Computational Modelling
Discrete Time Markov Chains

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Outline

1. Probabilities and Measures
2. Discrete Time Markov Chains

Outline

DTMC: DEFINITION

Let S be a countable state space with the discrete sigma-algebra. A stochastic matrix on S is an $|S| \times |S|$ matrix whose rows sum up to one.

DEFINITION

A sequence $(X_n)_{n\in\mathbb{N}}$ of random variables on S is called a (Discrete Time) Markov Chain (p, Π) with initial distribution p and transition matrix Π if

1. $\mathbb{P}(X_0 = s_j) = p_j$
2. $\mathbb{P}(X_{n+1} = s_{i_{n+1}} | X_0 = s_{i_0}, \ldots, X_n = s_{i_n}) = \pi_{i_n i_{n+1}}$.

The property 2 is known as the memoryless or Markov property (for time-homogeneous DTMC). In general, it is spelt

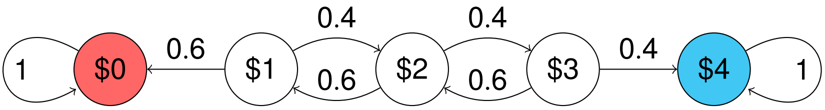
$$\mathbb{P}(X_{n+1} = s_{i_{n+1}} | X_0 = s_{i_0}, \ldots, X_n = s_{i_n}) = \mathbb{P}(X_{n+1} = s_{i_{n+1}} | X_n = s_{i_n})$$

EXAMPLE: GAMBLER'S RUIN

Consider a gambling game. On any turn you win \$1 with probability $p = 0.4$ or lose \$1 with probability $1 - p = 0.6$. You quit playing if your fortune reaches $\$N$ or \$0.

The state space is $S = \{0, 1, \ldots, N\}$.
The probability matrix, for $N = 4$ is:

$$\Pi = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0.6 & 0 & 0.4 & 0 & 0 \\ 0 & 0.6 & 0 & 0.4 & 0 \\ 0 & 0 & 0.6 & 0 & 0.4 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$



EXAMPLE: FLU

A person can be susceptible to flu, infected, or immune (usually after recovery). Susceptibles can be infected with probability 0.2, while infected individuals can recover and become immune with probability 0.4. Immunity is lost with probability 0.01.

State space $S = \{S, I, R\}$

$$\Pi = \begin{pmatrix} 0.8 & 0.2 & 0.0 \\ 0.0 & 0.6 & 0.4 \\ 0.01 & 0.0 & 0.99 \end{pmatrix}$$

QUESTION

Do the fraction of infected individuals stabilise? To which value?

EXAMPLE: BRANCHING PROCESS

Consider a population, in which each individual at each generation independently gives birth to k individuals with probability p_k. These will be the members of the next generation.

The state space is $S = \mathbb{N}$, hence infinite.
The transition matrix is defined by

$$\pi(i, j) = P(Y_1 + \ldots + Y_i = j) \text{ for } i > 0 \text{ and } j \geq 0$$

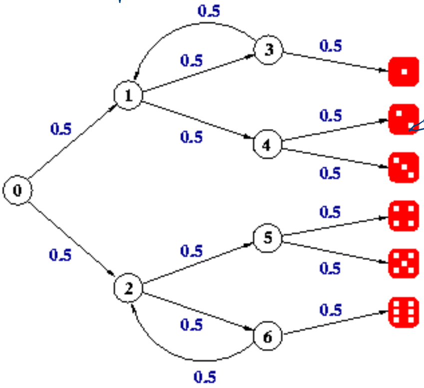
where Y_j is an independent random variable on $\mathbb{N}$ with $\mathbb{P}\{Y_j = k\} = p_k$.

QUESTION

What is the probability of extinction of the population?

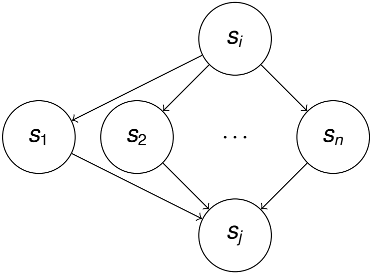
Example: simulating a dice with a coin (Knuth)

$$\frac{1}{4} + \frac{1}{4}\cdot\frac{1}{4} + \left(\frac{1}{4}\right)^2\cdot\frac{1}{4} + \dots + \left(\frac{1}{4}\right)^n\cdot\frac{1}{4} = \frac{1}{4}\cdot\sum_{n=0}^{\infty}\left(\frac{1}{4}\right)^n = \frac{1}{4}\cdot\frac{1}{1-\frac{1}{4}} = \frac{1}{4}\cdot\frac{4}{3} = \frac{1}{3}$$



