The Ising model in the canonical ensemble

- Introduction to the Ising model
- The Ising model in the canonical ensemble: application of Metropolis Monte Carlo Method
- Implementation in a code

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stochastic simulations of an interacting spin ensemble

We need a model for:

- energy

- evolution
  (to build the trajectory / the Markov chain)
stochastic simulations of an interacting spin ensemble

We need a model for:

- **energy** => Ising model

- **evolution**
  (to build the trajectory / the Markov chain)
Ising model

the simplest model of interacting spin on a lattice

\[ \mathcal{H}^{\text{spin}} = -J \sum_{i,j=1}^{N} s_i s_j \]

\[ s_i = \pm 1 \]

(next: where does it come from? ...)
2 interacting spins

Consider two spins $1/2$, $\vec{s}_1$, $\vec{s}_2$ (e.g. electrons in He), their sum $\vec{S}$, the basis $|S, S_z>$ of the Hilbert space in the coupled representation, and a hamiltonian $\mathcal{H} = h_1 + h_2 + V_{12}$:

The Pauli principle => the energy is affected by spin even if $\mathcal{H}$ does not depend explicitly on it:

$$<S = 0|\mathcal{H}|S = 0> = E_s$$ singlet
$$<S = 1|\mathcal{H}|S = 1> = E_t$$ triplet

We have:

$$<S = 0|\mathcal{H}|S = 0> - <S = 1|\mathcal{H}|S = 1> = E_s - E_t$$
2 interacting spins

Idea: write a model hamiltonian explicitly dependent on the spin giving the same energy difference between the eigenvalues of its eigenstates as the original one.

Consider the operator:

\[ \sum_{12} = s_1 \cdot s_2 = \frac{1}{2} S^2 - \frac{3}{4} \]

which is diagonal on the coupled basis, with eigenvalues:

\[ \langle S = 0 | \sum_{12} | S = 0 \rangle = -\frac{3}{4}, \quad \langle S = 1 | \sum_{12} | S = 1 \rangle = \frac{1}{4} \]

Consider then:

\[ \mathcal{H}^{\text{spin}} = -(E_s - E_t) \sum_{12} \]

We have:

\[ \langle S = 0 | \mathcal{H}^{\text{spin}} | S = 0 \rangle - \langle S = 1 | \mathcal{H}^{\text{spin}} | S = 1 \rangle = E_s - E_t \]
Heisemberg hamiltonian

\[ H^{spin} = -(E_s - E_t) \Sigma_{12} \]

is therefore OK! Defining: \( J \equiv E_s - E_t \), we have:

\[ H^{spin} = -J \vec{s}_1 \cdot \vec{s}_2 \]

\( J > 0 \) (\( E_s > E_t \)) \uparrow\uparrow \text{ spins favored } \Rightarrow \text{ ferromagnetic case} \n
\( J < 0 \) (\( E_s < E_t \)) \uparrow\downarrow \text{ spins favored } \Rightarrow \text{ antiferromagnetic case} \
Heisemberg Hamiltonian

Extension to the case of several spins:

\[ H^{spin} = - \sum_{i,j=1}^{N} J_{ij} \vec{S}_i \cdot \vec{S}_j \]
Ising model

Consider only the possibility: \( s_i = \pm 1 \) and nearest neighbor interaction only, with the same interaction constant \( J \)

\[
H_{\text{spin}} = -J \sum_{i,j=1}^{N} s_i s_j
\]

For \( J > 0 \) (\( E_s > E_t \)) \( \uparrow \uparrow \) spins favored \( \Rightarrow \) ferromagnetic case

For \( J < 0 \) (\( E_s < E_t \)) \( \uparrow \downarrow \) spins favored \( \Rightarrow \) antiferromagnetic case

\( E = -J \) \hspace{1cm} E = +J

(choosing the kind of interaction, we specify the energy model)
Ising model on a lattice

lattice containing $N$ sites

lattice site $i$ has associated with it a number $s_i$, where $s_i = +1$ for an “up” (↑) spin and $s_i = -1$ for a “down” (↓) spin. A particular configuration or microstate of the lattice is specified by the set of variables $\{s_1, s_2, \ldots, s_N\}$ for all lattice sites.

