Outline

TIME-INHOMOGENEOUS EXPONENTIAL

A exponential random variable $T \sim Exp(\lambda)$ has time inhomogeneous rate iff $\lambda = \lambda(t)$ is a function $\lambda : [0, \infty[\to \mathbb{R}^+$.

- Cumulative rate is $\Lambda(t) = \int_0^t \lambda(s)ds$
- Cdf is $\mathbb{P}(T < t) = 1 - e^{-\Lambda(t)}$
- Survival probability is $\mathbb{P}(T > t) = e^{-\Lambda(t)}$

density $p(t) = \lambda(t) e^{-\Lambda(t)}$

$\to$ (se $\lambda \equiv c$ constant $\Lambda(t) = c \cdot t$

$\underbrace{e^{-\Lambda(t)}}_{P(T>t)} = u \Rightarrow t = -\frac{1}{\lambda} \log(u)$

$P(T>t) = u \sim unif(0,1)$

To sample a time-inhomogeneous by inversion method $Exp(\lambda(t))$, one has to solve $e^{-\Lambda(t)} = U$, iff $\Lambda(t) = -\log U = \xi$, with $\xi \sim Exp(1)$.

$-\log U = \xi \Rightarrow t = \frac{1}{\lambda} \xi$

If λ is constant, then $\Lambda(t) = \lambda t$, and one has $t = -\frac{1}{\lambda} \log(U)$.

In general, one can either integrate $\lambda(t)$ or the equivalent ODE $\frac{d\Lambda(t)}{dt} = \lambda(t)$, and check for the root of $\Lambda(t) + \log(U)$ along the solution.

$\underline{\Lambda(t) = \xi}$ "$t = \Lambda^{-1}(\xi)$"

$\Lambda(t) - \xi = 0$

TIME-INHOMOGENEOUS POISSON PROCESS

POISSON POINT PROCESS

How many points

t_1 t_2 0 t $\lambda(t)$

A time-inhomogeneous Poisson process $\mathcal{N}_\lambda(0,t)$, $\lambda = \lambda(t)$, is a Poisson process with time-varying rate.

$\mathrm{Poisson}\left(\int_{t_1}^{t_2} \lambda(s)\,ds\right)$

$$0 \xrightarrow{\lambda(t)} 1 \xrightarrow{\lambda(t)} 2 \xrightarrow{\lambda(t)} 3 \xrightarrow{\lambda(t)} \cdots$$

It can be shown (same generating function argument as above) that the distribution of $\mathcal{N}_\lambda(0,t)$ is $\boxed{\mathit{Poisson}(\Lambda(t))}$, i.e. it is the r.v.

$$\mathcal{Y}(\Lambda(t)) = \mathcal{Y}\left(\underbrace{\int_0^t \lambda(s)ds}_{\Lambda(t)}\right).$$

TIME-INHOMOGENEOUS CTMC

$P(t_1,t_2) = P(t_1,\tau)\cdot P(\tau,t_2)$

$\tau \in (t_1,t_2)$

In general, if the rate matrix Q of a CTMC depends on time, $Q = Q(t)$, then the CTMC is time inhomogeneous.

The probability semigroup depends now also on the initial time: $P_{ij}(t_1, t_2) = \mathbb{P}\{X(t_2) = s_j \mid X(t_1) = s_i\}$.

FORWARD KOLMOGOROV EQUATION

$$\frac{\partial P(t_1,t_2)}{\partial t_2} = P(t_1,t_2)Q(t_2)$$

BACKWARD KOLMOGOROV EQUATION

$$\frac{\partial P(t_1,t_2)}{\partial t_1} = -Q(t_1)P(t_1,t_2)$$

POISSON REPRESENTATION

PCTMC

Population CTMC admit a simple description in terms of Poisson processes.
Essentially, we introduce variables $R_\eta(t)$ counting how many times each transition η has fired up to time t. Hence we can write:

$$\boxed{X(t) = X(0) + \sum_{\eta \in \mathcal{T}} \mathbf{v}_\eta R_\eta(t).}$$

It turns out that $R_\eta(t)$ is a time-inhomogeneous Poisson process with cumulative rate $\int_0^t r_\eta(X(s))ds$, independent from the other $R_{\eta'}$.
Hence, let $\mathcal{N}_\eta$ be independent Poisson processes. For each $t \geq 0$:

$$X(t) = X(0) + \sum_{\eta \in \mathcal{T}} \mathbf{v}_\eta \mathcal{N}_\eta \left(\int_0^t r_\eta(X(s))ds \right).$$

Equivalently, let $\mathcal{Y}_\eta$ be independent Poisson r.v. It holds:

$$\boxed{X(t) = X(0) + \sum_{\eta \in \mathcal{T}} \mathbf{v}_\eta \mathcal{Y}_\eta \left(\int_0^t r_\eta(X(s))ds \right).}$$