

Exercise n. II (Pseudo)random numbers with uniform distribution

1. Linear congruent method; periodicity

- (a) The *linear congruent* method consists in the following algorithm to generate pseudorandom numbers:

$$X_{i+1} = (aX_i + c) \bmod M = \text{remainder} \left(\frac{aX_i + c}{M} \right)$$

with arithmetic operations within the integer numbers. In the following there is a simple code (`random_lc.f90`) where this method is implemented.

```
!  
!   generation of pseudorandom numbers - linear congruent method  
!  
program random_lcm  
  implicit none  
  !  
  !   declaration of variables  
  integer :: i, number, old, seed, x, a,m,c  
  !  
  !   supply initial values of some variables:  
  !   seed:   to start;  
  !   number: how many numbers we want to generate  
  print*, ' seed, a ,m , c>'  
  read (*,*) seed, a, m, c  
  
  number = 20  
  !  
  OPEN(unit=1, file="random.dat", status="replace", action="write")  
  old = seed  
  !  
  do i = 1, number  
    x = mod ((a*old+c), m)  
    WRITE (unit=1,fmt=*) x  
    old = x  
  end do  
  close(1)  
  stop  
end program random_lcm
```

- (b) Test the program with $X_0=3$, $a=4$, $M=9$, $c=1$. Which is the interval over which the numbers are generated? Are ALL the numbers over the interval generated?
- (c) What do you observe in a sequence long enough? You should recognize that, at most after M numbers, the sequence repeats.
- (d) Test the program with $X_0=1$, $a=3$, $M=32$, $c=4$. Determine the period, that is, how many numbers are generated before the sequence repeats. Change and try with X_0 . Does the period depends on X_0 ?
- (e) Run the program with $X_0=10$, $a=57$, $M=256$, $c=1$. Determine the period. (how ? “by hands” ? Can the program itself do the job for you?)

2. Tests about randomness, uniformity, correlations

- (a) Using the linear congruent method of previous exercise, with parameters as in case (e), consider $r_i = X_i/M$ in order to have real pseudorandom numbers between 0 and 1.
- (b) Do a histogram with the sequence generated above and plot it using for instance **gnuplot** with the command `w[ith] boxes`. Is the distribution *uniform*?
Hint: to do the histogram, divide the range into a given number of channels of width Δr , then calculate how many points fall in each channel, $r/\Delta r$:

```
integer, dimension(20) :: histog
:
histog = 0
do n = 1, ndata
  i = int(r(n)/delta_r) + 1
  histog(i) = histog(i) + 1
end do
```

- (c) We can see now whether there is *correlation*. Take the sequence of random numbers obtained before and make a plot of successive points

$$(x_i, y_i) = (r_{2i-1}, r_{2i}) \quad i = 1, 2, 3, \dots$$

Do NOT connect the points with lines.

- (d) What do you see from the plot? How many different points (x_i, y_i) have been generated in the plot? You may “see” *correlations*. (Let the program write directly *pairs* of random numbers). If the numbers would be really random, how many points (different pairs) would you expect?

3. Intrinsic generators

Consider the subroutine that generates random numbers: e.g., in Fortran 90 `random_number()` is an *intrinsic* procedure generating real random numbers in the range $[0,1[$. The argument of the subroutine `random_number()` is `real`, has `intent out`, and can be a scalar or an array.

See for instance `rantest_intrinsic.f90` (in this exercise don't worry about the seed of the sequence). Note in the example the use of the dynamical allocation of memory (the instruction `allocatable`) for the array `x` and the use of `print` instruction with a specified format.

- (a) Produce a sequence (long enough) of random numbers in $[0,1[$ with the intrinsic generator of your favorite compiler. Test uniformity and correlation (see points b) e c) of Ex. 2) on this sequence. What do you see? Any difference with respect to Ex. 2)?

