1. Iterate (6.25) for $a = 1.4$ and $b = 0.3$ and plot 10^4 iterations starting from $x_0 = 0, y_0 = 0$. Make sure you compute the new value of y using the old value of x and not the new value of x. Do not plot the initial transient. Choose the `SET window` statement so that all values of the trajectory within the box drawn by the statement `BOX LINES -1.5,1.5,-0.45,0.45` are plotted. Make a similar plot beginning from the second initial condition, $x_0 = 0.63135448, y_0 = 0.18940634$. Compare the shape of the two plots. Is the shape of the two curves independent of the initial conditions?

2. Increase the scale of your plot so that all points with the box drawn by the statement `BOX LINES 0.50,0.75,0.15,0.21` are shown. Begin from the second initial condition and increase the number of computed points to 10^5. Then make another plot showing all points within the box drawn by `BOX LINES 0.62,0.64,0.185,0.191`. If patience permits, make an additional enlargement and plot all points within the box drawn by `BOX LINES 0.6305,0.6325,0.1889,0.1895`. (You have to increase the number of computed points to order 10^6.) What is the structure of the curves within each box? Does the attractor appear to have a similar structure on smaller and smaller length scales? Is there a region in the plane from which the points cannot escape? The region of points that do not escape is the basin of the Hénon attractor. The attractor itself is the set of points to which all points in the basin are attracted. That is, two trajectories that begin from different conditions will soon lie on the attractor. We will find in Section **??** that the Hénon attractor is an example of a strange attractor.

3. Determine if the system is chaotic, that is, sensitive to initial conditions. Start two points very close to each other and watch their trajectories for a fixed time. Choose different colors for the two trajectories.

4. It is straightforward in principle to extend the method for computing the Lyapunov exponent that we used for a one-dimensional map to higher-dimensional maps. The idea is to linearize the difference (or differential) equations and replace dx_n by the corresponding vector quantity $d\mathbf{r}_n$. This generalization yields the Lyapunov exponent corresponding to the divergence along the fastest growing direction. If a system has f degrees of freedom, it has a set of f Lyapunov exponents. A method for computing all f exponents is discussed in Project 6.22.

One of the earliest indications of chaotic behavior was in an atmospheric model developed by Lorenz. His goal was to describe the motion of a fluid layer that is heated from below. The result is convective rolls, where the warm fluid at the bottom rises, cools off at the top, and then falls down later. Lorenz simplified the description by restricting the motion to two spatial dimensions. This situation has been modeled experimentally in the laboratory and is known as a Rayleigh-Benard cell. The equations that Lorenz obtained are

$$\frac{dx}{dt} = -\sigma x + \sigma y \tag{6.26a}$$

$$\frac{dy}{dt} = -xz + rx - y \tag{6.26b}$$

$$\frac{dz}{dt} = xy - bz, \tag{6.26c}$$

where x is a measure of the fluid flow velocity circulating around the cell, y is a measure of the temperature difference between the rising and falling fluid regions, and z is a measure of the difference in the temperature profile between the bottom and the top from the normal equilibrium temperature profile. The dimensionless parameters σ, r, and b are determined by various fluid properties, the size of the Raleigh-Benard cell, and the temperature difference in the cell. Note that the variables x, y, and z have nothing to do with the spatial coordinates, but are measures of the state of the system. Although it is not expected that you will understand the relation of the Lorenz equations to convection, we have included these equations here to reinforce the idea that simple sets of equations can exhibit chaotic behavior.