DTMC: CHAPMAN-KOLMOGOROV EQUATIONS

$$\mathbb{P}\{X_j = s_j \mid X_i = s_i\} = \sum_{s \in S} \mathbb{P}\{X_j = s_j \mid X_k = s\}\mathbb{P}\{X_k = s \mid X_i = s_i\}$$



$$\mathbb{P}(X_0 = s_{i_0}, \ldots, X_n = s_{i_n}) = p_{i_0}\pi_{i_0 i_1} \cdots \pi_{i_{n-1} i_n}$$

$$\mathbb{P}\{X_n = s_{i_n} \mid X_0 = s_{i_0}\} = \sum_{s_1, \ldots, s_{n-1} \in S} \mathbb{P}\{X_n = s_{i_n} \mid X_{n-1} = s_{n-1}\} \cdots \mathbb{P}\{X_1 = s_1 \mid X_0 = s_{i_0}\}$$

If S is finite, $\mathbb{P}(X_n = s_j) = (p\Pi^n)_j$

EXAMPLE: FLU

QUESTION

What is the probability that an individual is initially infected, remains infected for one time unit, then recovers just before loosing immunity?

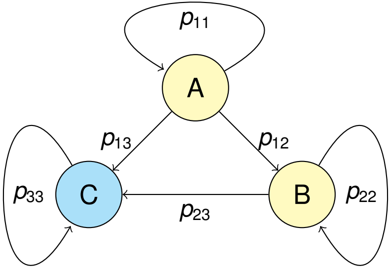
$$I \longrightarrow I \longrightarrow R \longrightarrow S$$

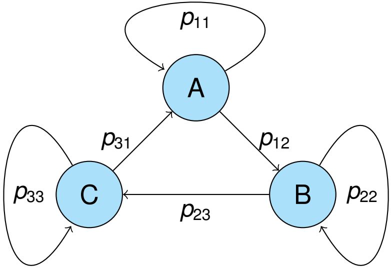
$$\Pi = \begin{pmatrix} 0.8 & 0.2 & 0.0 \\ 0.0 & 0.6 & 0.4 \\ 0.01 & 0.0 & 0.99 \end{pmatrix}$$

$$\mathbb{P}\{2 \to 2 \to 1 \to 3\} = p_2 \cdot \pi_{2,2} \cdot \pi_{2,3} \cdot \pi_{3,1} = 0.33 \cdot 0.6 \cdot 0.4 \cdot 0.01 = 0.000792$$

DTMC: COMMUNICATING CLASSES

- Support graph $G = (S, E)$ of a DTMC: $(s_i, s_j) \in E$ iff $\pi_{ij} > 0$.
- Communicating classes: strongly connected components of G.
- (p, Π) irreducible iff G strongly connected.





DTMC: Absorption probabilities

Let $A \subseteq S$. The absorption probability of A,

$$h_i^A = \mathbb{P}\{\textit{eventually } A \mid X_0 = i\},$$

is the least non-negative solution of

$$\begin{cases} h_i^A = 1 & \text{for } s_i \in A \\ h_i^A = \sum_{s_j \in S} p_{ij} h_j^A & \text{for } s_i \notin A \end{cases}$$

DTMC: Expected hitting times

Let $A \subseteq S$. The hitting time of A is a random variable on $\mathbb{N}$

$$\xi_i^A = \min\{n \mid X(n) \in A\},$$

$\mathbb{E}[\xi_i^A]$ is the least non-negative solution of

$$\begin{cases} \mathbb{E}[\xi_i^A] = 0 & \text{for } s_i \in A \\ \mathbb{E}[\xi_i^A] = 1 + \sum_{s_j \in S} p_{ij} \mathbb{E}[\xi_j^A] & \text{for } s_i \notin A \end{cases}$$

EXAMPLE: GAMBLER'S RUIN

QUESTION

What is the probability of the game eventually terminating?

It is the absorption probability of the set $A = \{0, N\}$. For $N = 4$:

$$\begin{cases} h_0^A = 1 \\ h_1^A = 0.6h_0^A + 0.4h_2^A \\ h_2^A = 0.6h_1^A + 0.4h_3^A \\ h_3^A = 0.6h_2^A + 0.4h_4^A \\ h_4^A = 1 \end{cases} \quad \text{with solution } h^A = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

EXAMPLE: GAMBLER'S RUIN

QUESTION

What is the probability of being ruined?