Lowest energy states of the 2D Ising model on a square lattice with ferromagnetic ($J>0$) and antiferromagnetic ($J<0$) interactions. Solid and open circles correspond to +1 and -1 spins, respectively.
Ising model: interesting quantities
Ising model: energy

lattice containing $N$ sites
No external magnetic field:

$$E = -J \sum_{i,j=\text{nn}(i)}^{N} S_i S_j$$

(nn=nearest neighbor)

Energy in presence of an external magnetic field:

$$E = -J \sum_{i,j=\text{nn}(i)}^{N} S_i S_j - H \sum_{i=1}^{N} S_i,$$

or, better, define an average energy per spin: $E/N$
Ising model: magnetization

\[ M = \sum_{i=1}^{N} s_i \]

“Order parameter”: total magnetization, or -better- average magnetization per spin:

\[ m = \frac{M}{N} = \frac{1}{N} \sum_{i=1}^{N} s_i \]

\[-1 \leq m \leq +1\]
Ising model: configurations and energy

$2^n$ different configurations for $n$ spins.

e.g.  $2^4 = 16$ spin configurations for $2 \times 2$ lattice

\begin{align*}
+ & + & - & - \\
+ & + & - & - \\
- & + & + & + \\
+ & + & + & - \\
- & - & - & - \\
+ & - & + & + \\
+ & - & + & + \\
- & - & - & - \\
- & - & - & - \\
+ & + & + & + \\
- & - & - & - \\
- & - & - & - \\
\end{align*}

For $J > 0$ the state of lowest energy is when all the spins are aligned. The state has macroscopic magnetization (ferromagnetic).

The ground state energy per spin (ferromagnetic case, thermodynamic limit (N large), no ext. field) is:

$$E_0/N = -2J$$
Ising model: dynamics?

Beside an **energy model**, we must define a **dynamics** in order to simulate the evolution of the system (i.e. to generate the trajectory in the phase space, to generate the configurations of the Markov chain).
Ising model: spin flip dynamics

Consider \( nn \) interactions, choose a random spin and flip it: it’s a new configuration (a microstate)

Apply **Metropolis Monte Carlo** method for evolution in the **canonical ensemble** (fix \( T \)). Evolution is driven by the **energy change** between the old and the new configuration, \( \Delta E \).

Remark: Is it sufficient to calculate only \( \Delta E \), not \( E \) at each new configuration!
Ising model: spin flip dynamics

The five possible transitions of the Ising model on the square lattice with spin flip
Ising model: boundary conditions

Of course we cannot simulate an infinite system (the thermodynamic limit). We have two choices for the simulation cell:

- free (open) boundary conditions
- periodic boundary conditions (PBC)
Energy per spin in the ground state converges to the value 
\[ E_0/N = -2J \] in the thermodynamic limit with behavior \( \sim 1/L \) in case of free boundaries.
The energy is a $2N$-term sum: each spin interacts with its NN within the simulation cell or with the NN images.

One of the $2^N$ possible configurations of a system of $N = 16$ Ising spins on a square lattice with periodic boundary conditions.
Ising model: PBC

We have always:

$$2^4 = 16$$ spin configurations for 2x2 lattice

but the energy for each configuration in case of free boundary conditions and PBC is different:

Two different configurations with 2 spins up

Energy per spin in the ground state is always equal to the value

$$E_0/N = -2J$$ in the thermodynamic limit
Ising model: phase transition

Low $T$: spin configuration minimizes energy
(if $J>0$: spins tend to align $\Rightarrow$ high (absolute) magnetization)

High $T$: spin configuration maximizes entropy
(=disorder) (spins tend to disalign $\Rightarrow$ low magnetization)

existence of a
Critical temperature $T_c$

in 2D the model has an analytical solution:

$T_c = 2.269 \text{ J}/\text{k}_B$
Apply **Metropolis Monte Carlo** method for evolution in the **canonical ensemble** (fix T):

![Magnetization plot](image)

*Magnetization* *(System : 10*10*)

Magnetization as a function of time for $T<T_c$ (solid) and $T>T_c$ (dashed curve)

Fluctuations! do, as usual, **temporal averages**: $\langle M \rangle/N$, $\langle E \rangle/N$
Magnetisation as a function of the temperature for the 2D Ising model.
Magnetization distribution for $T<T_c$ (solid) and $T>T_c$ (dashed curve)

\[ P(M) \]

(data collected during time evolution, at equilibrium)
Ising model: phase transition

\[ T_c \] also for energy, not only for magnetization:

\[ \frac{\langle E \rangle}{E_0} \]

\[ E = -J \sum_{i,j=nn(i)}^N s_i s_j \]

and also the energy fluctuates during time evolution...
**Ising model: fluctuations**

