CONTINUOUS RANDOM VARIABLES

def: σ-ALGEBRA ON Ω, $\mathcal{A} \subseteq 2^{\Omega}$ t.c.

1) $\Omega \in \mathcal{A}$
2) $A \in \mathcal{A} \Rightarrow \bar{A} \in \mathcal{A}$
3) $A_1, \dots, A_n, \dots \in \mathcal{A} \Rightarrow A_1 \cup \dots \cup A_n \cup \dots \in \mathcal{A}$

def $\sigma(\mathcal{E}) = \bigcap \{ \mathcal{A} \mid \mathcal{A}\ \sigma\text{-ALGEBRA and } \mathcal{A} \supset \mathcal{E} \}$ σ-ALGEBRA GENERATED BY $\mathcal{E}$

$\mathcal{E} \subseteq 2^{\Omega}$

def $\mathcal{E} = \{ [a,b] \mid a,b \in \mathbb{R}, a \leq b \}$, $\sigma(\mathcal{E}) = \mathcal{B}$ BOREL σ-ALGEBRA

def: PROBABILITY SPACE $(\Omega, \mathcal{A}, P)$ probability $\quad P: \mathcal{A} \longrightarrow [0,1]$

prob. sp $\quad$ σ-algebra

•) $P(\Omega) = 1$

σ-ADDITIVITY

•) if $A_1, \dots, A_n, \dots$ are disjoint $(A_i \cap A_j = \emptyset)$ and in $\mathcal{A}$ $\quad P(\bigcup A_i) = \sum P(A_i)$

$(\Omega, \mathcal{A}, P)$ $\qquad X : (\Omega, \mathcal{A}) \longrightarrow (\mathbb{R}, \mathcal{B})$

$X^{-1}(B) \in \mathcal{A}$ $\quad$ X is MEASURABLE

The LAW of X in $\mathbb{R}$ as $P_X(B) \left(= P(B)\right) = P\left(X^{-1}(B)\right)$

We care only of prob. in $\mathbb{R}$!

A distribution in $\mathbb{R}$ is ABSOLUTELY CONTINUOUS if it has

$p : \mathbb{R} \to \mathbb{R}_{\geq 0}$ t.c. $\forall B \in \mathcal{B}$ $\qquad P_X(B) = P(B) = \int_B p(x)\,dx$

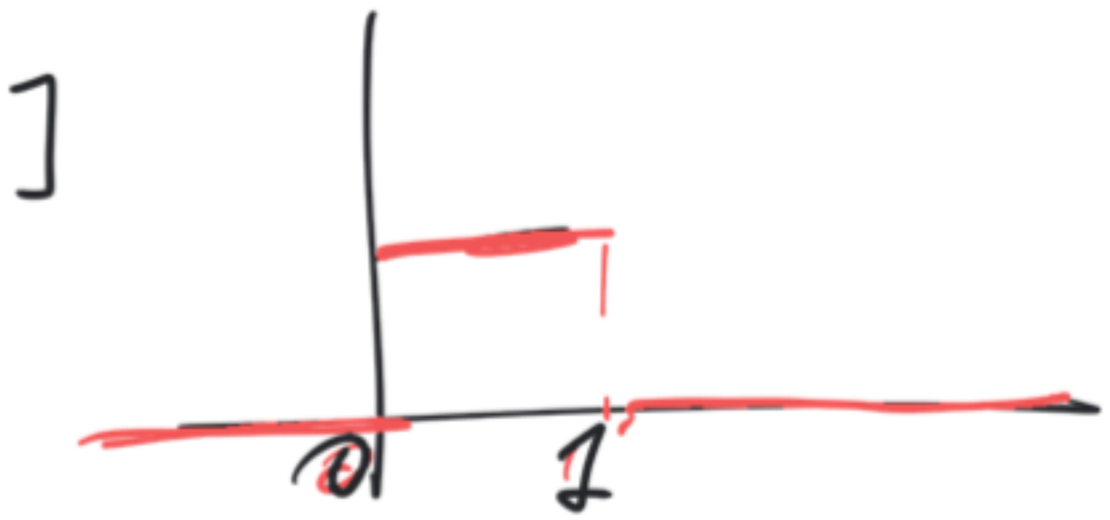
$P(X \in [a,b]) = \int_a^b p(x)\,dx$ $\qquad p(x)$ DENSITY (usually p continuous)

There is a "special density" δ_x, DELTA OF DIRAC, t.c.

$$\int_B \delta_x(y)\,dy = \begin{cases} 1, & \text{if } x \in B \\ 0, & \text{otherwise} \end{cases}$$

Esempi: UNIFORM $U \sim \mathrm{Unif}(0,1)$

$$p(x) = \begin{cases} 1, & \text{if } x \in [0,1] \\ 0, & \text{otherwise} \end{cases}$$

