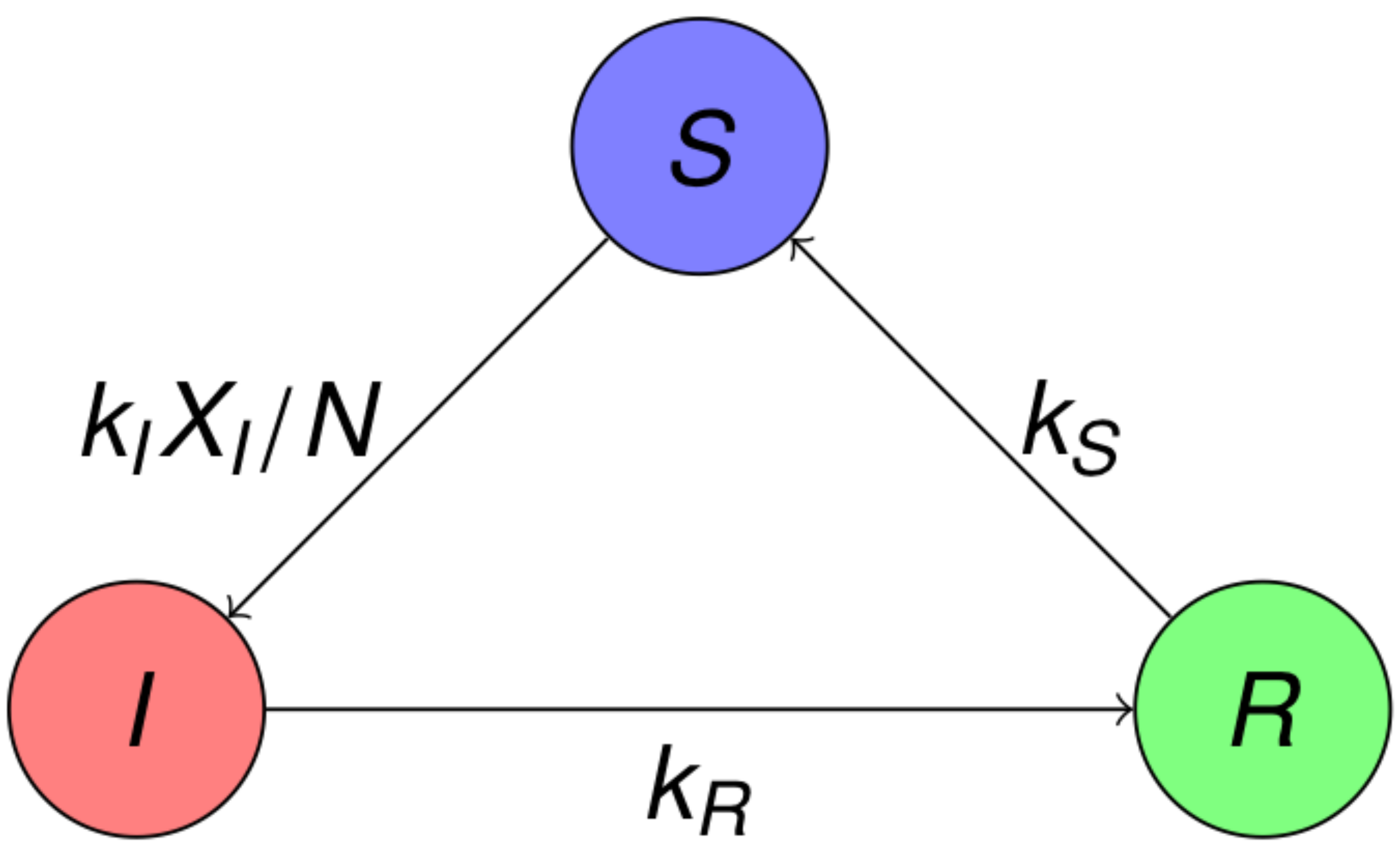
Outline

1. Continuous Time Markov Chains
   - Main concepts
   - Poisson Process
   - Time-inhomogeneous rates
2. Population Continuous Time Markov Chains
3. Simulation
   - SSA
   - Next Reaction Method
   - τ-leaping