Fluctuations are intrinsic to the system evolution and are important!
Linear response functions are related to equilibrium fluctuations:

(already proved): \( C = \frac{\partial \langle E \rangle}{\partial T}, \ C = \frac{1}{kT^2} \left( \langle E^2 \rangle - \langle E \rangle^2 \right) \)

but also: \( \chi = \lim_{H \to 0} \frac{\partial \langle M \rangle}{\partial H}, \ \chi = \frac{1}{kT} \left( \langle M^2 \rangle - \langle M \rangle^2 \right) \)

where \( \langle M \rangle \) and \( \langle M^2 \rangle \) are evaluated in zero magnetic fields.
Ising model:
fluctuations and phase transition

Rapid change in $\langle E \rangle$ and $\langle M \rangle \Rightarrow$ singularities in $C$ and $\chi$

(Large fluctuations near the phase transition:
Second Order phase transition)

specific heat:

$$C = \frac{\partial \langle E \rangle}{\partial T}$$

magnetic susceptibility:

$$\chi = \lim_{H \rightarrow 0} \frac{\partial \langle M \rangle}{\partial H}$$
Implementing
the Ising model
in the code
Implementing the Ising model

on a 2D square lattice in the canonical ensemble
zero-field, nearest neighbor interactions only

\[ \mathcal{H}^{spin} = -J \sum_{i,j=1}^{N} s_i s_j \quad s_i = \pm 1 \]

Input parameters are:

- L (linear lattice dimension, which gives the number of spins: N=L*L)
- \textit{nmcs} (number of total MC steps per spin)
- \textit{nequil} (number of equilibration MC steps per spin)
- T (temperature of the thermal bath).
Implementing the Ising model

program ising
  ! metropolis algorithm for the ising model on a square lattice
  use common
  integer :: imcs, ispin, jspin
  real (kind = double), dimension(5) :: cum
  call initial(nequil,cum)
  ! equilibrate system
  do imcs = 1, nequil
    call metropolis()
  end do
  ! accumulate data while updating spins
  do imcs = 1, nmcs
    call metropolis()
    call data(cum)
  end do
  call output(cum)
  ...... 
  ...... 
end program ising
Ising model on a lattice

$L : \text{linear lattice dimension}$

$N = L \times L : \text{number of spins}$

**a configuration** (a microstate) is the whole sequence of spins, i.e. the $L \times L$ array $\text{spin}(x,y)$

```fortran
module common
.....
   integer, public, dimension(:,,:), allocatable :: spin
.....
endmodule
```

```fortran
subroutine initial(nequil,cum)
.....
   allocate(spin(L,L))
.....
   spin(x,y) = 1
   else
      spin(x,y) = -1
.....
end subroutine
```

$s_i = \pm 1$
**Ising model: magnetization**

Total magnetization, or define an average magnetization per spin:

\[
M = \sum_{i=1}^{N} s_i
\]

\[
m = \frac{M}{N} = \frac{1}{N} \sum_{i=1}^{N} s_i
\]

\[-1 \leq m \leq +1\]

(Instead of the loop over \(x,y\), do also simply: \(M=\text{sum}(\text{spin})\))
Ising model: energy

\[ E = -J \sum_{i,j=\text{nn}(i)}^{N} s_i s_j \]

Here, \( s_i \) and \( s_j \) represent the spin variables at sites \( i \) and \( j \), respectively. The sum is over all nearest-neighbor pairs in the lattice.

In code:

```plaintext
! compute initial energy
E = 0.0_double
do y = 1,L
   ....
   do x = 1,L
      ....
      sums = spin(x,up) + spin(right,y)
   end do
   ! calculate the initial energy summing all over pairs
   ! (for a given spin, consider only the up NN and the right NN
   ! - NOT the down and the left NN - : each interaction is counted once
   E = E - spin(x,y)*sums
   end do
end do
```
Ising model: energy with PBC

do y = 1,L
   ! periodic boundary conditions
   if (y == L) then
      up = 1
   else
      up = y + 1
   end if
   do x = 1,L
      ! calculate the initial energy summing all over pairs
      ! (for a given spin, consider only the up NN and the right NN
      ! - NOT the down and the left NN - : each interaction is counted once
      !   E = E - spin(x,y)*sums
      sums = spin(x,up) + spin(right,y)
      do y = 1,L
         ! periodic boundary conditions
         if (y == L) then
            up = 1
         else
            up = y + 1
         end if
      end do
   end do
end do
Ising model: spin flip dynamics