- (b) For a *uniformity* quantitative test, calculate the moment of order k :

$$\langle x^k \rangle^{calc} = \frac{1}{N} \sum_{i=1}^N x_i^k,$$

that should correspond to

$$\langle x^k \rangle^{th} = \int_0^1 dx x^k p_u(x) = \frac{1}{k+1}$$

where $p_u(x)$ is the uniform distribution in $[0,1[$. For a given k (fix for instance $k=1, 3, 7$), consider the deviation of the calculated momentum from the expected one: $\Delta_N(k) = |\langle x^k \rangle^{calc} - \langle x^k \rangle^{th}|$, and study its behaviour with N (N up to ~ 100.000). It should be $\sim 1/\sqrt{N}$. (*a log-log plot could be useful*)

- (c) For a *correlation* quantitative test, calculate:

$$C(k)^{calc} = \frac{1}{N} \sum_{i=1}^N x_i x_{i+k}$$

that should correspond to

$$C^{th} = \int_0^1 dx \int_0^1 dy xy p_u(x)p_u(y) = \frac{1}{4}.$$

Consider the deviation of the calculated quantity from the expected one: $\Delta_N(k) = |C(k)^{calc} - 1/4|$ and study its behaviour with N (N up to ~ 100.000). It should be $\sim 1/\sqrt{N}$.

```

program rantest_intrinsic
!
! test program, call to intrinsic f90 random number generator
! generate random numbers in [0,1[ ; then,
! generate random integers between n_min and n_max.
!
implicit none
real :: rnd
real, dimension (:), allocatable :: x
integer :: L,i,n_min,n_max,ran_int

! generates ONE random number in [0,1[
call random_number(rnd)
print *, ' A real random number in [0,1[ is:',rnd

! generates L random numbers in [0,1[
print*, ' How many random numbers do you want to generate in [0,1[ ?'
print*, ' Insert the length of the sequence >'
read(*,*)L          ! length of sequence
do i = 1,L
call random_number(rnd)
print *,rnd
end do

! generates integer random numbers between n_min and n_max
print*, ' Generate ',L,' integer random numbers in [n_min,n_max[ ;'
print*, ' insert n_min, n_max >'
read(*,*),n_min,n_max
do i = 1,L
call random_number(rnd)
ran_int = (n_max - n_min + 1)*rnd + n_min
print *,ran_int
end do

! use array x to generate and store L random numbers with a unique call
print*, ' Generate other ',L,' real random numbers in [0,1[:'
allocate(x(L))
call random_number(x)
print*, x
deallocate(x)
end program rantest_intrinsic

```

4. Intrinsic generators - use of the seed

The subroutine `random_seed([size] [put] [get])` initializes the sequence (it can be useful to control the initialization in case you want for instance to reproduce exactly a sequence of random numbers), but it can also return informations on the random numbers generator. It does not need necessarily an argument, and you cannot use more than one. The output variable `SIZE` is a scalar integer and gives the dimension `N` of the integer array (`SEED`). `SIZE` is compiler and machine dependent. The input variable `PUT` is the array of integers supplied by the user and that are used to initialize the sequence. The output variable `GET` (which is also an integer array) reads the instantaneous value of `SEED`. If no argument is given, the seed is initialized depending on the processor.

The code `rantest_intrinsic_with_seed.f90` (a bit more complicate than the previous one) calls also `random_seed`, in a way which is valid for any dimension of the seed.

The relevant part is:

```
call random_seed (sizer)
allocate(seed(sizer))
allocate(seed_old(sizer))
...
call random_seed (put = seed )           ! Initialized by the user
call random_seed (get = seed_old )      ! Gives the present values
```

Here there is another example (`rantestbis_intrinsic.f90`), where you can see three different sequences (with changes of the seed):

```
program rantestbis_intrinsic
! test program, call to intrinsic random number f90 generator
! illustrate the use of "put" and "get" in call random_seed
implicit none
integer, dimension(:), allocatable :: seed, seed_old
integer :: L,i,sizer,n_min,n_max,ran_int
integer :: i, sizer
real, dimension(3) :: harvest

call random_seed(sizer)
allocate(seed(sizer))
allocate(seed_old(sizer))
print *, 'Here the seed has ',sizer,' components; insert them (or print "/" ) >'
read(*,*)seed

call random_seed(put=seed)
call random_seed(get=seed_old)
```

```

print*, "Old starting value: ",seed_old

call random_number(harvest)
print*,"3 random numbers: ",harvest

do i=1,3
  call random_seed(get=seed_old)
  print*,"Present values of seed: ",seed_old
  call random_number(harvest)
  print*,"Other 3 random numbers: ",harvest
  call random_number(harvest)
  print*,"and other 3 random numbers: ",harvest
end do

deallocate(x)
deallocate(seed)
deallocate(seed_old)

end program rantestbis_intrinsic