Population Processes

$S + I \rightarrow I + I$

SIR epidemics model
single individual



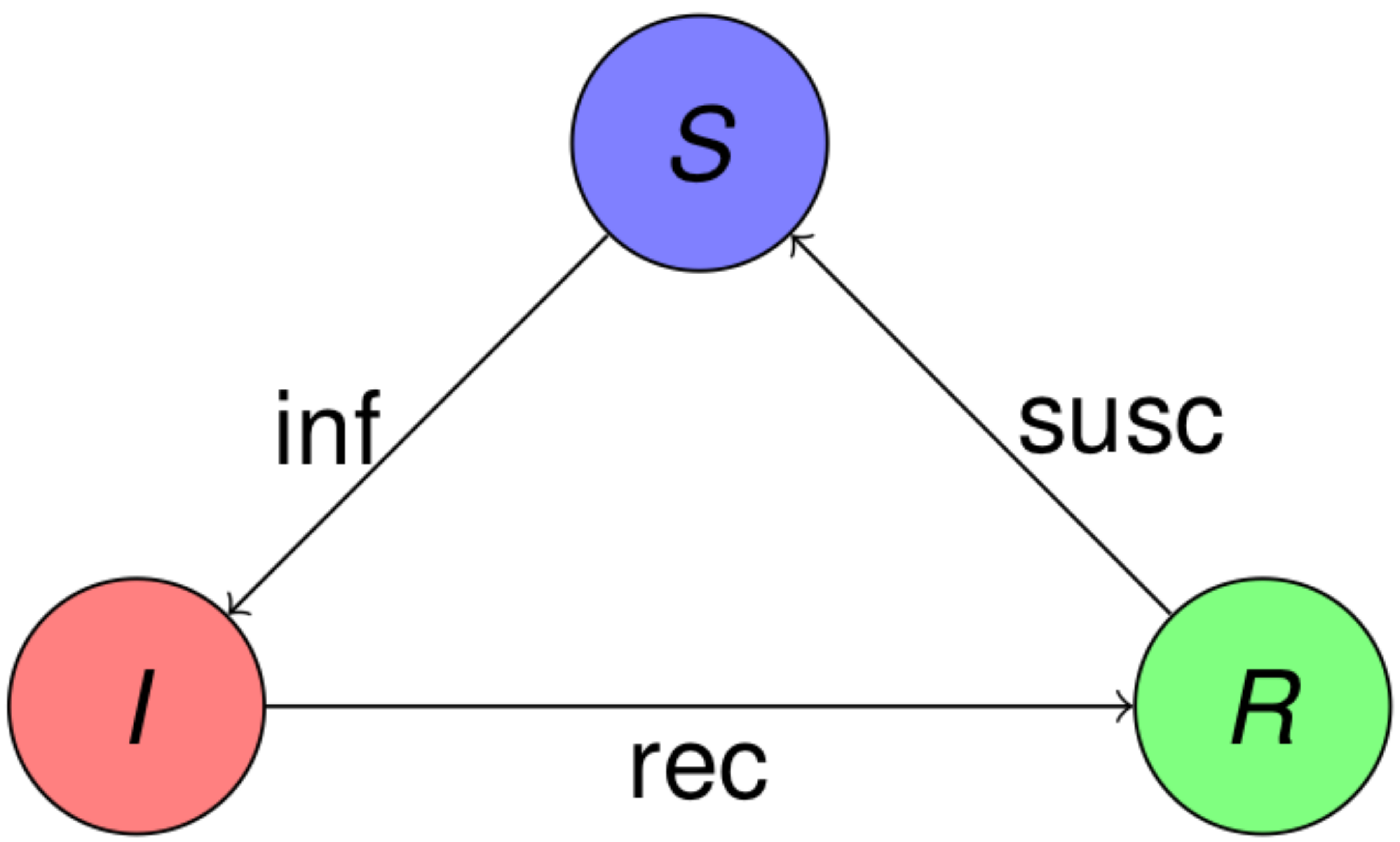
- Consider a CTMC model of a population epidemics in which each of N individuals can be in one of three states: susceptible (S), infected (I), and recovered (R);
- Infection rate depends on the density of infected individuals;
- The CTMC for N agents has 3^N states (if we distinguish the individuals) or $(N+1)^2$ states (if we just count them): *it's impossible to write down the Q matrix explicitly*.
- We need a better description of population CTMCs.

POPULATION CTMC

A population CTMC model is a tuple $\mathcal{X} = (\mathbf{X}, \mathcal{D}, \mathcal{T}, \mathbf{x_0})$, where:

1. $\mathbf{X}$ — vector of *variables* counting how many individuals in each state. $(X_S \; X_I \; X_R)$, $X_i \in \mathbb{N}$
2. $\mathcal{D} = \prod_i \mathcal{D}_i$ — (countable) state space.
3. $\mathbf{x_0} \in \mathcal{D}$ —*initial state*.
4. $\eta_i \in \mathcal{T}$ — *global transitions*, $\eta_i = (a, \phi(\mathbf{X}), \mathbf{v}, r(\mathbf{X}))$
   1. a — event name (optional).
   2. $\phi(\mathbf{X})$ — guard.
   3. $\mathbf{v} \in \mathbb{R}^n$ — *update vector* (from $\mathbf{X}$ to $\mathbf{X} + \mathbf{v}$)
   4. $r : \mathcal{D} \to \mathbb{R}_{\geq 0}$ — rate function.

