

OUTLINE

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TIME-INHOMOGENEOUS EXPONENTIAL

A exponential random variable $T \sim \text{Exp}(\lambda)$ has time-inhomogeneous rate iff $\lambda = \lambda(t)$ is a function $\lambda : [0, \infty[\rightarrow \mathbb{R}^+$.

• **Cumulative rate** is $\Lambda(t) = \int_0^t \lambda(s) ds$

• Cdf is $\mathbb{P}(T < t) = 1 - e^{-\Lambda(t)}$

• Survival probability is $\mathbb{P}(T > t) = e^{-\Lambda(t)}$

density $p(t) = \lambda(t) e^{-\Lambda(t)}$
 (se $\lambda = c$ constant $\Lambda(t) = c \cdot t$
 $e^{-\Lambda(t)} = u \Rightarrow t = -\frac{1}{c} \log(u)$
 $\mathbb{P}(T > t) = u \sim \text{unif}(0, 1)$

To sample a time-inhomogeneous by inversion method $\text{Exp}(\lambda(t))$, one has to solve $e^{-\Lambda(t)} = U$, iff $\Lambda(t) = -\log U = \xi$, with $\xi \sim \text{Exp}(1)$.

If λ is constant, then $\Lambda(t) = \lambda t$, and one has $t = -\frac{1}{\lambda} \log(U)$.

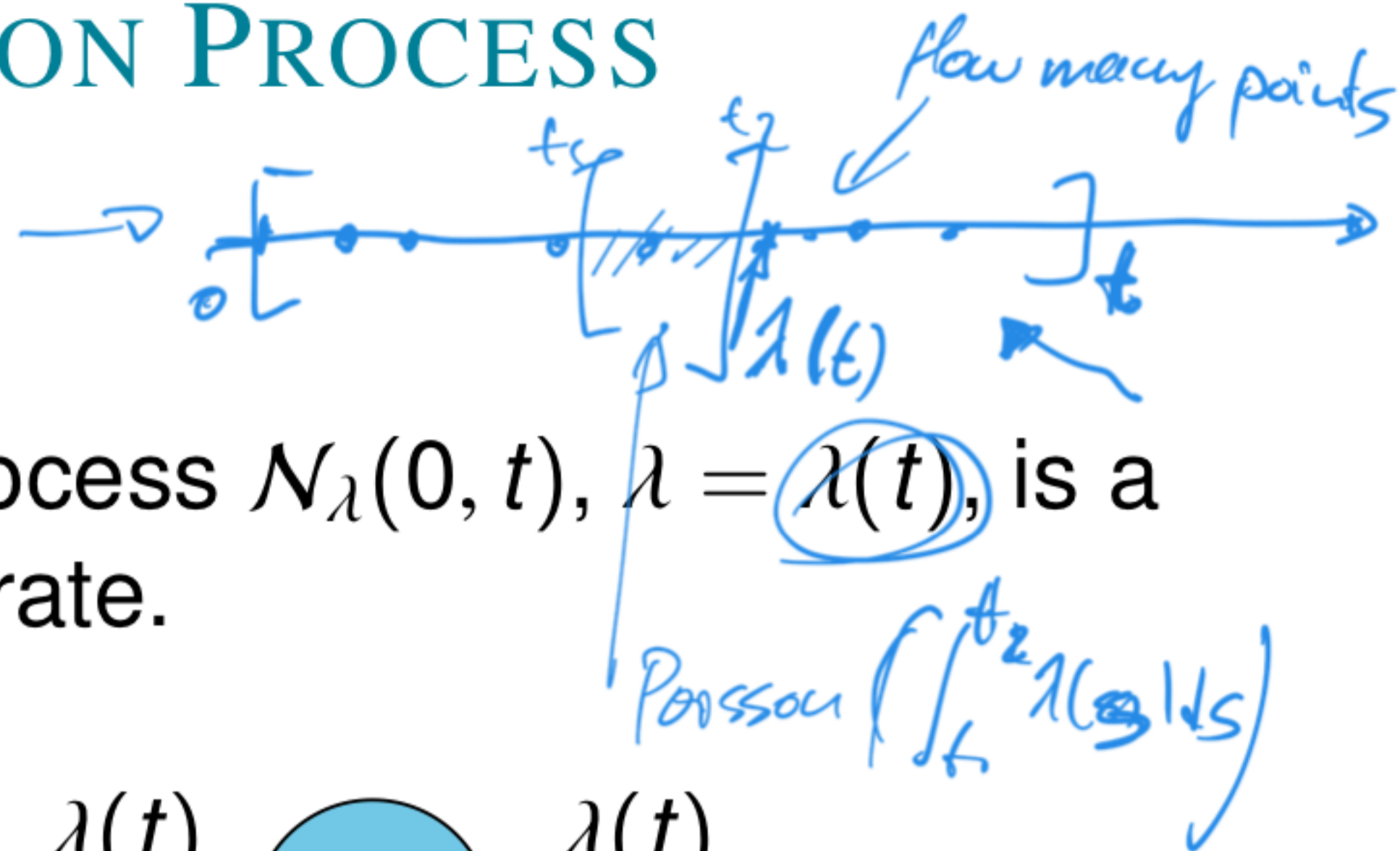
In general, one can either integrate $\lambda(t)$ or the equivalent ODE $\frac{d\Lambda(t)}{dt} = \lambda(t)$, and check for the root of $\Lambda(t) + \log(U)$ along the solution.

