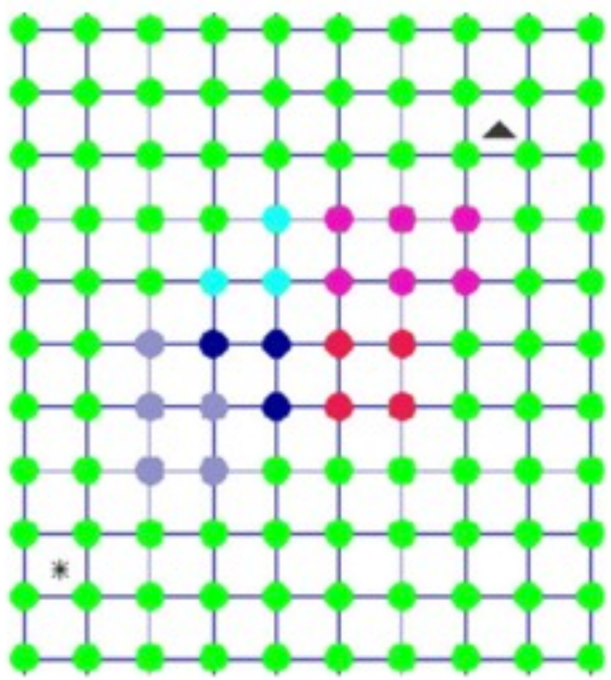
Propagation in laterally heterogeneous media: the modal summation approach



Basic Theory - 1D

- Seismic source
  - Green function, G_{ik}
  - Representation theorem
  - Shear dislocation $$u_i(\mathbf{x},t) = \iint_\Sigma [u_n]\nu_j c_{njkq} * \frac{\partial G_{ik}}{\partial \xi_q} d\Sigma$$
  - Equivalent body force $$u_i(\mathbf{x},t) = \iint_\Sigma m_{kq} * G_{ik,q} d\Sigma$$
  - Point source $$u_i(\mathbf{x},t) = M_{kq} * G_{ik,q}$$
  - Double couple $$M_{kq} = \mu A\left(\nu_k[u_q] + \nu_q[u_k]\right)$$

Basic Theory - 1D

- 1D problem
  - Vertically heterogeneous halfspace

$$\rho\ddot{\mathbf{u}} = (\lambda+\mu)\nabla(\nabla\cdot\mathbf{u}) + \mu\nabla^2\mathbf{u} +$$

$$+\frac{\partial\lambda}{\partial z}(\hat{\mathbf{z}}\nabla\cdot\mathbf{u}) + \frac{\partial\mu}{\partial z}[\hat{\mathbf{z}}\times(\nabla\times\mathbf{u}) + \nabla(\hat{\mathbf{z}}\cdot\mathbf{u})]$$

  - Solution

$$\mathbf{u}(\mathbf{x},t) = \mathbf{F}(z)e^{i(\omega t - kx)}$$

  - SH problem

$$\frac{\partial}{\partial z}\left(\mu\frac{\partial F_y}{\partial z}\right) + \left(\omega^2\rho - k^2\mu\right)F_y = 0$$

$$\left.\left(\mu\frac{\partial F_y}{\partial z}\right)\right|_{z=0} = 0$$

$$\lim_{z\to\infty} F_y = 0$$

  - P-SV problem...

Basic Theory - 1D

- Oscillation modes

$$G_{ik}^{L} = \sum_{m=1}^{\infty} G_{ik}^{mL}(\mathbf{x}, \mathbf{x}_0; t)$$

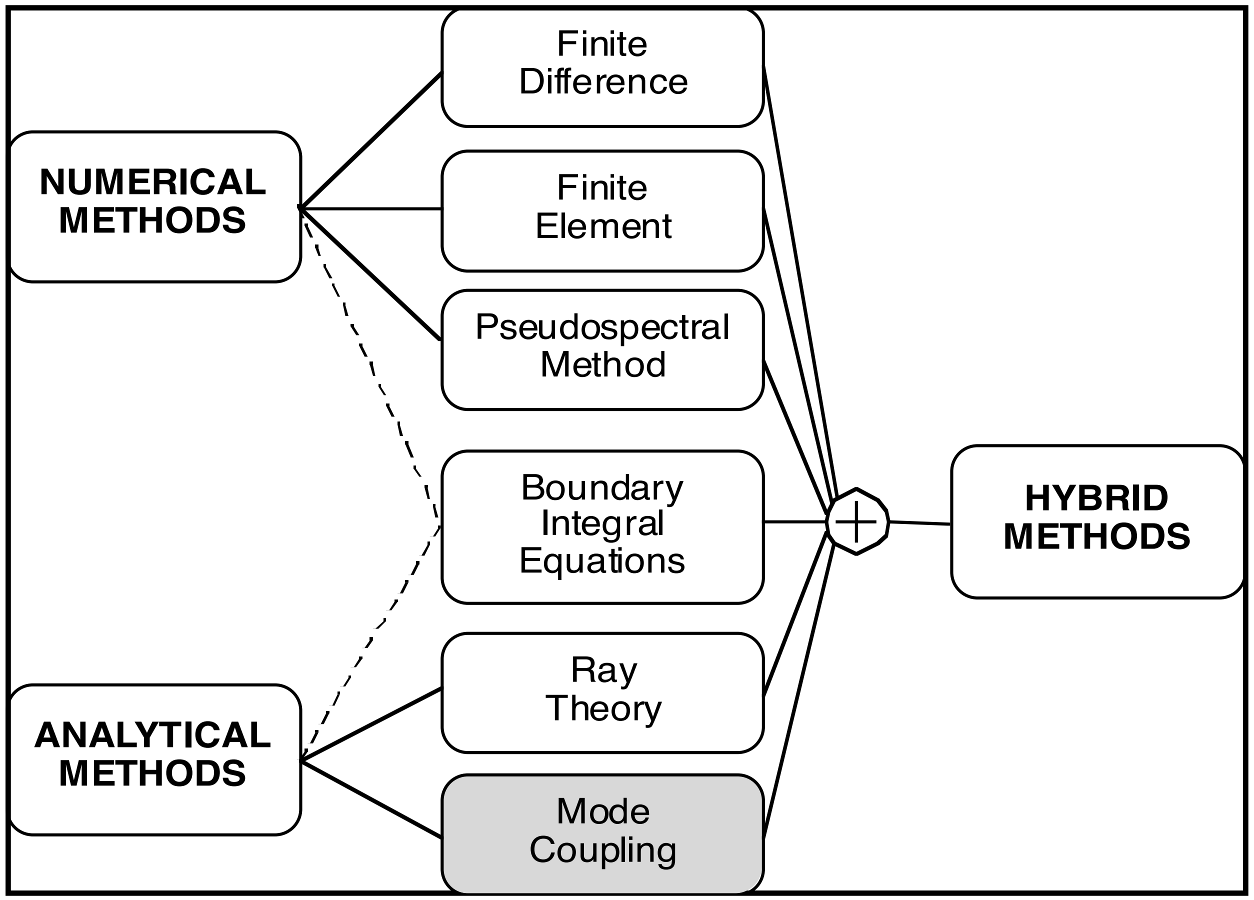
$$G_{ik}^{mL}(\omega) = \frac{e^{-i\pi/4}}{\sqrt{8\pi\omega}} \frac{e^{-ik_m x}}{\sqrt{x}} \frac{F_y^{(k)m}(h_s, \omega)}{\sqrt{c_m v_m I_m}} \frac{F_y^{(i)m}(z, \omega)}{\sqrt{v_m I_m}}$$