It is the absorption probability of the set $A = \{0\}$. For $N = 4$:

$$\begin{cases} h_0^A = 1 \\ h_1^A = 0.6h_0^A + 0.4h_2^A \\ h_2^A = 0.6h_1^A + 0.4h_3^A \\ h_3^A = 0.6h_2^A + 0.4h_4^A \\ h_4^A = h_4^A \end{cases} \quad \text{with solution } h^A = \begin{pmatrix} 1.0000 \\ 0.8769 \\ 0.6923 \\ 0.4154 \\ 0 \end{pmatrix}$$

EXAMPLE: GAMBLER'S RUIN

QUESTION

What is the probability of being a happy winner?

It is the absorption probability of the set $A = \{N\}$. For $N = 4$:

$$\begin{cases} h_0^A = h_0^A \\ h_1^A = 0.6h_0^A + 0.4h_2^A \\ h_2^A = 0.6h_1^A + 0.4h_3^A \\ h_3^A = 0.6h_2^A + 0.4h_4^A \\ h_4^A = 1 \end{cases} \quad \text{with solution } h^A = \begin{pmatrix} 0 \\ 0.1231 \\ 0.3077 \\ 0.5846 \\ 1.0000 \end{pmatrix}$$

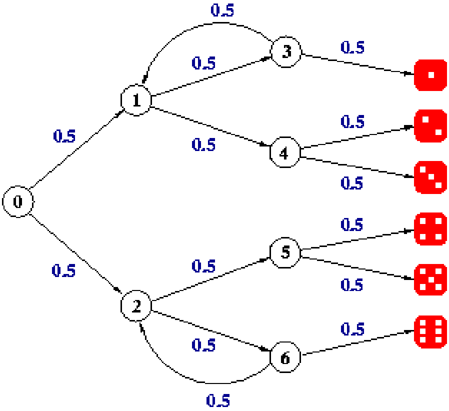
COMPUTING ABSORPTION PROBABILITIES

Consider the problem of computing the absorption probability of a set $A \subset S$ for a DTMC with transition matrix Π.

1. Make A-states absorbing, i.e. if $i \in A$, replace row i of Π by $\mathbf{e_i}$, the vector equal to 1 in position in i and zero elsewhere. Call Π_A the new transition matrix.
2. Let $\mathbf{h}^0$ be defined by $h_i^0 = 1$ if $i \in A$ and $h_i^0 = 0$ otherwise.
3. Iterate $\mathbf{h}^n = \Pi_A \cdot \mathbf{h}^{n-1}$, until $\|\mathbf{h}^n - \mathbf{h}^{n-1}\|_\infty < \varepsilon$

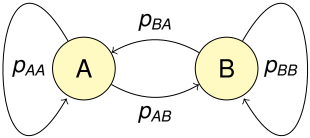
The correctness follows because $\lim_{n\to\infty} \mathbf{h}^n = \mathbf{h}_A$

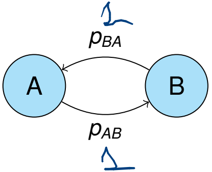
LIVE CODING: SIMULATING A DICE WITH A COIN



DTMC: RECURRENT AND APERIODIC STATES

- Return time in $s_i \in S$: $T_i = \min\{n > 0 \mid X_n = s_i, X_0 = s_i\}$
- A state is positive recurrent iff $\mathbb{E}[T_i] < \infty$.
- A state $s_i \in S$ is aperiodic if $\pi_{ii}^n > 0$ for $n \geq n_0$.





DTMC: INVARIANT MEASURES

EXISTENCE OF AN INVARIANT MEASURE

A measure μ is invariant iff $\mu\Pi = \mu$.
Let (p, Π) be irreducible. The following statements are equivalent:

1. Every state in S is positive recurrent.
2. Some state $s_i \in S$ is positive recurrent.
3. π has an invariant distribution μ. In this case, $\mathbb{E}[T_i] = 1/\mu_i$.

