PROBABILITÀ

SPAZIO CAMPIONARIO o DEGLI EVENTI

$\Omega = \{ \omega_1, \omega_2, \dots \}$ $\qquad \omega_j$: EVENTI ELEMENTARI

$A \subseteq \Omega$ EVENTO

<u>Example</u>: UN DADO A 6 FACCE $\quad \Omega = \{1,2,3,4,5,6\}$

$A = \{2,4,6\}$ ESCE UN NUMERO PARI

UNA MONETA LANCIATA n VOLTE (0 : TESTA, 1 : CROCE)

$$\Omega = \{ \underbrace{0 \dots 0}_{n}, \underbrace{0 \dots 1}_{n}, \dots \} = \{0,1\}^n$$

A: LA PRIMA TESTA ESCE AL TERZO LANCIO

$$\stackrel{!}{=} \{ 1\,1\,0\,[0,1]^{n-3} \} \subseteq \Omega$$

PROBABILITY

$$P : 2^{\Omega} \longrightarrow [0,1] \quad \text{t.c.}$$

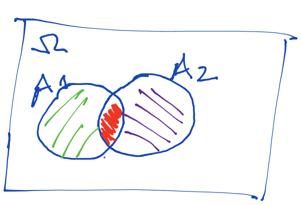
1) $P(\Omega) = 1$

2) Se $A_1, \dots, A_n, \dots \subseteq \Omega$ SONO DISGIUNTI $(A_i \cap A_j = \emptyset)$

allora $P(A_1 \cup \dots \cup A_n \cup \dots) = P_1(A_1) + P(A_2) + \dots + P(A_n) + \dots$

$$P\left(\bigcup_j A_j \right) = \sum_j P(A_j)$$

$$1 = P(\Omega) = P(\Omega \cup \emptyset) = P(\Omega) + P(\emptyset) \Rightarrow P(\emptyset) = 0$$



$$P(A_1 \cup A_2) = P(A \setminus B) + P(B \setminus A) + P(A \cap B) \quad \pm P(A \cap B)$$

$$= P(A) + P(B) - P(A \cap B)$$

$\Omega =$ "DADO" $\qquad A_1=\{2,4,6\}$, $A_2=\{5,6\}$

$$P(A_1 \cup A_2) = P(A_1)+P(A_2)-P(\{6\}) \overset{!}{=} 4/6 = 2/3$$

$A, B \subseteq \Omega$,

$P(B)>0 \quad P(A|B) = \dfrac{P(A\cap B)}{P(B)}$

$P(A)>0 \quad P(B|A) = \dfrac{P(A\cap B)}{P(A)}$

$$P(A\cap B) = P(B|A)\,P(A) \overset{!}{=} P(A|B)\,P(B)$$

$$\Downarrow$$

$$P(A|B) = \frac{P(B|A)\cdot P(A)}{P(B)}$$

$A = \{x$ HA UN CANCRO AI POLMONI$\}$ $\qquad P(A) = 72/200.000$

$B = \{x$ FUMA$\}$ $\qquad P(B) = 0.25$

$\bar{B} = \Omega - B = \{x$ non fuma$\}$.

$P(B|A) = 0.90$

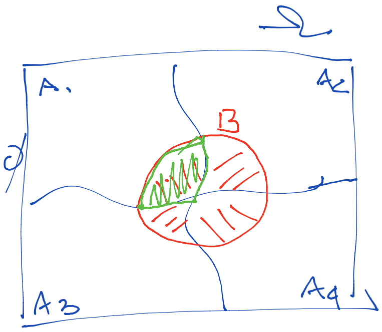
$P(A|B) = \dfrac{P(B|A)\cdot P(A)}{P(B)} = 0.0013$ $\qquad P(B) = P(B|A)\,P(A) + P(B|\bar{A})\,P(\bar{A})$

$P(A|\bar{B}) = \;.\;.\;.\;.\;.\; = 0.00005$

LEGGE DELLA PROB. TOTALE.

$A_1, \dots, A_n, \dots \subseteq \Omega$ t.c. $\bigcup_j A_j = \Omega$, $A_i \cap A_j = \emptyset$

$B = (B\cap A_1) \cup (B\cap A_2) \cup \dots \cup (B\cap A_n) \cup \dots$



$$P(B) = P(B\cap A_1) + \dots + P(B\cap A_n) + \dots = \sum_j P(B|A_j)\,P(A_j)$$

VARIABILI ALEATORIE (RANDOM VARIABLE)

$X: \Omega \to \mathbb{R}$ in Ω ho una Prob. $P: \wp(\Omega) \to [0,1]$

"$X(\Omega)$"