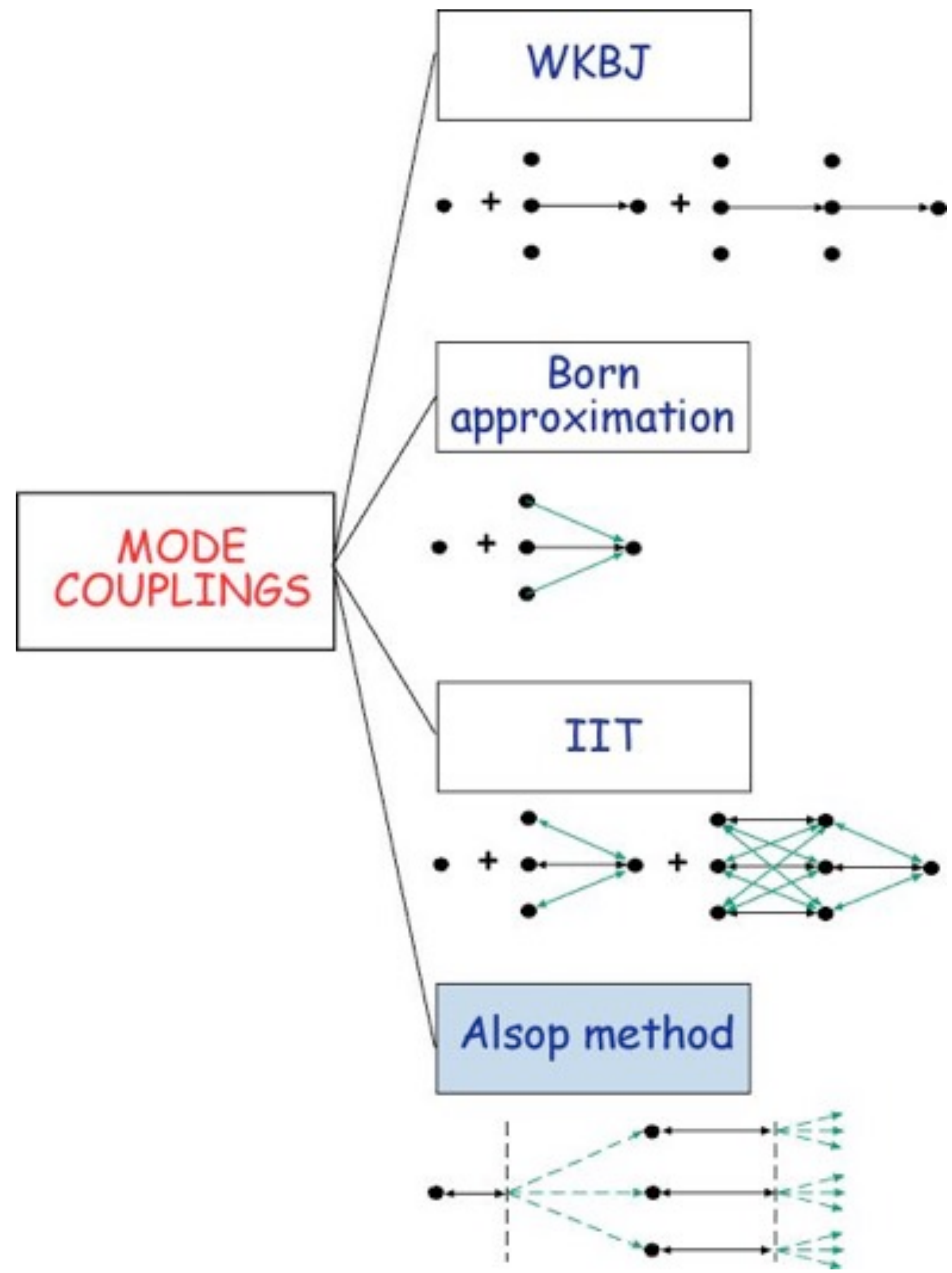
- Synthetic seismograms

$$u_i^{mL}(\omega) = \frac{e^{-i\pi/4}}{\sqrt{8\pi\omega}} \frac{e^{-ik_m x}}{\sqrt{x}} \frac{\chi_L^m(h_s, \varphi, \omega)}{\sqrt{c_m v_m I_m}} \frac{F_y^{(i)m}(z, \omega)}{\sqrt{v_m I_m}}$$

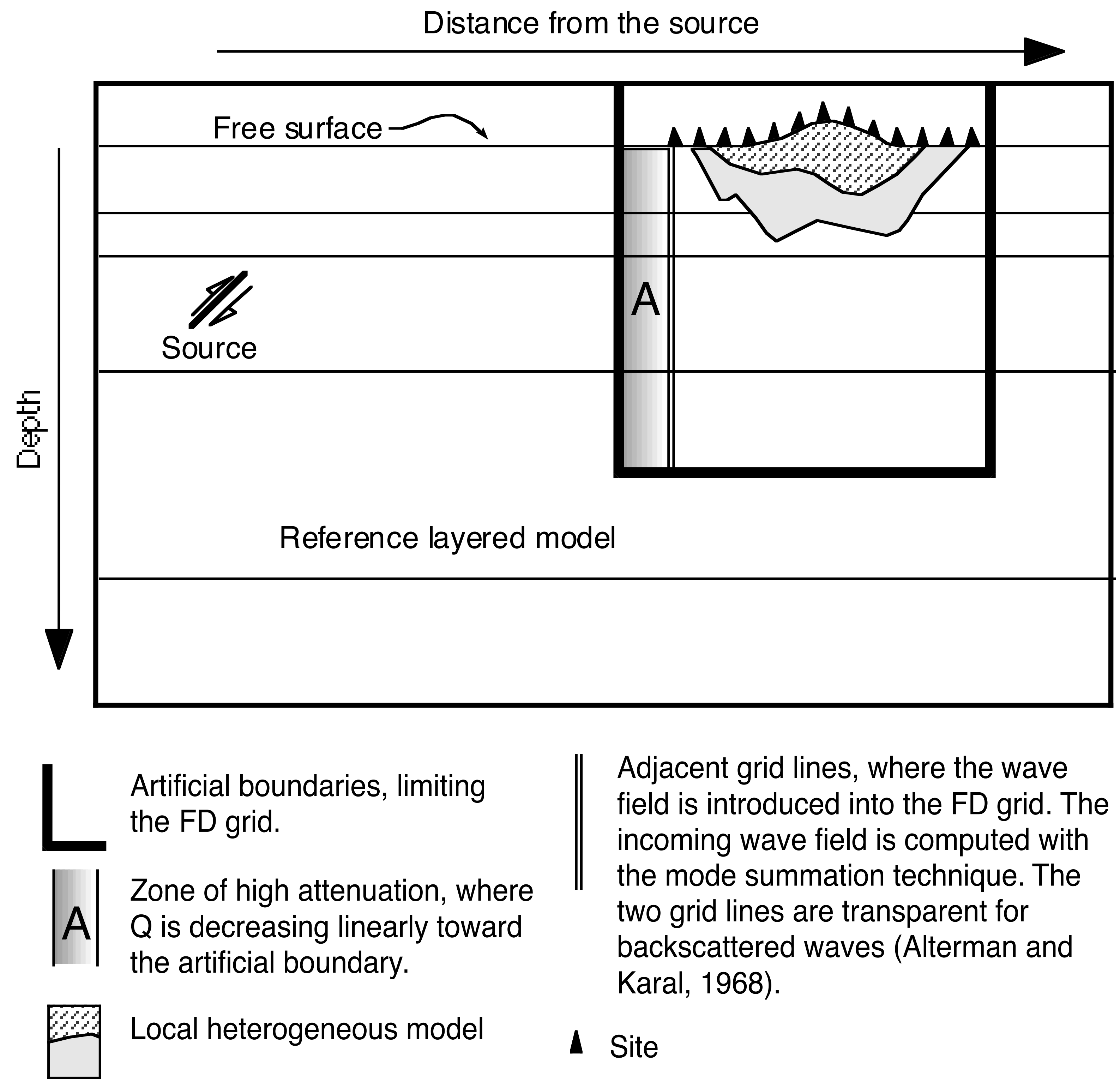
Propagation in Laterally Heterogenous Media

