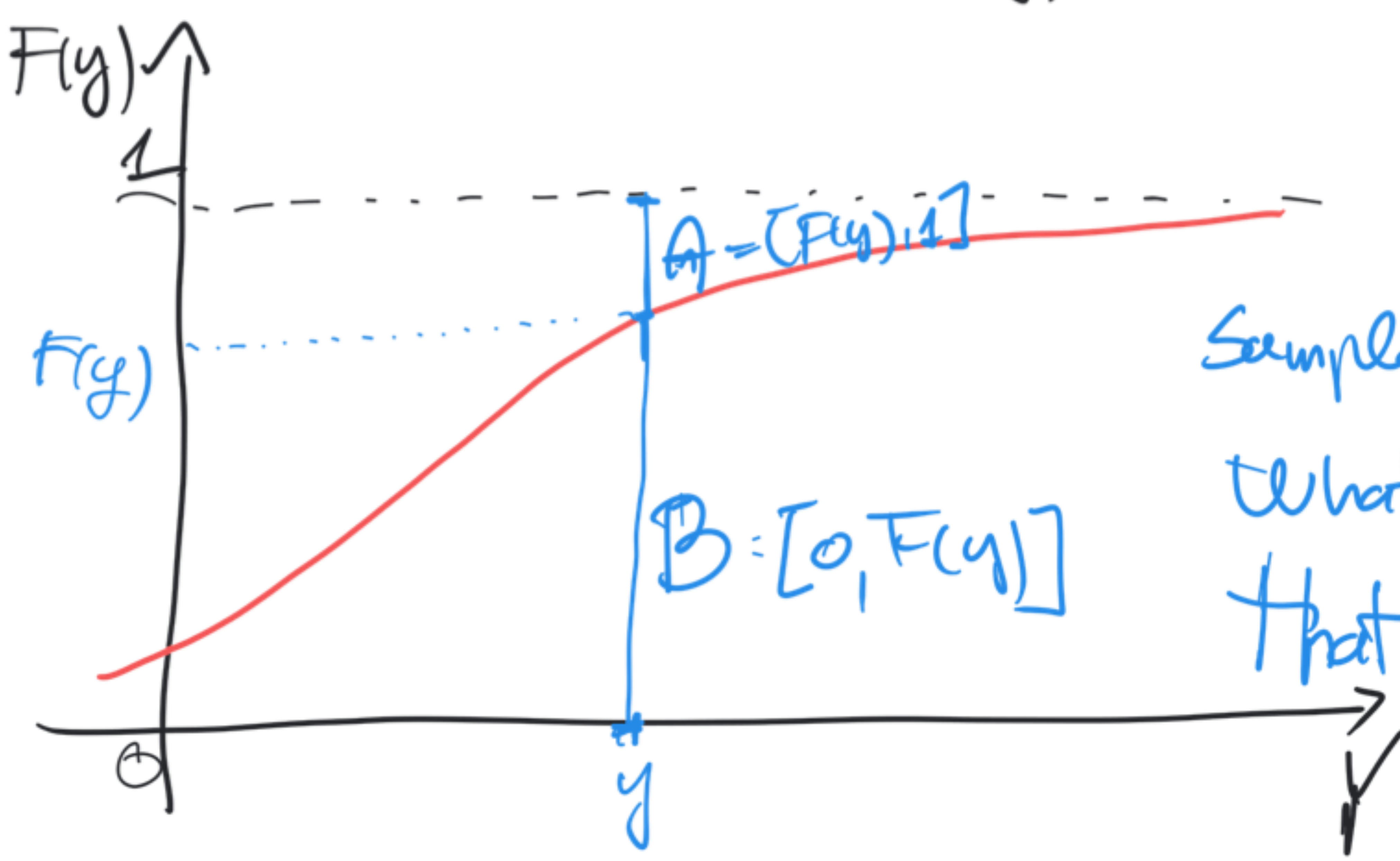
GENERATING RANDOM VARIABLES

STARTING POINT: PRNG $u \sim U(0,1)$ UNIFORM

(in fact $u \in \{0, \frac{1}{m}, \dots, \frac{m-1}{m}\}$ for $m \gg 0$

We have a RANDOM VARIABLE $Y \in \mathbb{R}$, we want to sample from Y

Suppose we know $F(y) = P(Y \leq y)$ c.d.f. $\left[F(y) = \int_{-\infty}^{y} p(y)\,dy\right]$