Choose a random spin and flip it: it’s a new configuration (a microstate)

\[
\text{do ispin} = 1, N \\
! \text{random x and y coordinates for trial spin} \\
\text{call random_number(rnd)} \\
x = \text{int}(L \times \text{rnd}) + 1 \quad 1 \leq x \leq L \\
\text{call random_number(rnd)} \\
y = \text{int}(L \times \text{rnd}) + 1 \quad 1 \leq y \leq L \\
.....
\]

Flip is: \( \text{spin}(x,y) = -\text{spin}(x,y) \)

but do it later, only if you decide to accept the flip (according to Metropolis)
Ising model: energy variations per spin flip

Evolution is driven by the energy change between the old and the new configuration (Metropolis MC).

dE = DeltaE(x,y) \quad \text{energy variation for spin(x,y) flip}

call random_number(rnd)

if (rnd <= w(dE)) then
    spin(x,y) = -spin(x,y)
    accept = accept + 1

function DeltaE(x,y) result (DeltaE_result)

DeltaE_result = 2*spin(x,y)*(left + right + up + down)
function DeltaE(x,y) result (DeltaE_result)
    ! periodic boundary conditions

    if (x == 1) then
        left = spin(L,y)
        right = spin(2,y)
    else if (x == L) then
        left = spin(L-1,y)
        right = spin(1,y)
    else
        left = spin(x-1,y)
        right = spin(x+1,y)
    end if
    if (y == 1) then
        up = spin(x,2)
        down = spin(x,L)
    else if (y == L) then
        up = spin(x,1)
        down = spin(x,L-1)
    else
        up = spin(x,y+1)
        down = spin(x,y-1)
    end if
    DeltaE_result = 2*spin(x,y)*(left + right + up + down)
end function DeltaE
Ising model: storage of Boltzmann’s coeff.

Choosing the interaction parameter $J=1$, possible energy variations per spin flip are $-8,-4,0,+4,+8$:

```plaintext
! do dE = -8,8,4
    w(dE) = exp(-dE/T)
end do
accept = 0
! do i = 1,5
    cum(i) = 0.0_double
end do
```

Convenient to store the Boltzmann’s coefficient for these discrete values of energy variations:

The five possible transitions of the Ising model on the square lattice with spin flip.
Ising model: updating energy and magnetization

subroutine metropolis()
! one Monte Carlo step per spin
......
do ispin = 1,N
......
dE = DeltaE(x,y)
call random_number(rnd)
if (rnd <= w(dE)) then
    spin(x,y) = -spin(x,y)
    accept = accept + 1
    $M = M + 2*spin(x,y)$  ! factor 2 is to account for the variation:
    $E = E + dE$             ! $(-(-)+(+))$
end if
end do
end subroutine metropolis

DO NOT CALCULATE EVERYTHING FROM THE SCRATCH!!

$\Delta E$ is already a variation
Spin flip dynamics: how to choose spin to flip?

Random ...

do ispin = 1,N
  ! random x and y coordinates for trial spin
  call random_number(rnd)
  x = int(L*c*rnd) + 1  \[1 \leq x \leq L\]
  call random_number(rnd)
  y = int(L*c*rnd) + 1  \[1 \leq y \leq L\]
  ....

or ordered (sequential) ...

do x = 1,L
  do y = 1,L
    ...  
    spin(x,y) = -spin(x,y)
Spin flip dynamics: how to choose spin to flip?

- ORDERED: in some cases, it could go more slowly towards equilibrium (see later: correlation time), but it depends...

- NO appreciable differences in the statistics at equilibrium
Measuring physical quantities: how to accumulate data?

subroutine data(cum)
  ! accumulate data after every Monte Carlo step per spin
  real (kind = double), dimension(5), intent (inout) :: cum
  cum(1) = cum(1) + E
  cum(2) = cum(2) + E*E
  cum(3) = cum(3) + M
  cum(4) = cum(4) + M*M
  cum(5) = cum(5) + abs(M)
end subroutine data

After one MC step per spin for all spins:

  do imcs = 1,nmcs
    call metropolis()   contains the loop over all the spins
    call data(cum)
  end do

Alternatively, do it after each MC step per individual spin...
Measuring physical quantities: how to accumulate data?