EXAMPLE: SIR EPIDEMICS



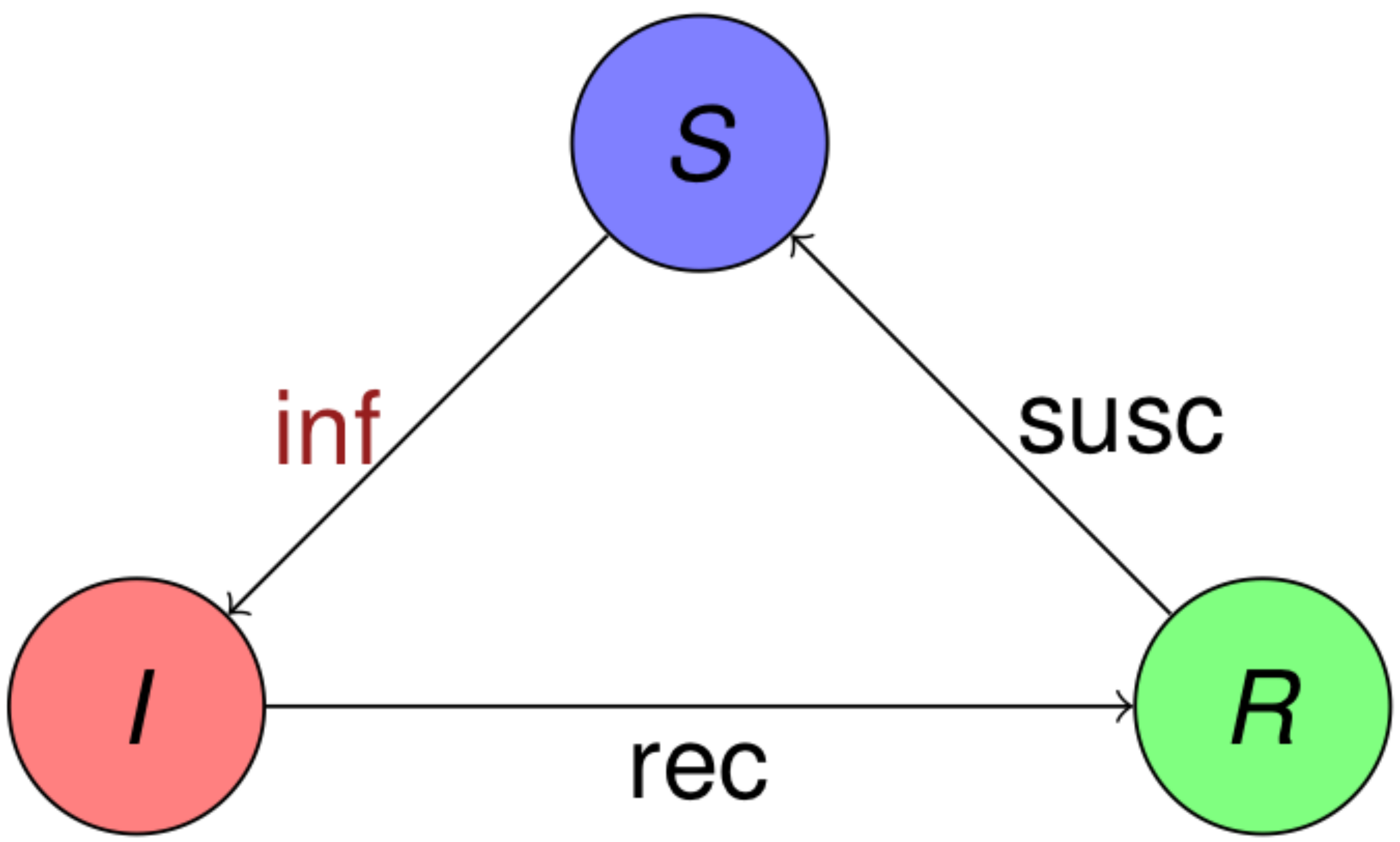
Three variables: X_S,X_I,X_R.
State space:
$\mathcal{D} = \{(n_1, n_2, n_3) \mid n_1 + n_2 + n_3 = N\} \subset \{0, \ldots, N\}^3$.

EXAMPLE: SIR EPIDEMICS

$\lambda_c \frac{X_I}{N} p_2$ K_F

$K_T \frac{X_I X_S}{N}$

$S + I \to I + I$



Three variables: X_S,X_I,X_R.
State space:
$\mathcal{D} = \{(n_1, n_2, n_3) \mid n_1 + n_2 + n_3 = N\} \subset \{0, \ldots, N\}^3$.