Sample $u \sim U(0,1)$

What is the probability that $u \in B$? It's $F(y)$!

$$P(\underbrace{U \in B}) = F(y)$$

$$\Updownarrow$$

$$P(U \leq F(y)) = F(y)$$

$$\Updownarrow$$

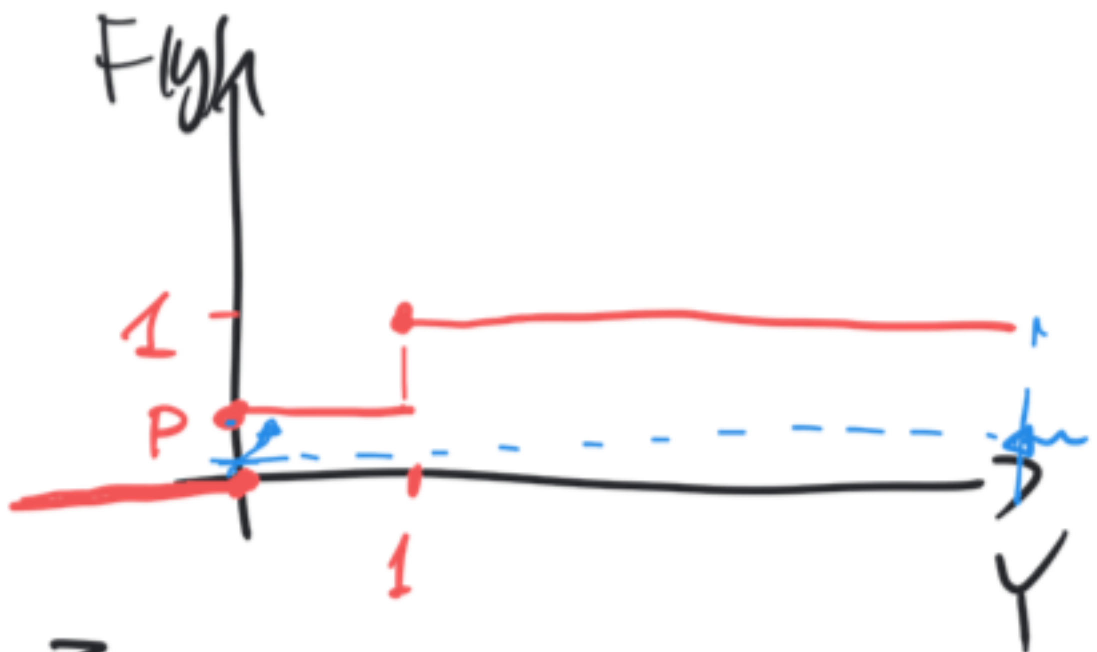
$$P(\underbrace{F^{-1}(U)} \leq y) = P(Y \leq y)$$

$\Rightarrow F^{-1}(U)$ and Y have the same distribution!