```

Play with the use of `random_seed` and its arguments. It is instructive to run the same code on different compilers/machines (or compare results with classmates). What about the *size* of the seed, for instance?

5. Intrinsic generators - initialisation of the seed

Check whether the seed is changed automatically or not using `random_seed()` subroutine in your computer (run the code for random number generation more than once). In general, you should see that the initialisation is processor dependent.

You can force a change using the system clock, like in the following subroutine. Do experiments!

```

SUBROUTINE init_random_seed
  INTEGER :: i, nseed, clock
  INTEGER, DIMENSION(:), ALLOCATABLE :: seed

  CALL RANDOM_SEED(size = nseed)
  ALLOCATE(seed(nseed))
  CALL SYSTEM_CLOCK(clock)

  seed = clock/2 + 37 * (/ (i - 1, i = 1, n) /)
  CALL RANDOM_SEED(PUT = seed)

  DEALLOCATE(seed)
END SUBROUTINE

```

6. Random numbers generators in libraries *(Optional)*

Use other “good” random number generators, that you can find for instance in the book *Numerical Recipes*; for Fortran 90 see:

<http://nrbook.com/a/bookf90pdf.php>.

Here there is an example of a call to `ran_func` using a module (`nrdemo_ran.f90`).

```
module ran_module
  implicit none
  public :: ran_func

contains
  FUNCTION ran_func(idum) result(ran)
    ! IMPLICIT NONE
    INTEGER, PARAMETER :: K4B=selected_int_kind(9)
    INTEGER(kind=K4B) , intent(inout) :: idum
    REAL :: ran
    ! "minimal" random number generator;
    ! returns a uniform random deviate between 0.0 and 1.0 (not endpoints).
    ! Fully portable, scalar generator;
    ! has the "traditional" (NOT Fortran 90) calling sequence with
    ! a random deviate as the returned function value:
    ! call with IDUM a NEGATIVE integer to initialize;
    ! thereafter, do not alter IDUM except to reinitialize.
    ! The period of this generator is about 3.1 * 1018
    INTEGER(kind=K4B), PARAMETER :: IA=16807,IM=2147483647,IQ=127773,IR=2836
    REAL, SAVE :: am
    INTEGER(kind=K4B), SAVE :: ix=-1,iy=-1,k
    if (idum <= 0 .or. iy < 0) then           ! initialize
      am=nearest(1.0,-1.0)/IM
      iy=ior(ieor(888889999,abs(idum)),1)
      ix=ieor(777755555,abs(idum))
      idum=abs(idum)+1                       ! set idum positive
    end if
    ix=ieor(ix,ishft(ix,13)) ! Marsaglia shift sequence, period 232-1
    ix=ieor(ix,ishft(ix,-17))
    ix=ieor(ix,ishft(ix,5))
    k=iy/IQ                                  ! Park-Miller sequence, period 231-2
    iy=IA*(iy-k*IQ)-IR*k
    if (iy < 0) iy=iy+IM
    ran=am*ior(iand(IM,ieor(ix,iy)),1) ! combine the two generators with
    END FUNCTION ran_func                   ! masking to ensure nonzero value

end module ran_module
```

```
program demo

  use ran_module
  implicit none
  integer :: i,idum
  real :: x

  print*, "idum (<0) = "
  read*,idum
  x =ran_func(idum)
  print*,"Random number: ",x

  do i=1,10
    x = ran_func(idum)
    print*,"Random number: ",x
  end do

end program demo
```