- Computational techniques



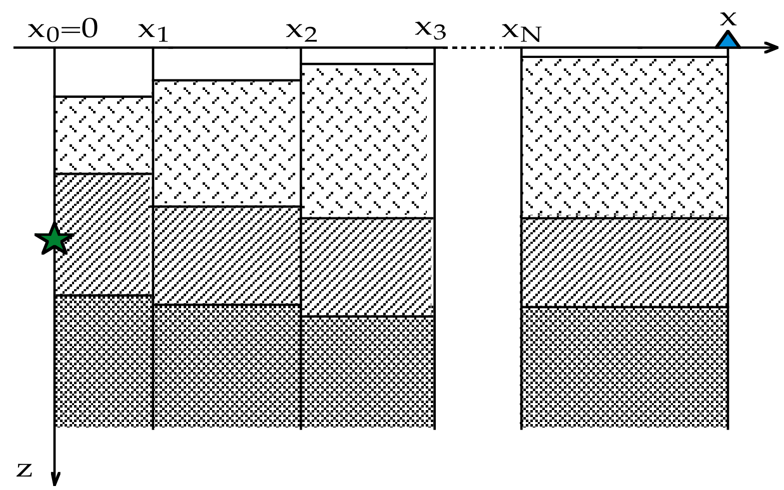


Propagation in LHM - Hybrid MS-FD



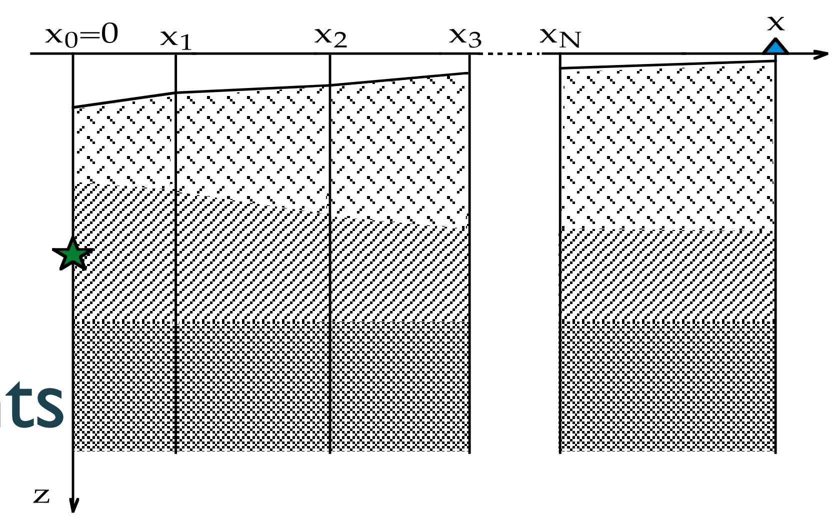
Propagation in LHM - The modal approach

a)



$$u_i^{mL}(\omega) = \frac{e^{-i\pi/4}}{\sqrt{8\pi\omega}} \frac{e^{-i\omega\sum_{j=1}^{N}\frac{x_j}{c_j}}}{\sqrt{x}} \frac{\chi_L^m(h_s,\varphi)}{\sqrt{c_m v_m I_m}}\Bigg|_S \prod_{j=1}^{N}\gamma_{mm}^{jL} \frac{F_y^{(i)m}(z)}{\sqrt{v_m I_m}}\Bigg|_R$$

b)



- Coupling coefficients

$$u_i^{mL}(\omega) = \frac{e^{-i\pi/4}}{\sqrt{8\pi\omega}} \frac{e^{-i\omega\int_S^R\frac{dx}{c}}}{\sqrt{J}} \frac{\chi_L^m(h_s,\varphi)}{\sqrt{c_m v_m I_m}}\Bigg|_S \frac{F_y^{(i)m}(z)}{\sqrt{v_m I_m}}\Bigg|_R$$

Basic Theory - 2D mode coupling

- Alsop approach
  - The 2-D model is built up by a set of layered quarterspaces in welded contact
  - The set of eigenfunctions, at a given frequency, form a complete set
  - One mode of the first structure is normally incident at the vertical interface

$$\phi_m^I + \sum_{n=1}^{\infty} R_{mn}^{I\,II} \phi_n^I = \sum_{k=1}^{\infty} T_{mk}^{I\,II} \phi_k^{II}$$

$$k_m^I \phi_m^I - \sum_{n=1}^{\infty} k_n^I R_{mn}^{I\,II} \phi_n^I = \frac{\mu_{II}}{\mu_I} \sum_{k=1}^{\infty} k_k^{II} T_{mk}^{I\,II} \phi_k^{II}$$

- Limits:
  The completeness assumption is wrong, as the Sturm-Liouville problem is singular

- Gregersen approach
  - The 2-D model is built up by a set of layered quarterspaces in welded contact
  - Definition of stress and displacement vectors and of their scalar product

$$\mathbf{A}_I = (0, v_I, 0, 0, \sigma_{xyI}, 0) \quad \mathbf{A}_{II} = (0, v_{II}, 0, 0, \sigma_{xyII}, 0)$$