Transitions:

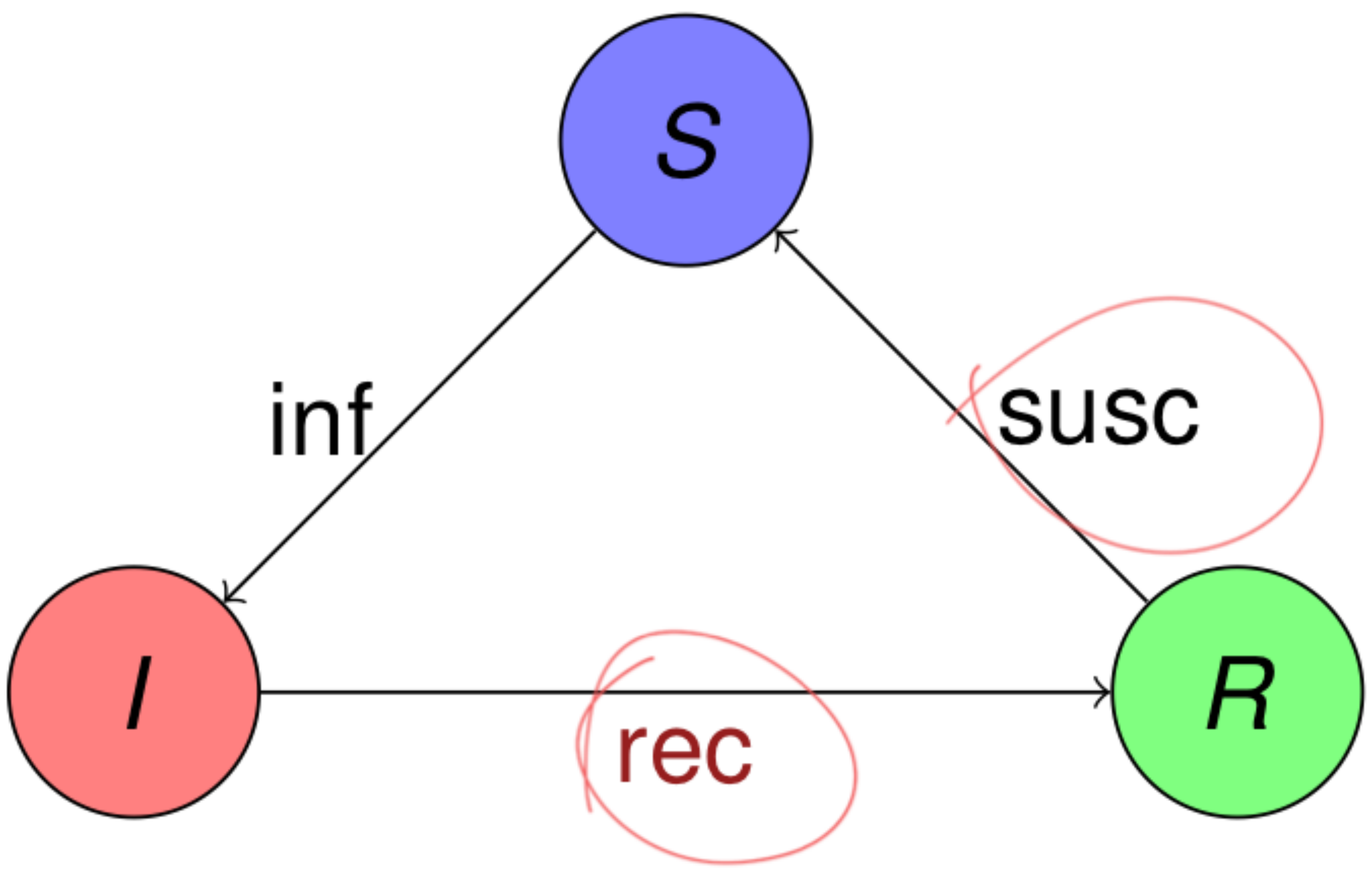
S I

- $(inf, \top, (-1, 1, 0) k_I \frac{X_I}{N} X_S)$

MASS ACTION

EXAMPLE: SIR EPIDEMICS

$I \to R \quad , \; k_R \cdot X_I$

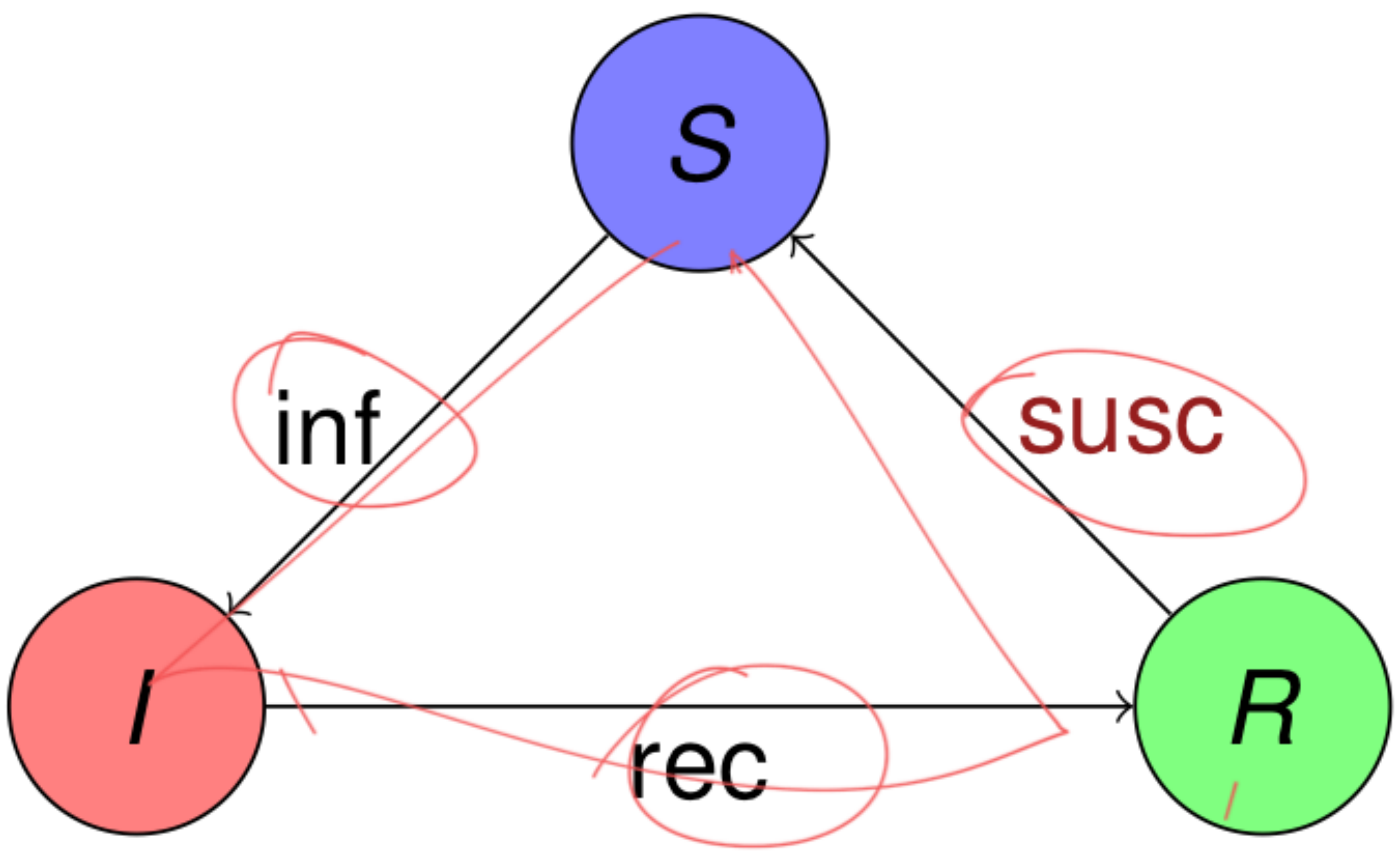


Three variables: X_S,X_I,X_R.
State space:
$\mathcal{D} = \{(n_1, n_2, n_3) \mid n_1 + n_2 + n_3 = N\} \subset \{0, \ldots, N\}^3$.

Transitions:

- $(inf, \top, (-1, 1, 0) k_I \frac{X_I}{N} X_S)$
- $(rec, \top, (0, -1, 1), k_R X_I)$