$$\Lambda(t) = \xi \quad \left| \quad t = \Lambda^{-1}(\xi) \quad \left| \quad \Lambda(t) - \xi = 0$$

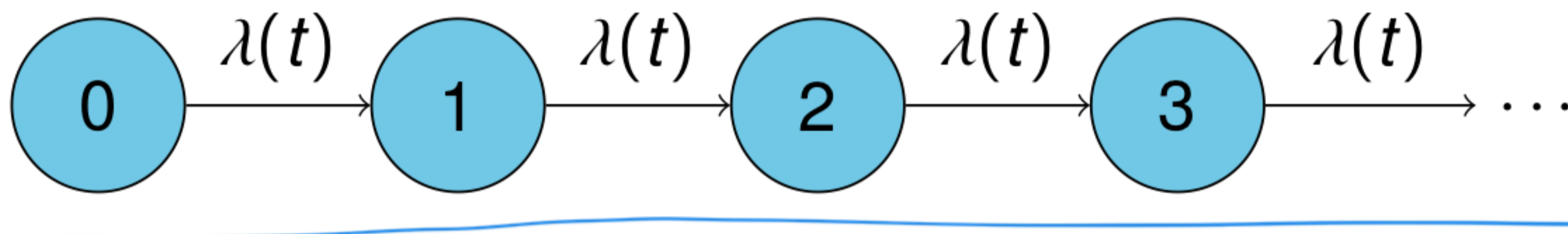
$$-\log U = \xi \Rightarrow t = -\frac{1}{\lambda} \xi$$

TIME-INHOMOGENEOUS POISSON PROCESS

POISSON
POINT
PROCESS



A time-inhomogeneous Poisson process $\mathcal{N}_\lambda(0, t)$, $\lambda = \lambda(t)$, is a Poisson process with time-varying rate.



It can be shown (same generating function argument as above) that the distribution of $\mathcal{N}_\lambda(0, t)$ is $Poisson(\Lambda(t))$, i.e. it is the r.v.

$$\mathcal{Y}(\Lambda(t)) = \mathcal{Y}\left(\int_0^t \lambda(s) ds\right).$$

$\Lambda(t)$

TIME-INHOMOGENEOUS CTMC



$$P(t_1, t_2) = P(t_1, \tau) \cdot P(\tau, t_2)$$

$\tau \in (t_1, t_2)$

In general, if the rate matrix Q of a CTMC depends on time, $Q = Q(t)$, then the CTMC is time inhomogeneous.

The probability semigroup depends now also on the initial time:

$$P_{ij}(t_1, t_2) = \mathbb{P}\{X(t_2) = s_j \mid X(t_1) = s_i\}.$$

FORWARD KOLMOGOROV EQUATION

$$\frac{\partial P(t_1, t_2)}{\partial t_2} = P(t_1, t_2) Q(t_2)$$

BACKWARD KOLMOGOROV EQUATION

$$\frac{\partial P(t_1, t_2)}{\partial t_1} = -Q(t_1) P(t_1, t_2)$$

POISSON REPRESENTATION PCTMC

Population CTMC admit a simple description in terms of Poisson processes.

Essentially, we introduce variables $R_\eta(t)$ counting how many times each transition η has fired up to time t . Hence we can write:

$$X(t) = X(0) + \sum_{\eta \in \mathcal{T}} \mathbf{v}_\eta R_\eta(t)$$

It turns out that $R_\eta(t)$ is a **time-inhomogeneous Poisson process** with cumulative rate $\int_0^t r_\eta(X(s)) ds$, independent from the other $R_{\eta'}$. Hence, let \mathcal{N}_η be independent Poisson processes. For each $t \geq 0$:

$$X(t) = X(0) + \sum_{\eta \in \mathcal{T}} \mathbf{v}_\eta \mathcal{N}_\eta \left(\int_0^t r_\eta(X(s)) ds \right)$$

Equivalently, let \mathcal{Y}_η be independent Poisson r.v. It holds:

$$X(t) = X(0) + \sum_{\eta \in \mathcal{T}} \mathbf{v}_\eta \mathcal{Y}_\eta \left(\int_0^t r_\eta(X(s)) ds \right)$$