CONVERGENCE TO THE INVARIANT MEASURE

Let Π be irreducible and aperiodic, and let μ be an invariant distribution for Π. Let $(X_n)_{n\in\mathbb{N}}$ be Markov(p, Π) for an arbitrary initial distribution p. Then

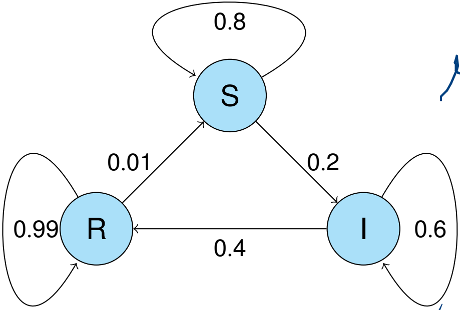
$$\forall s_j \in S,\ \mathbb{P}(X_n = s_j) \to \mu_j,\ n \to \infty.$$

Furthermore, μ is unique. Every finite irreducible and aperiodic chain has a unique positive invariant measure.

EXAMPLE: FLU SPREADING

QUESTION

Is there a steady state probability/ invariant measure for the flu example? What is it?



$$\mu \Pi = \mu$$

$$\Pi = \begin{pmatrix} 0.8 & 0.2 & 0.0 \\ 0.0 & 0.6 & 0.4 \\ 0.01 & 0.0 & 0.99 \end{pmatrix}$$

The chain is irreducible and aperiodic. Hence there is a unique invariant measure.

EXAMPLE: FLU SPREADING

QUESTION

Is there a steady state probability/ invariant measure for the flu infection example? What is it?

The invariant measure is given by the unique solution of

$$\mu \begin{pmatrix} 0.8 & 0.2 & 0.0 \\ 0.0 & 0.6 & 0.4 \\ 0.01 & 0.0 & 0.99 \end{pmatrix} = \mu, \text{ which is } \mu = \begin{pmatrix} 0.0465 \\ 0.0233 \\ 0.9302 \end{pmatrix}$$

COMPUTATION OF THE STEADY STATE

To compute the invariant/ steady state measure μ, one has to solve the following linear system:

$$\mu\Pi = \mu,$$

for the unknowns $\mu_1, \ldots, \mu_n$. This is equivalent to:

$$\mu(\Pi - I_n) = \mathbf{0}.$$

NUMERICALLY...

For stability reasons (the matrix $\Pi - I_n$ is singular), it is better to solve the following linear system:

$$\mu([\Pi - I_n, \mathbf{1}]) = [\mathbf{0}, 1]$$

i.e. adding a column of ones to $\Pi - I_n$.

DTMC: INVARIANT MEASURES

CONVERGENCE TO THE INVARIANT MEASURE

Let Π be irreducible and aperiodic, and let μ be an invariant distribution for Π. Let $(X_n)_{n\in\mathbb{N}}$ be Markov(p, Π) for an arbitrary initial distribution p. Then

$$\forall s_j \in S,\ \mathbb{P}(X_n = s_j) \to \mu_j,\ n \to \infty.$$

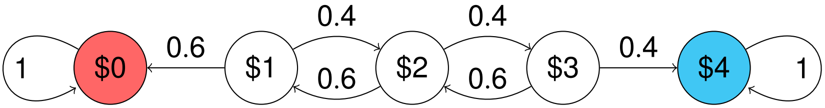
Furthermore, μ is unique. Every finite irreducible and aperiodic chain has a unique positive invariant measure.

DECOMPOSITION FOR FINITE DTMC

If S is finite, let B_j be the bottom s.c.c. of G, and suppose they are aperiodic. Let μ_{B_j} be their unique invariant measure. Then

$$\lim_{n\to\infty} \mathbb{P}(X_n \mid X_0 = s_i) = \sum_j h_i^{B_j} \mu_{B_j}.$$

Example: gambler's ruin



In the gambler's ruin model, we have two single-state bottom s.c.c.: 0 and N.

Hence we have the following steady state (conditional on the initial state):

$$\mu(\cdot \mid 0) = (1, 0, 0, 0, 0)$$
$$\mu(\cdot \mid 1) = (0.8769, 0, 0, 0, 0.1231)$$
$$\mu(\cdot \mid 2) = (0.6923, 0, 0, 0, 0.3077)$$
$$\mu(\cdot \mid 3) = (0.4154, 0, 0, 0, 0.5846)$$
$$\mu(\cdot \mid 4) = (0, 0, 0, 0, 1)$$

References

- J.R. Norris. Markov Chains, Cambridge University Press, 1998.
- R. Durrett, Essentials of Stochastic Processes, Springer-Verlag, 1998.