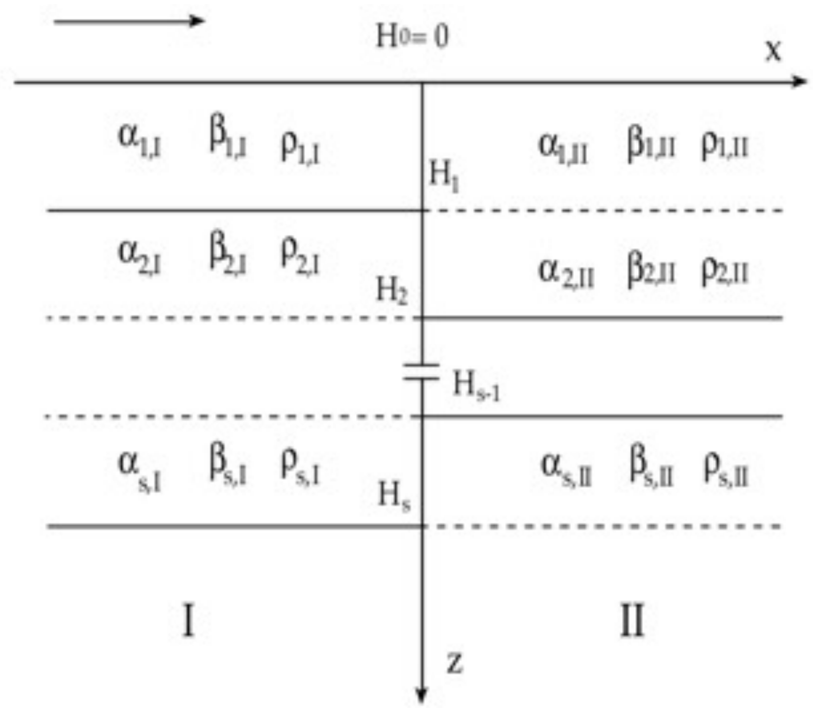
$$\langle \mathbf{A}_I, \mathbf{A}_{II} \rangle = \frac{1}{2i} \int_0^{\infty} \left[v_I \bar{\sigma}_{xyII} - \bar{v}_{II} \sigma_{xyI} \right] dz$$

  - One mode, at a given frequency, of the first structure is normally incident at the vertical interface
  - Definition of the coupling coefficients

$$\gamma_T^{(m,m')} = \frac{\langle \mathbf{A}_T^{(m)}, \mathbf{A}_{II}^{(m')} \rangle}{\langle \mathbf{A}_I^{(m)}, \mathbf{A}_I^{(m)} \rangle^{1/2} \langle \mathbf{A}_{II}^{(m')}, \mathbf{A}_{II}^{(m')} \rangle^{1/2}}$$

Basic Theory - 2D mode coupling

- 2D model scheme
  - Two layered quarterspaces in welded contact



  - Stress-Displacement vectors for a given Love mode, at a given frequency

$$\mathbf{A}_{\mathrm{I}} = (0, v_{\mathrm{I}}, 0, 0, \sigma_{xy\mathrm{I}}, 0) \quad \mathbf{A}_{\mathrm{II}} = (0, v_{\mathrm{II}}, 0, 0, \sigma_{xy\mathrm{II}}, 0)$$

  - Scalar product

$$\langle \mathbf{A}_{\mathrm{I}}, \mathbf{A}_{\mathrm{II}} \rangle = \frac{1}{2i} \int_0^\infty \left[v_{\mathrm{I}} \bar{\sigma}_{xy\mathrm{II}} - \bar{v}_{\mathrm{II}} \sigma_{xy\mathrm{I}} \right] dz$$

  - Coupling coefficients

$$\gamma_{\mathrm{T}}^{(m,m')} = \frac{\langle \mathbf{A}_{\mathrm{T}}^{(m)}, \mathbf{A}_{\mathrm{II}}^{(m')} \rangle}{\langle \mathbf{A}_{\mathrm{I}}^{(m)}, \mathbf{A}_{\mathrm{I}}^{(m)} \rangle^{1/2} \langle \mathbf{A}_{\mathrm{II}}^{(m')}, \mathbf{A}_{\mathrm{II}}^{(m')} \rangle^{1/2}}$$

$$\gamma_{\mathrm{R}}^{(m,m')} = \frac{\langle \mathbf{A}_{\mathrm{R}}^{(m)}, \mathbf{A}_{\mathrm{I}}^{(m')} \rangle}{\langle \mathbf{A}_{\mathrm{I}}^{(m)}, \mathbf{A}_{\mathrm{I}}^{(m)} \rangle^{1/2} \langle \mathbf{A}_{\mathrm{I}}^{(m')}, \mathbf{A}_{\mathrm{I}}^{(m')} \rangle^{1/2}}$$

$$\langle \mathbf{A}_{\mathrm{I}}^{(m)}, \mathbf{A}_{\mathrm{I}}^{(m')} \rangle = \frac{(k_{\mathrm{I}}^{(m)} + k_{\mathrm{I}}^{(m')})}{2} \int_0^\infty \mu_{\mathrm{I}} (v_{\mathrm{I}}^{(m)} v_{\mathrm{I}}^{(m')}) dz =$$

$$= \delta_{mm'} k_{\mathrm{I}}^{(m)} \int_0^\infty \mu_{\mathrm{I}} (v_{\mathrm{I}}^{(m)})^2 dz = \delta_{mm'} \omega U_{\mathrm{I}}^{(m)} I_{\mathrm{I}}^{(m)}$$

$$\gamma_{\mathrm{T}}^{(m,m')} = \frac{(k_{\mathrm{T}}^{(m)} + k_{\mathrm{II}}^{(m')})}{2} \frac{\int_0^\infty \mu_{\mathrm{II}} (v_{\mathrm{T}}^{(m)} v_{\mathrm{II}}^{(m')}) dz}{\sqrt{\omega U_{\mathrm{I}}^{(m)} I_{\mathrm{I}}^{(m)}} \sqrt{\omega U_{\mathrm{II}}^{(m')} I_{\mathrm{II}}^{(m')}}}$$

Displacement

- The expression that describes the displacement due to Love wave modes propagating in a layered halfspace has been generalized to laterally heterogeneous structures by Levshin (1985), so asymptotic expression of the FT of the transverse component of displacement, ${}^{m}U_y$, associated with the incoming Love mode m and transmitted into the mode m', at a distance r from the source can be written as

$$
{}^{m,m'}U_y(r,z,\omega) = \frac{\exp(-i3\pi/4)}{\sqrt{8\pi}} \left[\frac{\chi_L(h_s,\varphi)S(\omega)}{c_L\sqrt{v_{gL}I_{1L}}} \right]_m
$$

$$
\left[\frac{\exp\left[-i(k_L d + k'_L d') - \omega(dC_{2L} + d'C'_{2L})\right]}{\sqrt{d/k_L + d'/k'_L}} \gamma_{TL}^{(mm')} \right]_{mm'} \left[\frac{u_y(z,\omega)}{\sqrt{v_{gL}I_{1L}}} \right]_{m'}
$$