Further remarks...

- Use statistically INDEPENDENT configurations
- Calculate therefore the CORRELATION TIME by considering the autocorrelation functions:
  \[ C_M(t) = \langle M(t)M(0) \rangle - \langle M \rangle^2, \quad C_E(t) = \langle E(t)E(0) \rangle - \langle E \rangle^2 \]
  \[ (C_M(0) \propto \chi, \quad C_E(0) \propto C_V) \]
  \[ C_M(t) \to 0 \text{ and } C_E(t) \to 0 \text{ exponentially for } t \to \infty \]
  with a certain decay time \( \tau \): consider intervals longer than \( \tau \) for statistical averages
  
  (NOTE: ”critical slowing down” for \( T \to T_C \))
Measuring physical quantities: how to accumulate data?
Further remarks...

- see also CORRELATION LENGTH between magnetic domains, $\zeta(T)$
- close to $T_c$, also the correlation length increases (spin alignments are more correlated), up to divergence
Measuring physical quantities: which errors?

- necessary to give the ERROR ESTIMATE corresponding to the measured physical quantity !!!
- (see Tab. 1 of D.P. Landau, PRB 13, 2997 (1976), “Finite size behavior of the Ising square lattice”)
- do also BLOCKING (called “coarse grained technique” in that paper)
How to do efficiently simulations as a function of $T$?

- Sometimes EQUILIBRATION time is long...
- IDEA: for $T'$ close to $T$, choose as starting point the equilibrated output of $T$
Ising model: size problems

We cannot simulate an INFINITE system!

The temperature dependence of the specific heat $C$ (per spin) of the Ising model

$L = 8$ and $L = 16$
Ising model: size problems

SURFACE EFFECTS:
example of energy for HALF UP/HALF DOWN configurations:

<table>
<thead>
<tr>
<th>L</th>
<th>E</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>-1</td>
</tr>
<tr>
<td>8</td>
<td>-1.5</td>
</tr>
<tr>
<td>16</td>
<td>-1.75</td>
</tr>
<tr>
<td>20</td>
<td>-1.8</td>
</tr>
<tr>
<td>32</td>
<td>-1.875</td>
</tr>
</tbody>
</table>

... for an infinite system: $E = -2$

We have a (“surface”) term proportional to $1/L$
Ising model: alternative dynamics

- in the SPIN FLIP dynamics the order parameter is not conserved ($M$ changes during evolution)
- alternative: NN spin exchange (Kawasaki dyn.) (exchange two NN spins picked at random; $M$ is conserved; this is equivalent to LATTICE GAS MODELS with fixed number of particles)
Ising model: Kawasaki dynamics

Fixed magnetization: change of thermodynamical ensemble

No modification of the equilibrium properties except phase separation
T=10, starting from random configuration

By HeMath - Own work, CC BY-SA 4.0, https://commons.wikimedia.org/w/index.php?curid=37327967
Ising model: other generalizations

• SPINS: XY, Heisenberg, Potts...

• LATTICES: Square, Triangle, Cubic, Honeycomb, Kagome....

• INTERACTIONS: Magn. Field, Antiferrom., Next Nearest Neighbor (NNN)....
Universality and critical exponents

Reduced temperature: \( \Delta T = (T - T_c)/T_c \)

\[ \begin{align*}
C & \sim |\Delta T|^{-\alpha} \\
\langle M \rangle & \sim |\Delta T|^{\beta} \quad \text{for } \Delta T < 0 \\
\chi & \sim |\Delta T|^{-\gamma} \\
\xi & \sim |\Delta T|^{-\nu}
\end{align*} \]
Program:

on
$/home/peressi/comp-phys/IX-ising/
[do: $cp /home/peressi/.../IX-ising/* . ]

ising.f90
Exercise

(a) Choose L=30, T=2, and initially spin=±1 randomly. Calculating and plotting the energy \( < E > /N \) and the magnetization \( < M > /N \) per particle as a function of Metropolis-MC steps, how much time (i.e. how many \( nequil \) MC steps) is it necessary to equilibrate the system?

Hint:
- Since initially spin=±1 randomly, \( E/N \) and \( M/N \) initially will be far from the expected equilibrium average value.
First, set \( nequil=0 \) and plot instantaneous values of \( E/N \) and \( M/N \).
Estimate \( nequil \) from that plot!!! Visualization is important!!!!