$a \in \mathbb{R}$ $(a \in X(\Omega))$ $P(\{\omega \mid X(\omega) = a\}) = P(X^{-1}(a))$

$$P_X(a) = \boxed{P(X=a)} := P(X^{-1}(a))$$

"ci si dimentica Ω" e si lavora con P_X

$\Omega = \{1, \dots, 6\} \times \{1, \dots, 6\}$

$X: \Omega \to \mathbb{R}$ $\quad X(\omega_1, \omega_2) = \omega_1 + \omega_2$

$$P(X=5) = P(X^{-1}(5)) = P(\{(1,4),(3,2)\,(2,3)\,(4,1)\}) = \frac{4}{36} = \frac{1}{9}$$

$$P(X>8) = P(X^{-1}(\underbrace{\{9,10,11,12\}}_{\{x \mid x>8\}}))$$

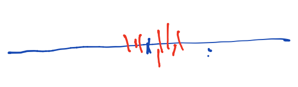
$F(a) = P(\underbrace{X \le a})$

EXPECTATION

$$E[X] = \sum_{y \in X(\Omega)} y \cdot P(X=y)$$

moneta: $E[X] = 0 \cdot \frac{1}{2} + 1 \cdot \frac{1}{2} = \frac{1}{2}$

$E[D] = 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + \dots + 6 \cdot \frac{1}{6} = \frac{21}{6} = 3.5$



MISURE DI DISPERSIONE: VARIANZA

$$VAR[X] = E[(X - E[X])^2] \qquad STD[X] = \sqrt{VAR[X]}$$

Esercizio: $I_A : \Omega \to \mathbb{R}$, $A \subseteq \Omega$

$$I_A(\omega) = \begin{cases} 1, & \text{se } \omega \in A \\ 0, & \text{se } \omega \notin A \end{cases} \qquad E[I_A] = ? \quad (P(A))$$

$\sim\sim$

$X, Y : \Omega \to \mathbb{R}$

$P(X=x, Y=y)$ ← joint prob. distribution!

$$\left(P(X=x) = \sum_y P(X=x, Y=y) \right)$$

LAW OF SUM $\quad P(x) = \sum_y P(x, y)$ ← marginalization

LAW OF PRODUCT $\quad P(x, y) = P(x|y)\,P(y)$

$$\left(P(X=x, Y=y) = P(X=x \mid Y=y)\, P(Y=y) \right)$$

$\sim\sim$

$X, Y : \Omega \to \mathbb{R}$

$Z = X + Y \qquad P(Z=z) = \sum_{x,y \mid x+y=z} P(X=x, Y=y)$

$Z = f(X, Y) \qquad P(Z=z) = \sum_{(x,y) \mid f(x,y)=z} P(X=x, Y=y)$

$\sim\sim$ $\quad a, b \in \mathbb{R}$

$E[aX+b] = aE[X] + b$

$E[X+Y] = E[X] + E[Y]$

$E[XY] = E[X]E[Y]$ iff X, Y INDEPENDENT

$$P(X=x, Y=y) = P(X=x)P(Y=y)$$

$$\left(P(X=x \mid Y=y) = P(X=x) \right)$$

X, Y

$\to COV[X,Y] = E[XY] - E[X]E[Y]$, $COV[X,Y]=0$ se X, Y indipendenti

$VAR[X+Y] = VAR[X] + VAR[Y] + 2\,COV[X,Y]$

$VAR[aX+b] = a^2\,VAR[X]$

BINOMIALE: $X \sim Bin(p, N)$

X CONTA IL NUMERO DI SUCCESSI IN N ESPERIMENTI CON PROBABILITÀ di SUCCESSO p

$\Omega = \{0,1\}^N$ $\quad X : \Omega \to \mathbb{R}$

$X(b_1 \dots b_N) = \sum_i b_i \quad b_i \in \{0,1\}$

$P(X=m) = \binom{N}{m} p^m (1-p)^{N-m}$ $\quad \sum_m \binom{N}{m} p^m (1-p)^{N-m} = 1$

$0 \leq m \leq N$

$E[X] =$

$VAR[X]$

$Z_1, \dots, Z_N \quad Z_i \in \{0,1\} \quad P(Z=1) = p$

$X = Z_1 + \dots + Z_N \quad E[X] = \sum_j E[Z_j] = N \cdot E[Z] = N \cdot p$

$VAR[X] = N(1-p)p$

GEOMETRICA

$X \sim Geom(p)$ NUMERO DI ESPERIMENTI FINO AL PRIMO SUCCESSO

$$P(X=m) = (1-p)^{m-1}\, p \qquad \underbrace{0 \cdots 0}_{m-1}1$$

$$E[X] = \frac{1}{p} \qquad VAR[X] = \frac{1-p}{p}$$

POISSON DISTRIBUTION !

$$P(X=k) = \frac{\lambda^k}{k!} e^{-\lambda}, \quad \lambda > 0 \quad (\text{RATE})$$

$$E[X] = \sum_{k=0}^{\infty} k \cdot \frac{\lambda^k}{\cancel{k}!} e^{-\lambda} = e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda \cdot \lambda^{k-1}}{(k-1)!} = \lambda e^{-\lambda} \underbrace{\sum_{n=1}^{\infty} \frac{\lambda^n}{n!}}_{e^{\lambda}}$$

$$= \lambda e^{-\lambda} e^{\lambda} = \lambda$$

$VAR[X] = ?$ ← Vateje