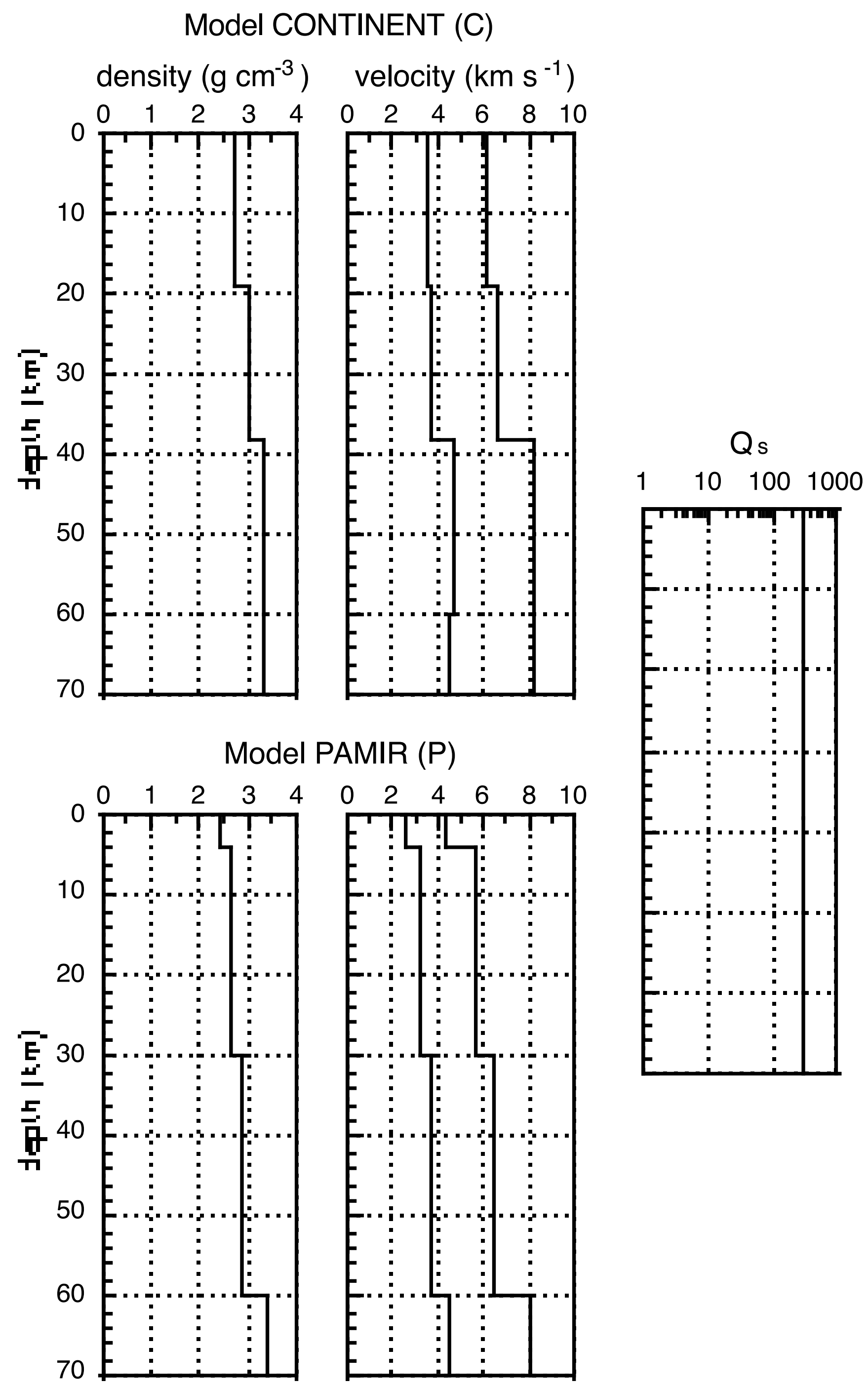
where the subscript L refers to Love modes, prime-indexed quantities are related to medium II (the medium with the receiver) and those without index refer to medium I (the medium with the source), and d and d' indicate the distances travelled in medium I and II, respectively.

Displacement

- The expression represents the contribution of one single mode m generated by a point-source placed in medium I, transmitted across the vertical interface and recorded as mode m' in medium II at a distance r=d+d'. If medium I and medium II are equal, it reduces exactly to the expression valid for a layered halfspace.

