So far we have performed our Ising model simulations on the square lattice. How do the critical temperature and the critical exponents depend on the symmetry and the dimension of the lattice? Based on your experience with the percolation transition in Chapter 13, you might have a good idea what the answer is.

from Gould-Tobochnik

Problem 17.11. The effects of symmetry and dimension on the critical properties of the Ising model

a. The nature of the triangular lattice is discussed in Chapter 8 (see Fig. 8.5). The main difference between the triangular lattice and the square lattice is the number of nearest neighbors. Make the necessary modifications in your Ising program, for example, determine the possible transitions and the values of the transition probabilities. Compute C and χ for different values of T in the interval $[1, 5]$. Assume that $\nu = 1$ and use finite size scaling to estimate T_c in the limit of an infinite triangular lattice. Compare your estimate of T_c to the known value $kT_c/J = 3.641$ (to three decimal places). The simulation of Ising models on the triangular lattice is relevant to the understanding of the experimentally observed phases of materials that can be absorbed on the surface of graphite.

b. No exact results are available for the Ising model in three dimensions. Write a Monte Carlo program to simulate the Ising model on the simple cubic lattice (six nearest neighbors). Compute C and χ for T in the range $3.2 \leq T \leq 5$ in steps of 0.2 for different values of L. Estimate $T_c(L)$ from the maximum of C and χ. How do these estimates of $T_c(L)$ compare? Use the values of $T_c(L)$ that exhibit a stronger L dependence and plot $T_c(L)$ versus $L^{-1/\nu}$ for different values of ν in the range 0.5 to 1 (see (17.27)). Show that the extrapolated value of $T_c(L = \infty)$ does not depend sensitively on the value of ν. Compare your estimate for $T_c(L = \infty)$ to the known value $kT_c/J = 4.5108$ (to four decimal places).

c. Compute $|m|$, C, and χ at $T = T_c \approx 4.5108$ for different values of L on a simple cubic lattice. Do a finite size scaling analysis to estimate β/ν, α/ν, and γ/ν. The best known values of the critical exponents for the three-dimensional Ising model are given in Table 13.1. For comparison, published Monte Carlo results in 1976 for the finite size behavior of the Ising model on the simple cubic Ising lattice are for $L = 6$ to $L = 20$; 2000–5000 Monte Carlo steps per spin were used for calculating the averages after equilibrium had been reached.

Problem 17.12. Critical slowing down

a. Consider the two-dimensional Ising model on a square lattice with $L = 16$. Compute $C_M(t)$ and $C_E(t)$ and determine the correlation times τ_M and τ_E for $T = 2.5, 2.4,$ and 2.3. Determine the correlation times as discussed in Problem 17.7d. How do these correlation times compare with one another? Show that τ increases as the critical temperature is approached, a physical effect known as *critical slowing down.*

b. We can define the dynamical critical exponent z by the relation

$$\tau \sim \xi^z. \tag{17.33}$$

On a finite lattice we have the relation $\tau \sim L^z$ at $T = T_c$. Compute τ for different values of L at $T = T_c$ and make a very rough estimate of z. (The value of z for the two-dimensional Ising model with spin flip dynamics is still not definitely known, but appears to be slightly greater than 2.)