EXAMPLE: SIR EPIDEMICS

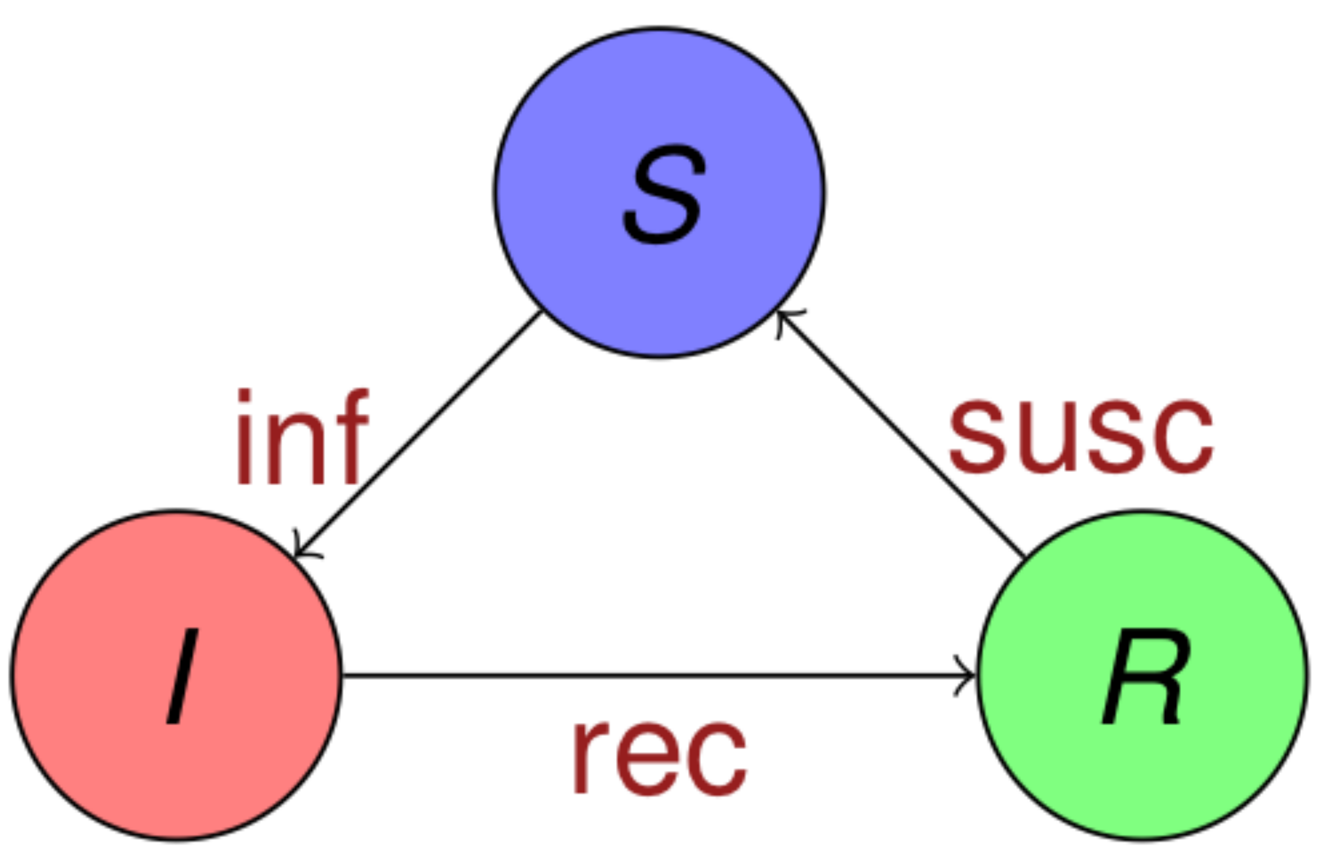


Three variables: X_S,X_I,X_R.
State space:
$\mathcal{D} = \{(n_1, n_2, n_3) \mid n_1 + n_2 + n_3 = N\} \subset \{0, \ldots, N\}^3$.

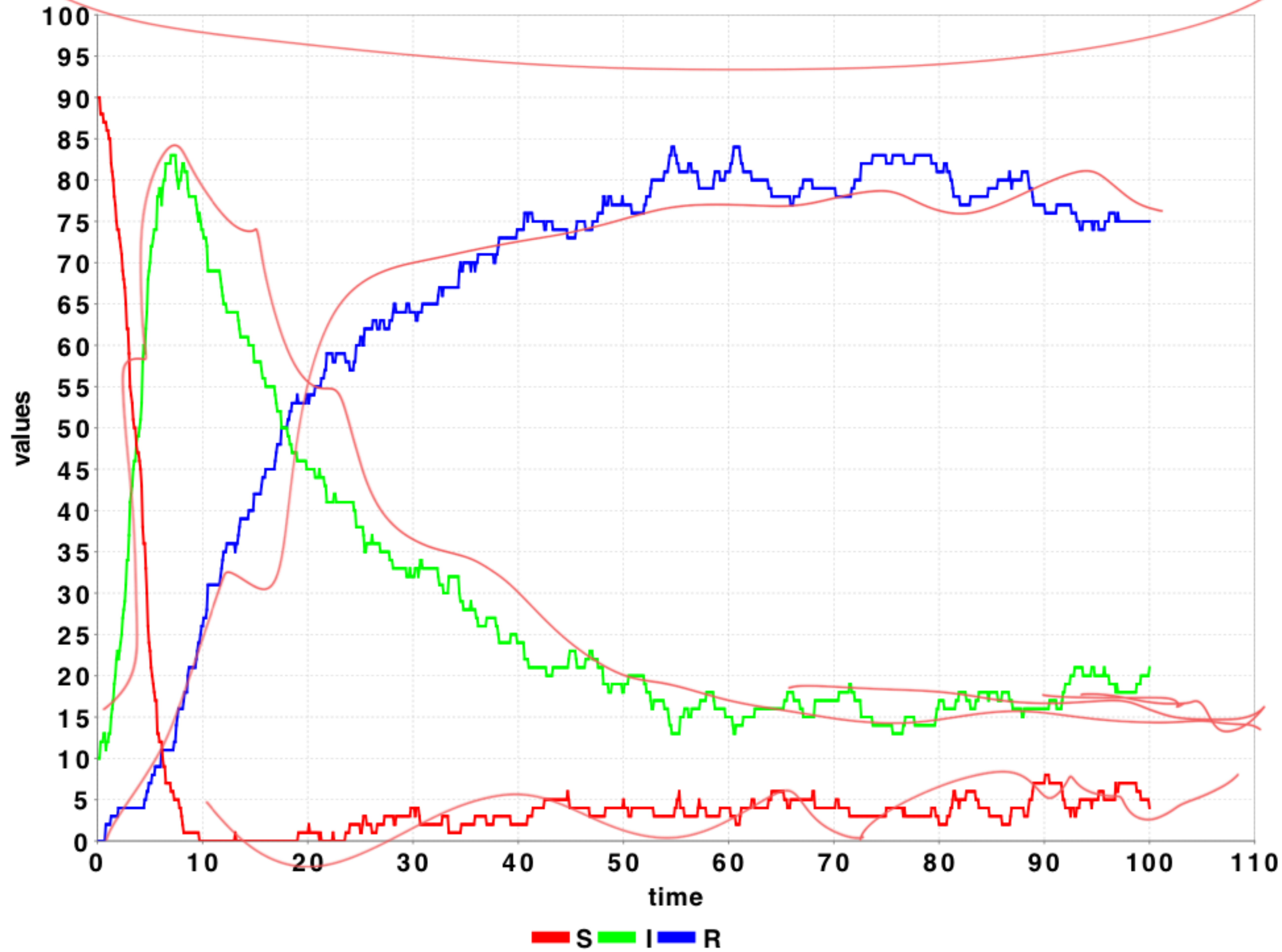
Transitions:

- $(inf, \top, (-1, 1, 0), k_I \frac{X_I}{N} X_S)$
- $(rec, \top, (0, -1, 1), k_R X_I)$
- $(susc, \top, (1, 0, -1), k_S X_R)$

EXAMPLE: SIR EPIDEMICS

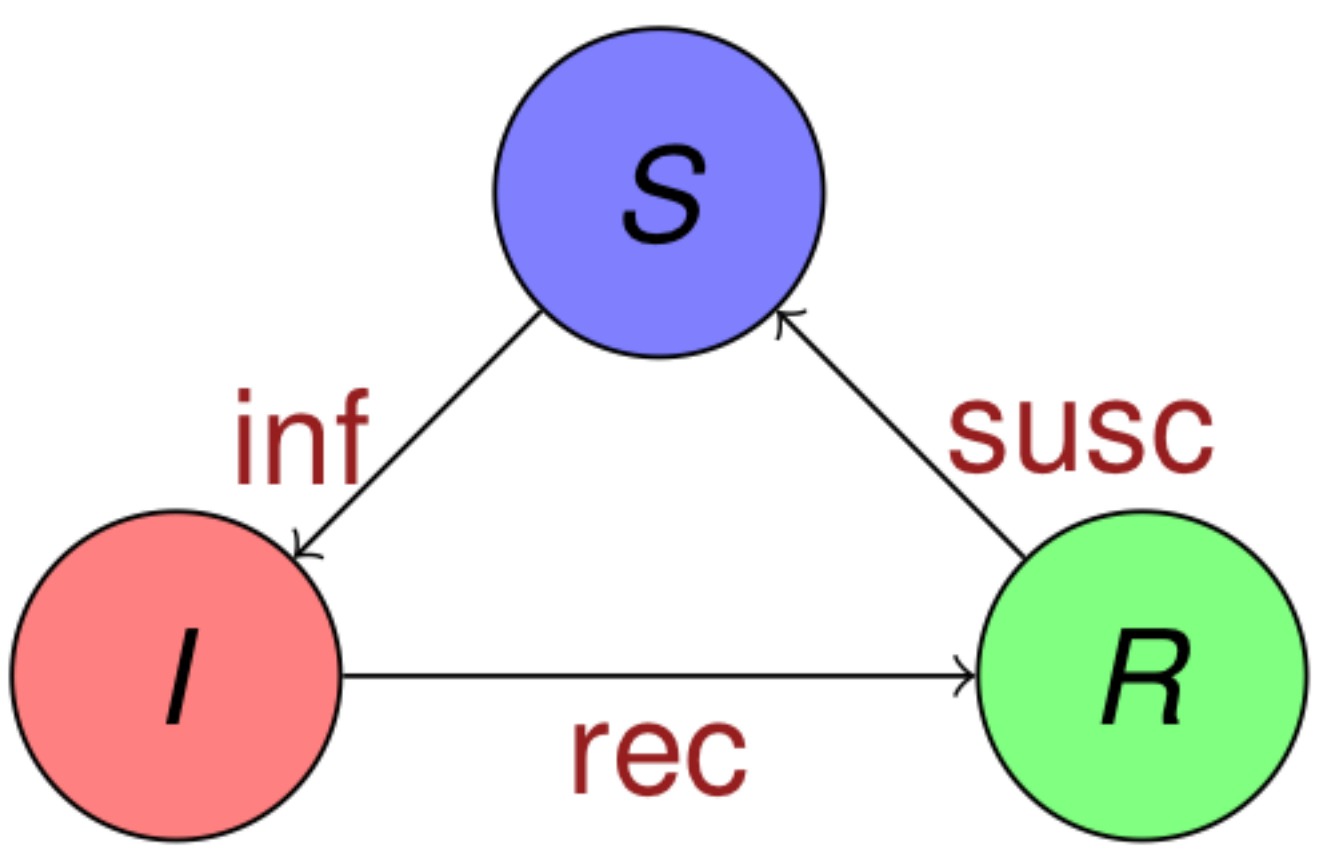


$N = 100,\ k_I = 1,\ k_R = 0.05,\ k_S = 0.01$



(1 run)