Couplings - Example

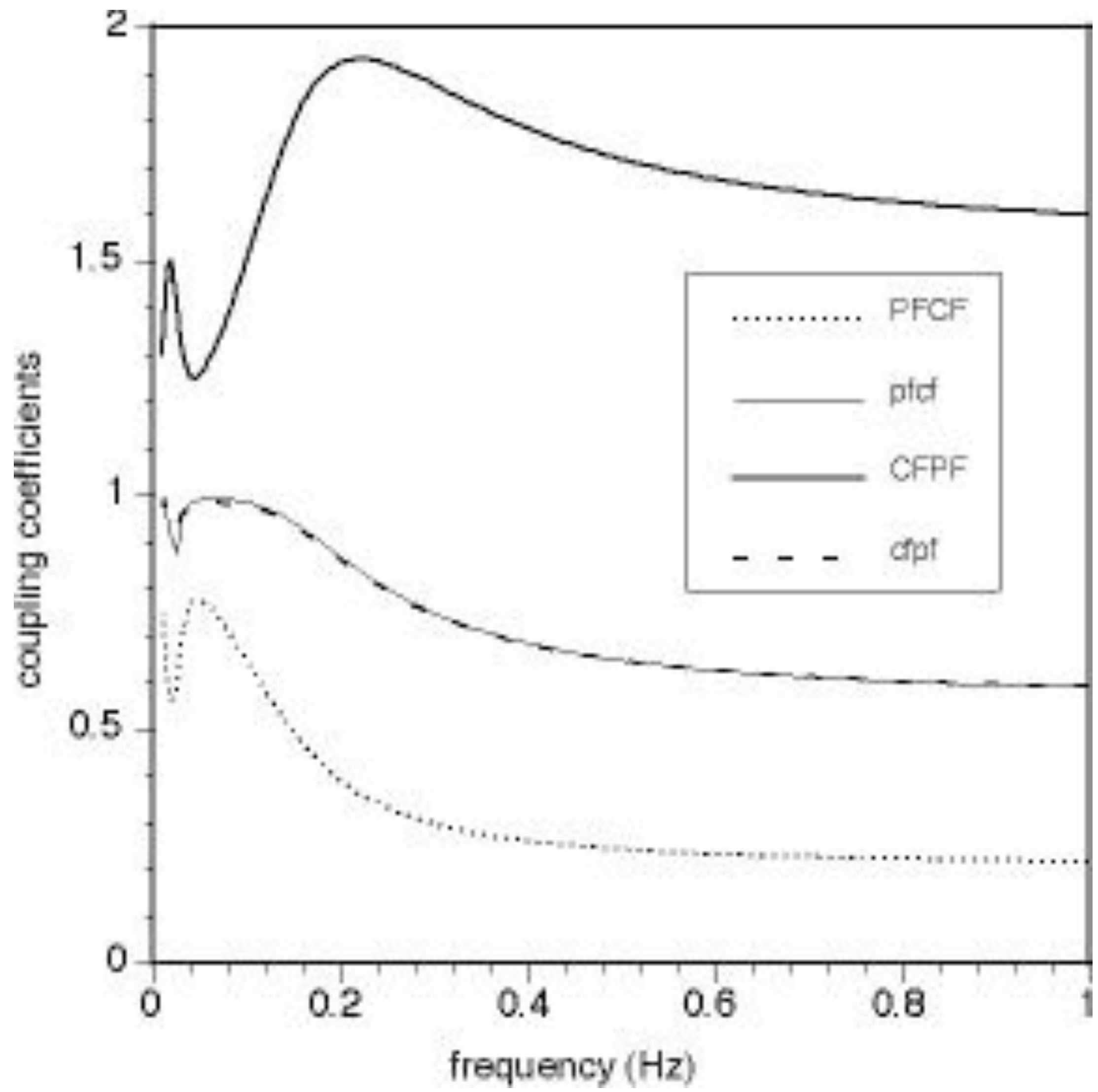
- Structures C and P



Elastic and anelastic parameters of model C and model P.
Qs is the quality factor common to both structures