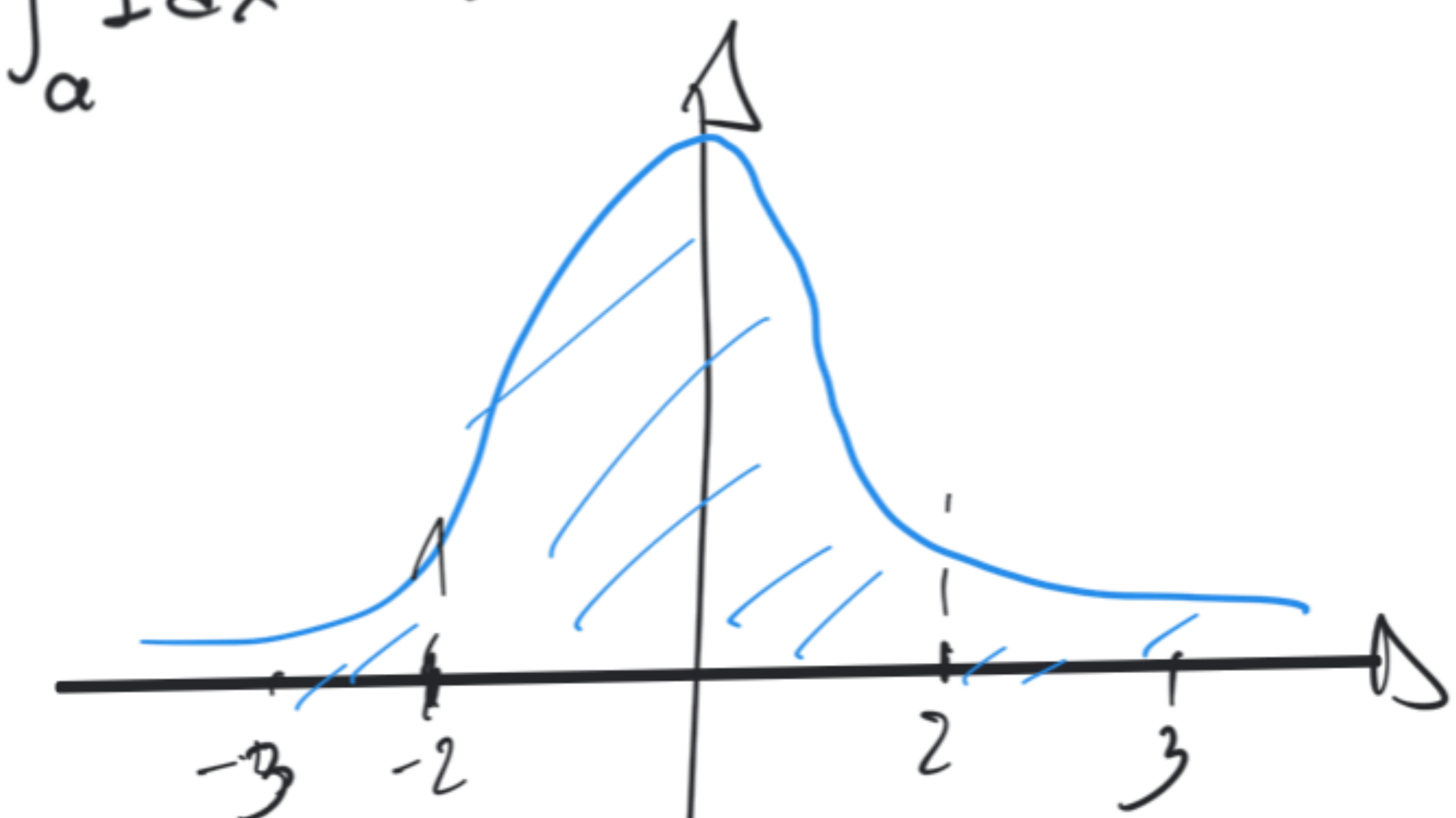


$a,b,\ a<b,\ a,b \in [0,1]$

$$P(U \in [a,b]) = b-a = \int_a^b 1\,dx \quad \checkmark$$

NORMAL: $X \sim \mathcal{N}(0,1)$

$$p(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$



EXPONENTIAL λ = rate,

$T \sim \mathrm{Exp}(\lambda)$ $\quad p(t) = \lambda e^{-\lambda t}$.

$T \in \mathbb{R}_{\geq 0}$

$$\int_0^t \lambda e^{-\lambda s}\,ds = -e^{-\lambda s}\Big|_0^t = 1 - e^{-\lambda t}$$

• Cumulative Distribution Function CDF , X with density $p(x)$

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} p(y)\,dy$$

(Exponential $p(t) = \lambda e^{-\lambda t}$, $F(t) = 1 - e^{-\lambda t}$)

EXPECTATIONS $f(X)$

$$E[f(X)] = \int_{-\infty}^{+\infty} f(x)\,p(x)\,dx$$

$$E[X] = \int_{-\infty}^{+\infty} x\,p(x)\,dx$$

$$VAR[X] = \int_{-\infty}^{+\infty} (x - E[X])^2\,p(x)\,dx$$

Exponential T

$$E[T] = \int_{0}^{+\infty} t\lambda e^{-\lambda t}\,dt = \frac{1}{\lambda}$$

$VAR[T]$ = exercise (per parts)

Conditional continuous R.V. $X, Y \in \mathbb{R}$

joint distribution $p(x,y)$

$$P(X\in A, Y\in B) = \int_{A\times B} p(x,y)\,dx\,dy$$

conditional distributions $p(x|y) = \dfrac{p(x,y)}{p(y)}$

Marginal $p(x) = \int_{\mathbb{R}} p(x,y)\,dy$

$$\mathbb{P}(X\in A \mid Y=y) = \int_A p(x|y)\,dx$$

Bayes for densities $p(x|y) = \dfrac{p(y|x)\,p(x)}{p(y)}$

Obs: X, Y can take values in $\mathbb{R}^n$.

Then $p(x)$ is such that $p: \mathbb{R}^n \to \mathbb{R}$