EXAMPLE: SIR EPIDEMICS



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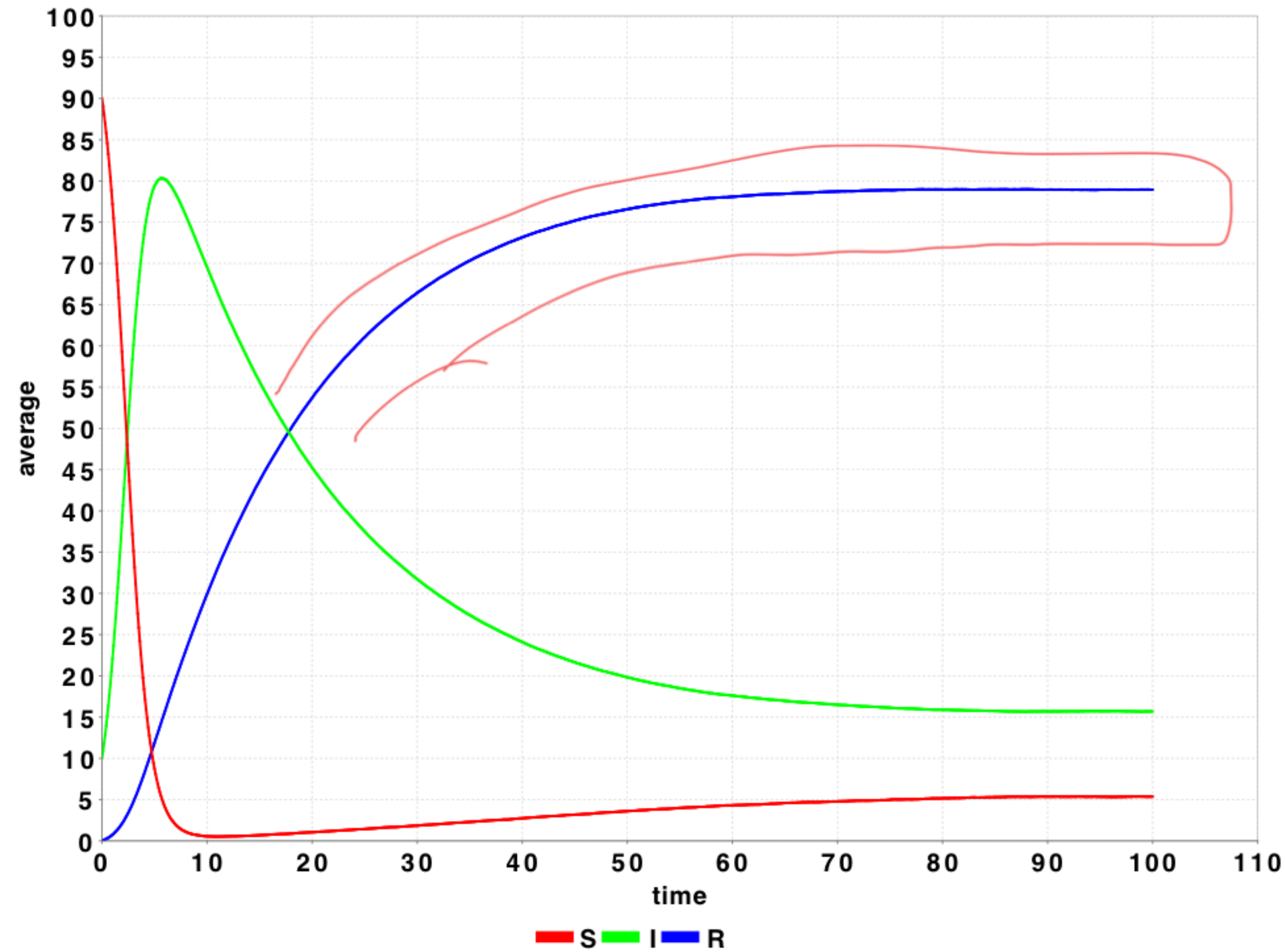

(average)

MASTER EQUATION

The Kolmogorov equation in the context of Population Processes is often know as master equation.

There is one equation per state $\mathbf{x} \in \mathcal{D}$, for the probability mass $P(\mathbf{x}, t)$, which considers the inflow and outflow of probability at time t.

$x - v_\eta \to x$

$$\frac{dP(\mathbf{x},t)}{dt} = \underbrace{\sum_{\eta \in \mathcal{T}} r_\eta(\mathbf{x} - \mathbf{v}_\eta) P(\mathbf{x} - \mathbf{v}_\eta, t)}_{\text{INFLOW}} - \underbrace{\sum_{\eta \in \mathcal{T}} r_\eta(\mathbf{x}) P(\mathbf{x}, t)}_{\text{OUTFLOW}}$$

POISSON REPRESENTATION

Population CTMC admit a simple description in terms of Poisson processes.