Couplings - Coupling coefficients

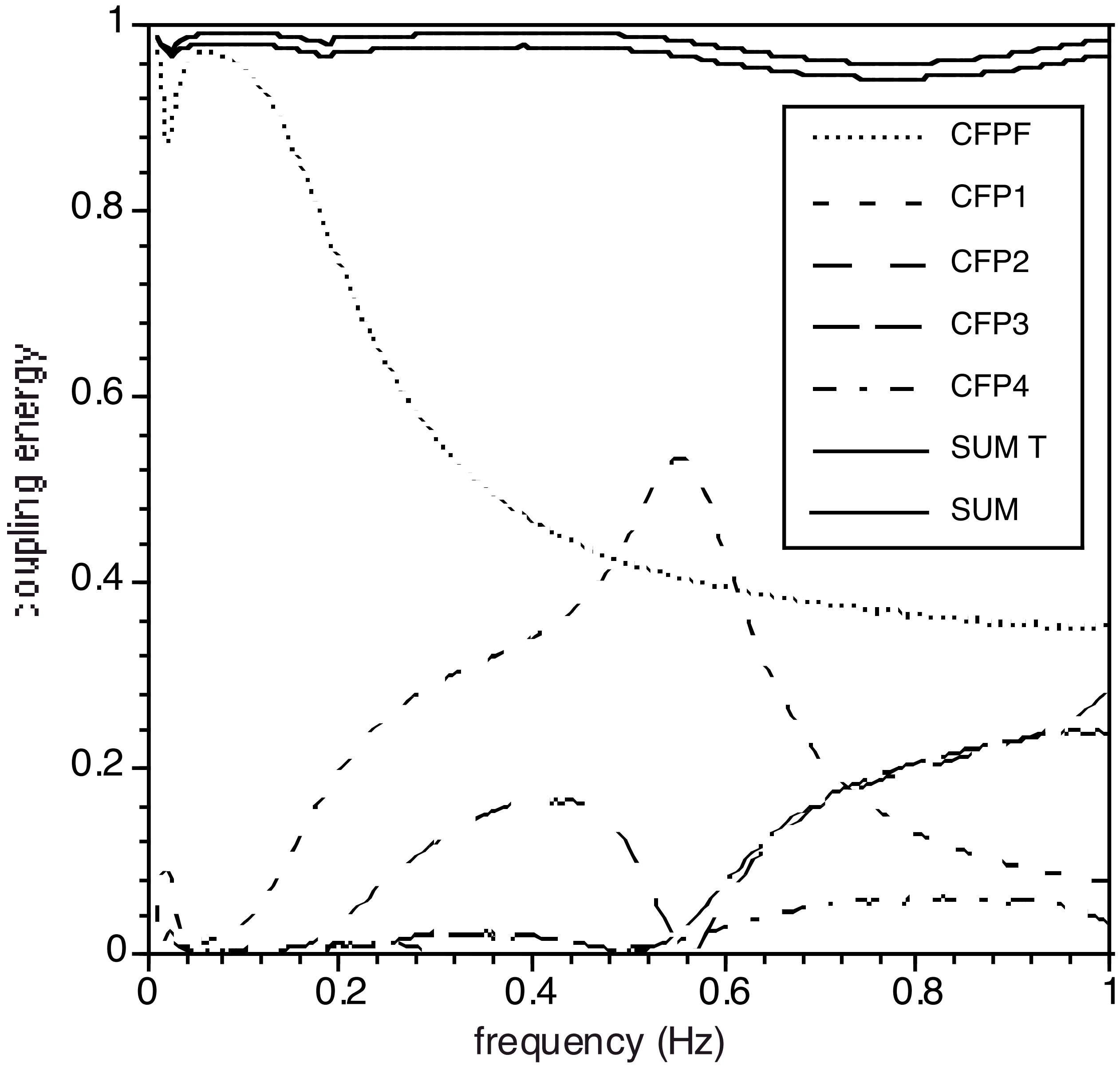
- Fundamental mode incoming



CFPF: from C to P
PFCF: from P to C

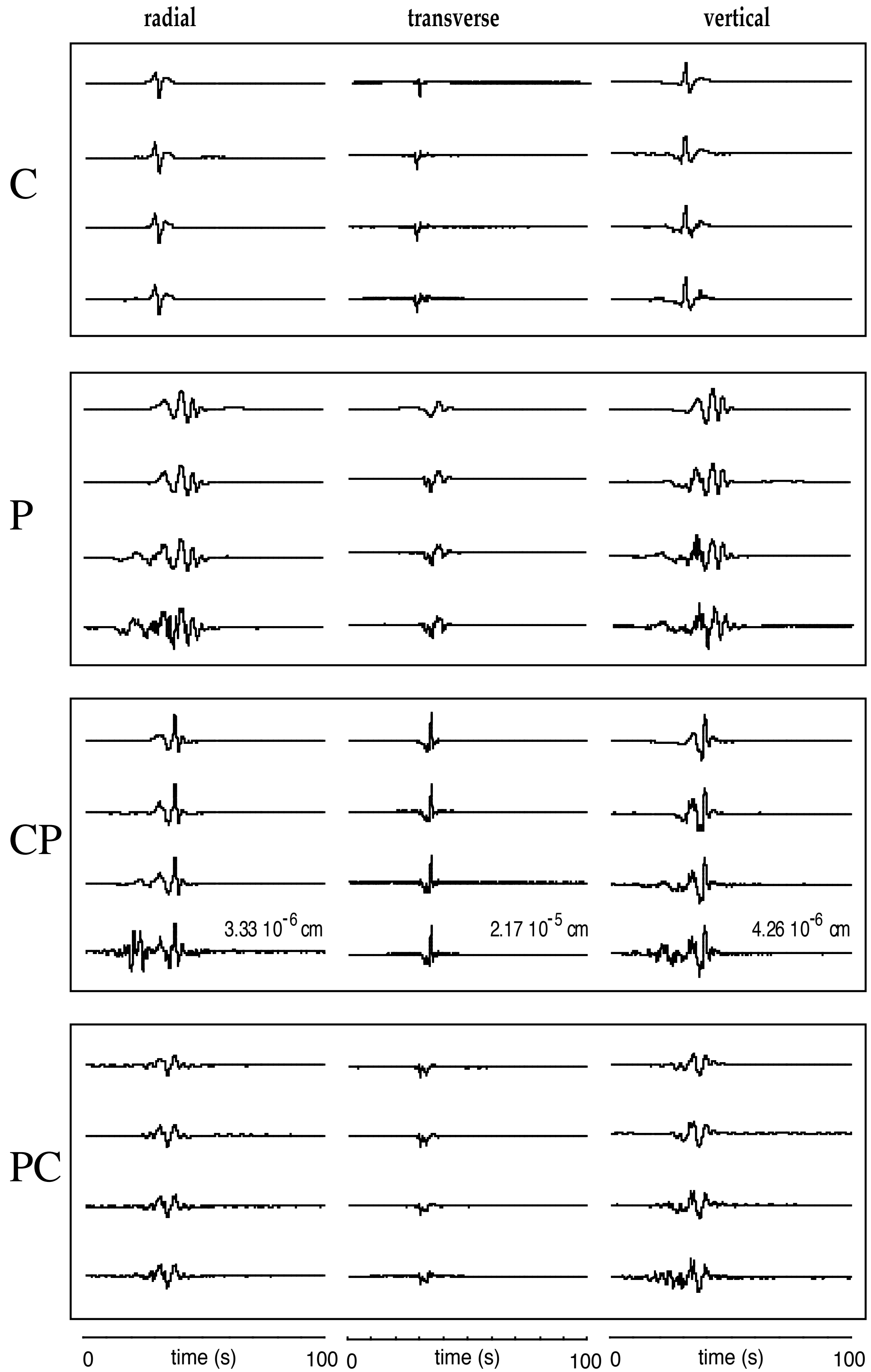
Coupling - Coupling energy

- Energy transmission and reflection

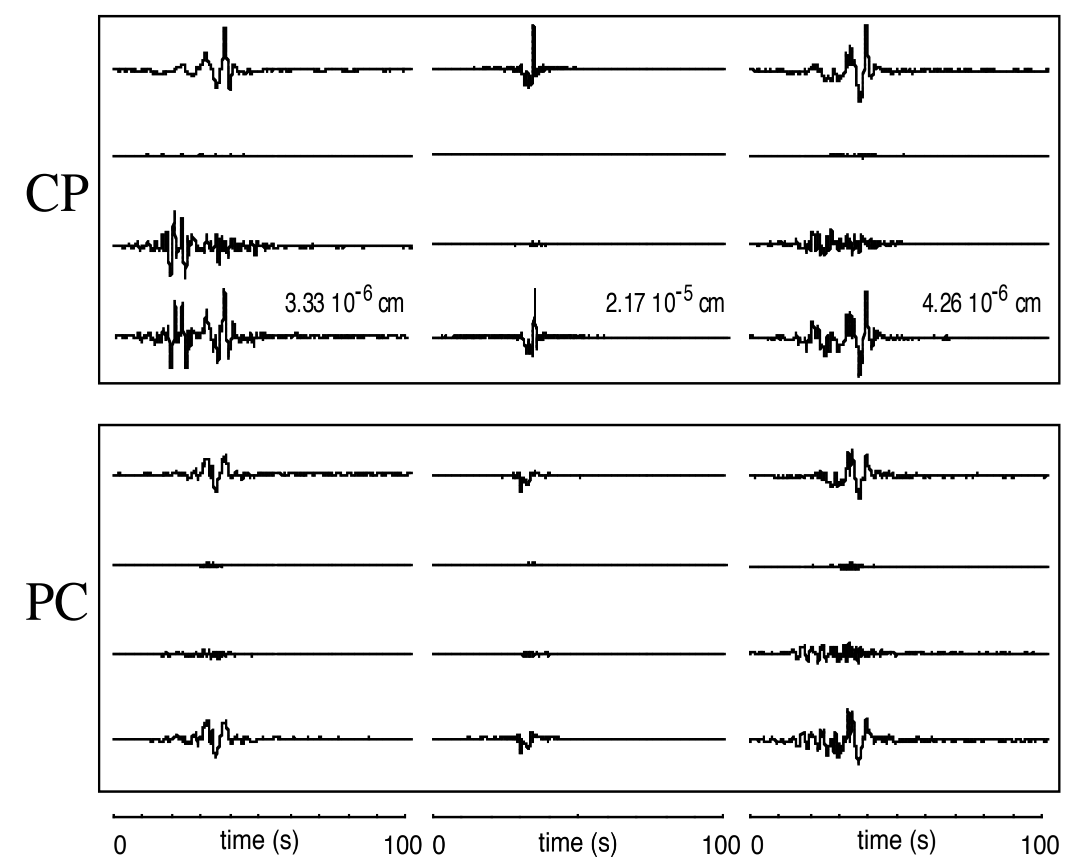


Transmission of energy from incident fundamental mode of model C to the first five modes of model P. SUM T and SUM are the curves describing the outgoing energy, without and with the inclussion of reflected energy

Couplings - Seismograms



An instantaneous point-source is placed at the depth of 10 km, with a scalar seismic moment of 10^{13} Nm. strike-receiver angle = 60°, 90°, dip = 90°, rake = 180°



Basic Theory - 2D WKBJ

- Introduction of the methodology
- The Wentzel-Kramers-Brillouin-Jeffreys approximation
- 2D model
- 3D model
- Computational scheme
- Validation study
- Conclusions

WKBJ approximation - 2D model

- The aim is to compute synthetic seismograms in anelastic media that present heterogeneity both in horizontal and vertical directions (2D model)
- The technique is largely used in seismology and is based on an assumption of regularity of the lateral variations of the elastic parameters
- A laterally varying model can be defined representing the elastic moduli and the density as

$$\lambda = \lambda(\varepsilon x, \varepsilon y, z) \quad \mu = \mu(\varepsilon x, \varepsilon y, z) \quad \rho = \rho(\varepsilon x, \varepsilon y, z)$$