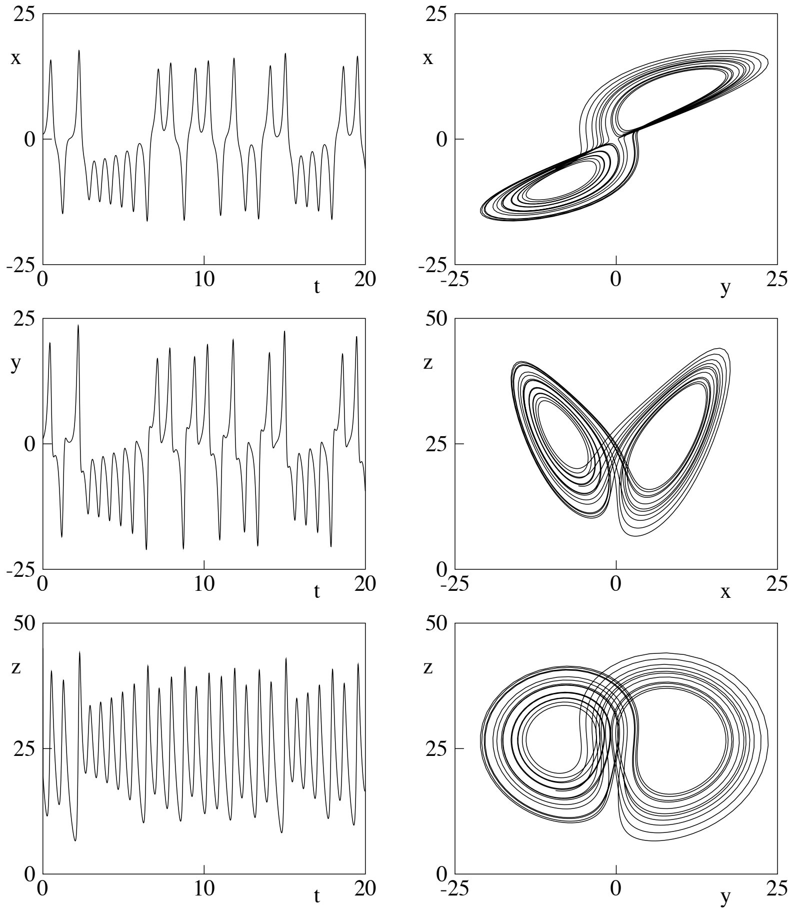

Figure 6.10: A trajectory of the Lorenz model with $\sigma = 10$, $b = 8/3$, and $r = 28$ and the initial condition $x_0 = 1, y_0 = 1, z_0 = 20$. A time interval of $t = 20$ is shown with points plotted at intervals of 0.01. The fourth-order Runge-Kutta algorithm was used with $\Delta t = 0.0025$.

Problem 6.13. The Lorenz Model

1. Use either the simple Euler algorithm or one of the Runge-Kutta methods (see Appendix 5A) to obtain a numerical solution to the Lorenz equations (6.26). Generate plots of x versus y, y versus z, and x versus z as in Figure 6.10 or use a separate graphics program to make three-dimensional plots. Explore the basin of the attractor with $\sigma = 10$, $b = 8/3$, and $r = 28$.

2. Determine qualitatively the sensitivity to initial conditions. Start two points very close to each other and watch their trajectories for approximately 10,000 time steps.

3. Let z_m denote the value of z where z is a relative maximum for the mth time. You can determine the value of z_m by finding the average of the two values of z when the right-hand side of (6.26) changes sign. Plot z_{m+1} versus z_m and describe what you find. This procedure is one way that a continuous system can be mapped onto a discrete map. What is the slope of the z_{m+1} versus z_m curve? Is its magnitude always greater than unity? If so, then this behavior is an indication of chaos. Why?

The application of the Lorenz equations to weather prediction has led to a popular metaphor known as the *butterfly effect*. That is, if the conditions are such that the atmosphere displays chaotic behavior, then even a small effect such as the flapping of a butterfly's wings would make our long-term predictions of the weather meaningless. This metaphor is made even more meaningful by inspection of Figure 6.10.

6.8 Forced Damped Pendulum

We now consider the dynamics of nonlinear mechanical systems described by classical mechanics. The general problem in classical mechanics is the determination of the positions and velocities of a system of particles subjected to certain forces. For example, we considered in Chapter 4 the celestial two-body problem and were able to predict the motion at any time. We will find that we cannot make long-time predictions for the trajectories of nonlinear classical systems when these systems exhibit chaos.

A familiar example of a nonlinear mechanical system is the simple pendulum (see Chapter 5). To make its dynamics more interesting, we assume that there is a linear damping term present and that the pivot is forced to move vertically up and down. Newton's second law for this system is (cf. McLaughlin or Percival and Richards)

$$\frac{d^2\theta}{dt^2} = -\gamma\frac{d\theta}{dt} - [{\omega_0}^2 + 2A\cos\omega t]\sin\theta, \tag{6.27}$$

where θ is the angle the pendulum makes with the vertical axis, γ is the damping coefficient, ${\omega_0}^2 = g/L$ is the natural frequency of the pendulum, and ω and A are the frequency and amplitude of the external force. Note that the effect of the vertical acceleration of the pivot is equivalent to a time-dependent gravitational field.

How do we expect the driven, damped simple pendulum to behave? Because there is damping present, we expect that if there is no external force, the pendulum would come to rest. That is, $(x = 0, v = 0)$ is a stable attractor. As A is increased from zero, this attractor remains stable for sufficiently small A. At a value of A equal to A_c, this attractor becomes unstable. How does the driven nonlinear oscillator behave as we increase the amplitude A? Because we are mainly