1) generate $u \sim U(0,1)$

2) compute $y = F^{-1}(u)$, then $y \sim Y$

$$Y=\begin{cases}0, & \text{con prob } p\\ 1, & \text{with prob } 1-p\end{cases}$$



If $u\in[0,p]$, then $\min\{y \mid F(y)\geq u\}=0$

if $u\in[p,1]$, then $F^{\leftarrow}(u)=1$

$$y=\min\{y \mid F(y)\geq u\}=F^{\leftarrow}(u)$$

Example: EXPONENTIAL DISTRIBUTION T, $p(x)=\lambda e^{-\lambda t}$, $T\sim Exp(\lambda)$, $T\geq 0$

$F(t)=1-e^{-\lambda t}$. To apply the INVERSION METHOD, we need to solve

$$u=1-e^{-\lambda t} \iff e^{-\lambda t}=1-u \iff -\lambda t=\log(1-u) \iff t=-\frac{1}{\lambda}\log(u)$$

(if $u\sim U(0,1)$, then $1-u\sim U(0,1)$)

$$T=-\frac{1}{\lambda}\log(U)$$

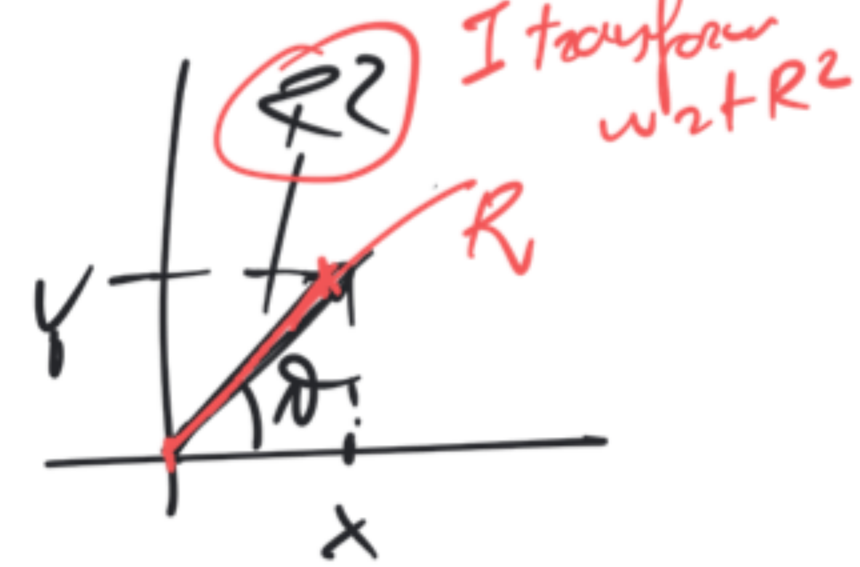
TRASFORMATA di BOX-MULLER

GEORGE BOX
"ALL MODELS ARE WRONG, BUT SOME ARE USEFUL"

(X, Y) two independent $\mathcal{N}(0,1)$, then $p(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$

$$p(x,y) = \frac{1}{2\pi} e^{-(x^2+y^2)/2}$$

Change of coordinates from CARTESIAN to POLAR (ϑ, R^2)



$$\begin{cases} R^2 = X^2 + Y^2 = d \\ \vartheta = \arctan \frac{Y}{X} \end{cases} \qquad J = \begin{pmatrix} \frac{\partial d}{\partial x} & \frac{\partial d}{\partial y} \\ \frac{\partial \vartheta}{\partial x} & \frac{\partial \vartheta}{\partial y} \end{pmatrix} = \dots \quad \det(J) = 2$$

$$p(d, \vartheta) = p(x,y) \cdot \det(J)^{-1} = \frac{1}{4\pi} \cdot e^{-d/2} = \underbrace{\frac{1}{2\pi}}_{\text{Unif}(0,2\pi)} \cdot \underbrace{\frac{1}{2} e^{-d/2}}_{\text{Exp}\left(\frac{1}{2}\right)}$$

$$\begin{cases} \vartheta \sim \text{Unif}(0, 2\pi) \\ R^2 \sim \text{Exp}\left(\frac{1}{2}\right) \end{cases}$$

sample U_1, U_2 and transform

$$\begin{cases} X = R\cos\vartheta = \sqrt{-2\log U_1} \cdot \cos(2\pi U_2) \\ Y = R\sin\vartheta = \sqrt{-2\log U_1} \sin(2\pi U_2) \end{cases}$$

$\leadsto$ Box-Muller transform!