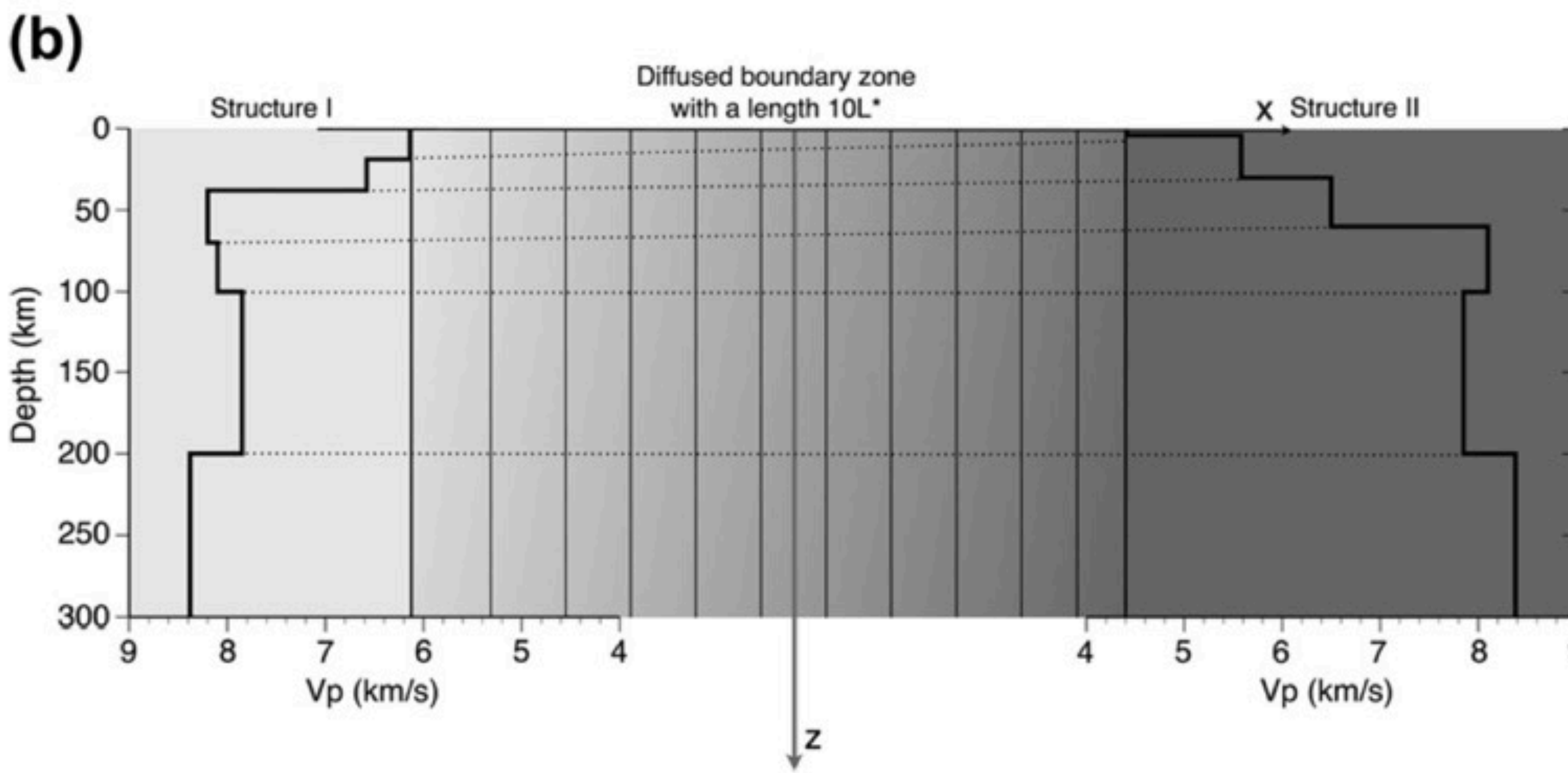
where ε is a small parameter such that if $\varepsilon = 0$ the medium becomes a laterally homogeneous layered structure (1D model)

- The parameter ε must be so small that the relative lateral variation of the elastic moduli and the density should be small over distances the order of a wavelength, i.e.

$$|\nabla_\perp \lambda| << \frac{\omega}{c}\lambda \quad |\nabla_\perp \mu| << \frac{\omega}{c}\mu \quad |\nabla_\perp \rho| << \frac{\omega}{c}\rho$$

Methodology: WKBJ approximation - 2D model

- If we focus our attention on a model made up by only two structures, this problem is solved introducing between them a set of sub-structures that have the objective to "smooth" the gradient of the lateral variation, so that the new laterally varying model presents weak lateral heterogeneities, where weak is meant in the sense of the w



2D model with a diffused boundary zone that replaces the sharp vertical boundary between the two initial quarter-spaces. The solid vertical lines are the boundaries between the interpolating structures, whose width depends on the impedance contrast. The dotted lines represent trajectories in the parameters' space.

Methodology: WKBJ approximation - 2D model

- In the condition of regularity previously mentioned, it can be assumed that the energy carried out by each propagation mode is neither transferred to other modes or reflected. This means that the modes are un-coupled: each mode propagates with a wave number determined only by the local elastic properties, i.e. by the local structure.

$$U(x,y,z;\omega)=\sum_k \frac{\exp(-i\pi/4)}{\sqrt{8\pi}}\frac{\exp(-i\omega\tau_k-\omega\gamma_k)}{\sqrt{J_k\omega}}\left.\frac{V_k(z,\omega)}{\sqrt{u_k I_{0k}}}\right|_R\left.\frac{W_k(h,\omega)}{\sqrt{c_k u_k I_{0k}}}\right|_S$$

- The sum is over the modes k, c is the phase velocity, u is the group velocity, τ is the travel time and γ is the attenuation factor

$$\tau=\int_S^r c^{-1}(x,y)\,ds \qquad \gamma=\int_S^r \varsigma(x,y)\,ds$$

Modal summation in 3D media

- We treat the problem of the derivation of the surface wave field far from the seismic source (compared with the wavelength) in a LHM, where, from now on, the heterogeneity is considered in both horizontal directions.
- As in 2D case, if the heterogeneity is not so severe, it can be reviewed as a small perturbation, i.e. within a wavelength, of a reference lateral homogeneous model and a procedure based on the ray method can be used to construct an approximate solution corresponding to the wave field (Woodhouse, 1974; AA.VV. 1989; Dahlen and Tromp, 1998).
- The principal quantities that ray methods use are travel time and geometrical spreading, which are characteristics of rays.
- Treating the propagation of waves from the point of view of ray theory requires that the minimum wavenumber must be much larger than the modulus of the ratio of the lateral gradient of elastic parameter and the value of this parameter.

Modal summation in 3D media

- Starting from available models, e.g. cellular models given in Panza et al. (2007) and Brandmayr et al. (2010), the 3D model is determined by distributing a set of vertically heterogeneous sections on a regular grid in such a way that the WKBJ approximation is satisfied: each element of the grid is occupied by a vertically heterogeneous anelastic structure (1D structure).
- A Cartesian reference framework is associated with the grid itself. The grid step is determined in such a way the lateral heterogeneity is small within a wavelength. So, the grid step is chosen as the maximum length that still allows the relative variation of the lateral gradient of the elastic parameters to be smaller than the shortest wavelength.