\( nequil \) of course depends on \( T \) and on the initial situation

Then, set \( nequil \) not zero and calculate the time average \( < E > /N \) and \( < M > /N \);
increasing the total \( nmcs \), the two quantities should converge...
Exercise

(a) Choose $L=30$, $T=2$, and initially spin=$\pm 1$ randomly. ...

Plot a snapshot of the spin pattern: does the system appear ordered or disordered?

it should appear ordered...

```
p 'ising-up.dat' ps 3 pt 7,'ising-down.dat' ps 3 pt 7
```

Plotting “ising-up.dat” and “ising-down.dat” which contain the coordinates of spin up and down respectively, one should get something like that:

![Diagram](image)

Calculate also $c$ and $\chi$. 
Exercise

(a) Choose $L=30, T=2$, and initially spin=$\pm 1$ randomly. ...

Calculate also $c$ and $\chi$.

(b) Choose $T=1$ and repeat (a)...

Phase transitions in magnetics

Phase transitions are observed in surprisingly simple systems, e.g. on a lattice of interacting spins $s_i$ (magnet vectors). Interaction energy of nearest neighbours pair is $E_{ij} = -J (s_i s_j)$.

Total energy $E$ and magnetization $M$ for a spins configuration $\{s_1, s_2, ..., s_n\}$ is obtained by summation throughout the lattice. In the Heisenberg model every spin can take arbitrary direction. In the XY model spins rotate in a plane.

In the Ising model spins have only two possible states $\pm 1$ (up or down). As since every spin takes two values, therefore there are $2^n$ different configurations for $n$ spins. You see below $2^4 = 16$ spin configurations for $2 \times 2$ lattice.

For $J > 0$ the state of lowest energy is when all spins are aligned. The state has macroscopic magnetization, i.e. it is ferromagnetic. The system is degenerate as since several configurations have the same energy. Entropy $S(E)$ is minimal when spins are aligned and it grows with increasing of $E$ (and hence degeneracy).

It is supposed that spins interact too with thermostat at temperature $T$. In thermal equilibrium any system minimizes the $F = E - T S$ value. Therefore at low temperature Ising spins minimize energy. Interaction aligns all spin vectors in the same direction, giving huge total magnetic fields. At high temperature the system maximizes entropy (and disorder). Thermal fluctuations break this order. The randomness of the spin configuration tends to wash out the large scale magnetism.

In the 2D Ising model there is a phase transition at $T_c = 2.269$ from disordered (non-magnetic) to ordered magnetic state (see Fig. 1).
Raw data: traces, covariance and autocorrelation time

Trace: magnetization for $T = 2.27 J/k_B \approx T_C$ (10^5 sweeps)
Magnetization ($10^5$ sweeps)

\[ \frac{\langle |M| \rangle}{\langle N \rangle} \]

- $L=04$ (red)
- $L=08$ (green)
- $L=14$ (blue)
- $L=20$ (magenta)
- $L=30$ (cyan)
- $L=40$ (black)

$k_B T / J$ vs. $\frac{\langle |M| \rangle}{\langle N \rangle}$
Energy \((10^5\) sweeps\)
Magnetic susceptibility ($10^5$ sweeps)
Magnetic susceptibility near $T_c$ ($10^6$ sweeps)
Specific heat ($10^5$ sweeps)

![Graph showing specific heat vs. k_B T/J for different system sizes L=4, 8, 14, 20, 30, 40. The graph plots $c_v k_B/J^2 = L^2 (\langle E^2 \rangle - \langle E \rangle^2) / (k_B T^2)$ against $k_B T / J$. The x-axis represents $k_B T / J$ ranging from 1 to 5, and the y-axis represents $c_v k_B/J^2$ ranging from 0 to 2.2. Each curve corresponds to a different system size, with the legend indicating L=4 (red), L=8 (green), L=14 (blue), L=20 (purple), L=30 (cyan), and L=40 (black).]
Specific heat near $T_c$ ($10^6$ sweeps)

\[ c_V \frac{k_B}{J^2} = L^2 \left( \frac{\langle E^2 \rangle - \langle E \rangle^2}{k_B T^2} \right) \]

- $L=04$ (red)
- $L=08$ (green)
- $L=14$ (blue)
- $L=20$ (purple)
- $L=30$ (cyan)
- $L=40$ (brown)

$k_B T / J$ ranges from 2.2 to 2.4.